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ENSURING A FINITE GROUP IS SUPERSOLUBLE

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A special case of the main result is the following. Let G be a finite, non-supersoluble group in which from arbitrary subsets X, Y of cardinality n we can always find $x \in X$ and $y \in Y$ generating a supersoluble subgroup. Then the order of G is bounded by a function of n. This result is a finite version of one line of development of B.H. Neumann's well-known and much generalised result of 1976 on infinite groups.

1. BACKGROUND

A group is Abelian, of course, if every pair of its elements generates an Abelian subgroup. In [10] Neumann generalised this showing that a group is centre-by-finite if in every infinite subset of it there is a pair of elements that generates an Abelian subgroup. This result has itself been generalised by many authors, although usually proving results that are vacuous in a finite group. The present article gives a finite version of some of the post-Neumann work.

Our motivation includes the following results. Firstly Lennox and Wiegold [7] proved, among other things, that a finitely generated soluble group is finite-by-nilpotent if and only if in every infinite subset of it there is a pair generating a nilpotent subgroup; and Groves [5] showed that this result remains valid when 'nilpotent' is replaced by 'supersoluble'. Spiezia [12] and Longobardi, Mai and Rheumtulla [8] strengthened Neumann's basic hypothesis; and this is used in Edimioni [4] to prove a result having the following as a corollary: a finitely generated soluble group G is nilpotent if, whenever X, Y are infinite subsets of G, there exists $x \in X$ and $y \in Y$ so that $\langle x, y\rangle$ is nilpotent. Earlier Lennox [6] had shown the weaker result that a finitely generated soluble group is nilpotent if *every* two-generator subgroup of it is nilpotent; and subsequently Trabelsi [14] was able to replace Endimioni's 'nilpotent' by 'nilpotent-by-finite'. In [13] Tomkinson showed that, given a positive integer n, a finitely generated soluble group has hypercentre of index bounded by a function of n if every subset of cardinality n of the group contains a pair generating a nilpotent subgroup.

Our main result is Theorem 6 given at the beginning of Section 3. Corollaries of this show that for each of the properties supersolubility, nilpotence and Abelianness there is

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a function f on the positive integers with the following property. Suppose G is a finite group in which, whenever X, Y are subsets of G of cardinality n, there exists $x \in X$ and $y \in Y$ for which $\langle x, y \rangle$ is supersoluble, nilpotent or Abelian: then, whenever $|G| > f(n)$, G is supersoluble, nilpotent or Abelian, as the case may be.

Many articles in the literature build on the work of Lennox and Wiegold [**7**]; a search for references in this area could usefully start with those given in Trabelsi's articles [**14**] and [**15**]. Our notation and terminology is generally that of Doerk and Hawkes [**3**].

2. Preliminary definitions and results

For a class $\mathfrak{Y}$ of finite groups and a positive integer n we define the class of finite groups $\mathfrak{Y}^{[n]}$ as follows. A group G is in $\mathfrak{Y}^{[n]}$ if, whenever X, Y are subsets of cardinality n in G, there exists $x \in X$ and $y \in Y$ for which $\langle x, y \rangle \in \mathfrak{Y}$. This definition is motivated by that of Spiezia [**12**] in the context of infinite groups. We adopt the convention that $\mathfrak{Y}^{[n]}$ contains all groups of order less than n. Then $\mathfrak{Y}^{[n]}$ is S-closed and is Q-closed whenever $\mathfrak{Y}$ is.

In Theorem 6 we show that, for certain classes $\mathfrak{Y}$ and for all positive integers n, $\mathfrak{Y}^{[n]}$ is 'almost' equal to $\mathfrak{Y}$ in the sense that groups in $\mathfrak{Y}^{[n]} \setminus \mathfrak{Y}$ have order bounded by a function of n.

The classes $\mathfrak{Y}$ we consider are formations with two extra properties. Firstly a finite group G is in $\mathfrak{Y}$ whenever every pair of its elements generates a $\mathfrak{Y}$-group: this is the property Doerk and Hawkes call $\mathfrak{G}_2$-completeness, $\mathfrak{G}_2$ being the class of 2-generator groups ([**3**, p. 516]). Secondly there is a unique Sylow p-subgroup in an $\mathfrak{Y}$-group when p is the largest prime dividing its order. That is $\mathfrak{Y}$ is contained in the class $\mathfrak{T}_>$ of Sylow tower groups with the inverse order on the set of primes ([**3**, pp. 358–359]). A formation $\mathfrak{Y}$ with these two, extra, properties we shall term a *star* class. The classes $\mathfrak{A}$, $\mathfrak{N}$, of finite Abelian and finite nilpotent groups respectively, are star classes, being formations satisfying the extra properties. $\mathfrak{U}$, the class of finite supersoluble groups, is a star class by a result of Carter, Fischer and Hawkes [**2**]. (In [**2**] groups are soluble; however the proof of Corollary 2 below shows that a finite group is necessarily soluble if every pair of its elements generates a supersoluble subgroup.) A proof by induction using another result from [**2**] – see [**3**, 6.15 on p. 523] – shows that product classes $\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\dots\mathfrak{S}_{p_r}$ are $\mathfrak{G}_2$-complete, so are star classes whenever the sequence of primes (p_i) is decreasing. that the minimal simple groups are all two-generator.

In what follows we shall appeal often to the lemmas of this section. Throughout ϕ denotes the Euler totient, function and $o(x)$ is the order of the element x of a group.

Lemma 1. *Let $\mathfrak{X}$ be a star class, G a group in $\mathfrak{X}^{[n]}$ and $x, y \in G$ with $\phi(o(x)) \geqslant n$. Then $\langle x, y \rangle \in \mathfrak{X}$.*

Proof: There are in $\langle x \rangle$ distinct generators $x_1, x_2, \dots, x_n$ and the elements

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$x_i y$ $(1 \leqslant i \leqslant n)$ are also distinct. Because $G \in \mathfrak{X}^{[n]}$ there are integers i, j for which $\mathfrak{X} \ni \langle x_i, x_j y \rangle = \langle x, y \rangle$. $\square$

COROLLARY 2. *Let $\mathfrak{X}$ be a star class and suppose that $G \in \mathfrak{X}^{[n]}$. If p is the largest prime dividing $|G|$ and if $p > n$ then G has unique Sylow p-subgroup.*

PROOF: Let $x, y \in G$ be p-elements. Then, since $\phi(o(x)) \geqslant p - 1 \geqslant n$, $H := \langle x, y \rangle \in \mathfrak{X}$ by Lemma 1. But p is the largest prime dividing $|H|$ so $x, y \in O_p(H)$. Consequently xy is a p-element of G. It follows that the set of p-elements of G is a subgroup, so G has a unique Sylow p-subgroup. $\square$

LEMMA 3. *Suppose that $\mathfrak{X}$ is a star class, that $G \in \mathfrak{X}^{[n]}$ and that $N \trianglelefteq G$ with $|N| > n$. Then $G/N \in \mathfrak{X}$.*

PROOF: Choose elements $g_1 N, g_2 N \in G/N$. Since $|g_1 N| = |g_2 N| > n$ it follows that, for some elements $n_1, n_2 \in N$, $K := \langle g_1 n_1, g_2 n_2 \rangle \in \mathfrak{X}$. Hence, in G/N,

$$\langle g_1 N, g_2 N \rangle = KN/N \cong K/K \cap N \in \mathfrak{X}$$

because $\mathfrak{X}$ is Q-closed. As $\mathfrak{X}$ is a star class, therefore, $G/N \in \mathfrak{X}$. $\square$

The next result enables us in Section 3 to reduce to the case of soluble groups; it relies on the classification of finite simple groups.

PROPOSITION 4. *Let $\mathfrak{X}$ be a star class of soluble groups and n a positive integer. The number of isomorphism classes of insoluble groups in $\mathfrak{X}^{[n]}$ is bounded.*

PROOF: Let $G \in \mathfrak{X}^{[n]} \setminus \mathfrak{S}$, where $\mathfrak{S}$ is the class of finite soluble groups, and suppose that $|G| > n(n!)$. Define $N := G^{\mathfrak{X}}$ and note that $N \neq \{1\}$ so that there is a chief factor N/M of G. It follows from Lemma 3 that $|M| \leqslant n$. Write $C := C_G(M)$. Then, from $|G| = |C| \cdot |G : C| \leqslant |C| \cdot n!$, we deduce that $|C| > n$. By Lemma 3 again, $G/C \in \mathfrak{X}$ so $N \leqslant C$. In particular $M \leqslant Z(N)$ so M is Abelian. Both M and G/N are soluble so N/M is insoluble. Hence $N/M = S_1 \times S_2 \times \cdots \times S_t$ where $S_i \cong S_1$ $(1 \leqslant i \leqslant t)$ and S_1 is a non-Abelian simple group.

Let $C_0 := C_G(N/M)$ and note that $C_0 \cap N = M$. G/C_0 is insoluble, so not in $\mathfrak{X}$. Hence, by Lemma 3, $|C_0| \leqslant n$. If $t > 1$, Lemma 3 shows that $|N/M| \leqslant n^2$ so $|G| = |C_0| \cdot |G/C_0| \leqslant n \cdot (n^2)!$. In the case $t = 1$ we invoke Corollary 2 to conclude that all primes dividing $|S_1|$ are less than n. By the classification of finite simple groups there is a number $s(n)$ bounding the order of such groups. Then $|G| = |C_0| \cdot |G/C_0| \leqslant n \cdot s(n)!$ and it follows that

$$|G| \leqslant \max\{n \cdot (n^2)!, n \cdot s(n)!\}.$$

This shows that the orders of insoluble groups in $\mathfrak{X}^{[n]}$ are bounded, establishing the Proposition. $\square$

It will be convenient to have an explicit bound for a natural number in terms of its Euler function value.

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Lemma 5. *For every positive integer m, $m \leqslant 2\phi(m)^2$.*

Proof: Suppose $m = \prod_{i=1}^{r} p_i^{\alpha_i}$ where the p_is are distinct primes in increasing order and with $\alpha_i \geqslant 1$ $(1 \leqslant i \leqslant r)$. Then, on the one hand,

$$\phi(m) = m\prod_{i=1}^{r}\Bigl(1 - \frac{1}{p_i}\Bigr) \geqslant m\Bigl(1 - \frac{1}{2}\Bigr)^r = \frac{m}{2^r},$$

so $m \leqslant 2^r\phi(m)$; and on the other

$$\phi(m) = \prod_{i=1}^{r}\phi(p_i^{\alpha_i}) = \prod_{i=1}^{r}p_i^{\alpha_i-1}(p_i - 1) \geqslant 2^{r-1},$$

giving $m \leqslant 2\phi(m)^2$.

3. The main theorem

Theorem 6. *Let $\mathfrak{X}$ be a subgroup closed and saturated star class of groups contained in $\mathfrak{N}\mathfrak{A}$. There is a function $f_{\mathfrak{X}} : \mathbb{N} \to \mathbb{N}$ for which groups in $\mathfrak{X}^{[n]} \setminus \mathfrak{X}$ have order at most $f_{\mathfrak{X}}(n)$.*

The classes $\mathfrak{N}$, $\mathfrak{U}$, of nilpotent and supersoluble groups respectively, satisfy the hypotheses of this theorem so we have the following corollaries immediately.

Corollary 7. *There is a function $f_{nilp} : \mathbb{N} \to \mathbb{N}$ for which groups in $\mathfrak{N}^{[n]} \setminus \mathfrak{N}$ have order at most $f_{nilp}(n)$.*

For soluble groups this is consistent with Tomkinson's result **[13]** in the following sense. Every subset of $2n$ elements of a group in $\mathfrak{N}^{[n]}$ has a pair that generates a nilpotent subgroup so, by **[13]**, soluble groups in $\mathfrak{N}^{[n]}$ have hypercentre of bounded index; and Corollary 7 ensures this.

Corollary 8. *There is a function $f_{ssol} : \mathbb{N} \to \mathbb{N}$ for which groups in $\mathfrak{U}^{[n]} \setminus \mathfrak{U}$ have order at most $f_{ssol}(n)$.*

The class $\mathfrak{A}$ is not saturated so does not satisfy the the hypotheses of Theorem 6. Nevertheless a similar result holds for it.

Corollary 9. *There is a function $f_{ab} : \mathbb{N} \to \mathbb{N}$ for which groups in $\mathfrak{A}^{[n]} \setminus \mathfrak{A}$ have order at most $f_{ab}(n)$.*

We now begin the proof of the Theorem.

Proposition 4 reduces the question to showing that, for each $n \geqslant 1$, soluble groups in $\mathfrak{X}^{[n]} \setminus \mathfrak{X}$ have bounded order. We suppose the theorem to be false and derive a contradiction. That is, we suppose that, for some positive integer n, there are groups of arbitrarily large order in $\mathfrak{H}_n := (\mathfrak{X}^{[n]} \setminus \mathfrak{X}) \cap \mathfrak{S}$.

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Our first step is to show that $\mathfrak{H}_n$ contains arbitrarily large primitive groups. To this end let ξ be an arbitrary natural number and let $G \in \mathfrak{H}_n$ have order at least $n\xi$. Write $N := G^{\mathfrak{X}}$; it is non-trivial so there is a chief factor N/M of G. Now $\mathfrak{X}$ is saturated so $N/M \not\leqslant \Phi(G/M)$ and N/M has a complement C in G; that is $G = CN$, $C \cap N = M$. C is a maximal subgroup of G as N/M is an Abelian chief factor of G. The factor group $G/\mathrm{core}_G(C)$ is primitive with a stabiliser $C/\mathrm{core}_G(C)$, and kernel (that is, unique minimal normal subgroup) $N\mathrm{core}_G(C)/\mathrm{core}_G(C)$. The latter is the (non-trivial) $\mathfrak{X}$-residual of $G/\mathrm{core}_G(C)$. By Lemma 3, $|\mathrm{core}_G(C)| \leqslant n$ so that $|G/\mathrm{core}_G(C)| \geqslant \xi$. That is, writing $\mathfrak{P}_n$ for the class of primitive groups in $\mathfrak{H}_n$ with stabilisers in $\mathfrak{X}$,

(1) there are arbitrarily large groups in $\mathfrak{P}_n$.

We observe that $\mathfrak{P}_n$ contains groups with arbitrarily large kernels because, if $G = KU \in \mathfrak{P}_n$ where K is a stabiliser and U is the kernel, $|K| \leqslant |U|!$.

Using the same notation suppose that $G = KU \in \mathfrak{P}_n$ is a Frobenius group with K generated by two elements, a, b say. If $|U| > 2n$ there are disjoint subsets U_1, U_2 of U of cardinality n and $|a^{U_1}| = |b^{U_2}| = n$. Then, as $G \in \mathfrak{H}_n$, for some $u_1 \in U_1$ and $u_2 \in U_2$, $H_0 := \langle a^{u_1}, b^{u_2} \rangle \in \mathfrak{X}$. Note that $H_0U = G$ so $H_0 \cap U \trianglelefteq G$. Since U is minimal normal in G it follows that either $H_0 = G$ or $H_0 \cap U = \{1\}$. But $G \notin \mathfrak{X}$ so $H_0 \cap U = \{1\}$. Now $H_0 \cap K^{u_1} \neq \{1\} \neq H_0 \cap K^{u_2}$ so, since G is Frobenius, $K^{u_1} = K^{u_2}$ thus $u_1 u_2^{-1} \in U \cap N_G(K) = \{1\}$ contradicting that $u_1 \neq u_2$. That is $|U|$, and therefore $|G|$, is bounded. In particular groups in $\mathfrak{P}_n$ with Abelian, and that is cyclic, stabilisers are Frobenius so have bounded orders.

Now let ξ be arbitrary and suppose that $H := KU \in \mathfrak{P}_n$ where K is a non-Abelian stabiliser, and U the kernel of order greater than $n\xi$. We denote by p, a prime, the exponent of U; then $O_p(K) = \{1\}$ and, since K is nilpotent-by-Abelian, K' is a p'-group. Let L be a minimally non-Abelian subgroup of K. Note that $J := LU \in \mathfrak{H}_n$. Since L' acts faithfully by conjugation on U, $C_U(L') \leqslant U_1 < U$ where U/U_1 is a chief factor of J and L' has no non-identity fixed points in U/U_1 so $[L', U]U_1 = U$. We have $J/U_1 \notin \mathfrak{X}$ or else $J/U_1 \in \mathfrak{NA}$ which leads to $[L', U]U_1 = [L', U, L']U_1 = U_1$, a contradiction. It follows from Lemma 3 that $|U_1| \leqslant n$ and hence $|U/U_1| > \xi$. Also LU_1/U_1 is maximal in J/U_1; its core intersects $L'U/U_1$ trivially; and modulo its core it is minimally non-Abelian. It follows that J has a primitive factor group in $\mathfrak{P}_n$ with kernel U/U_1 and minimally non-Abelian stabilisers. Let $\mathfrak{P}_n^*$ be the subclass of $\mathfrak{P}_n$ of non-Frobenius groups with minimally non-Abelian stabilisers. Since a minimally non-Abelian group is 2-generator the upshot of (1), this paragraph and the last is that

(2) there are groups in $\mathfrak{P}_n^*$ with arbitrarily large kernels.

Our aim now is to show that, on the contrary, the groups in $\mathfrak{P}_n^*$ do have uniformly bounded orders, thus contradicting our assumption that the theorem is wrong. To this

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end let $G = KU \in \mathfrak{P}_n^*$ using the usual notation. We first show that

(3) the exponent of U is at most n.

Let p be the exponent of U. If $p \mid |K|$ then, since $O_p(K) = \{1\}$, G has no normal Sylow p-subgroup so, by Corollary 2, $p \leqslant n$. If, on the other hand, $p \nmid |K|$, there is a p'-element $a \in L \setminus \{1\}$ and an element $u \in U \setminus \{1\}$ for which $au = ua$. Because non-identity central elements of K have no fixed points in U, $a \notin Z(K)$ so, for some $b \in K$, $\langle a, b\rangle = K$. Then $\langle au, b\rangle = \langle a, u, b\rangle = G \notin \mathfrak{X}$. Hence, by Lemma 1,

$$n > \phi\big(o(au)\big) = \phi\big(o(a)o(u)\big) = \phi\big(o(a)\big)\phi\big(o(u)\big) \geqslant \phi\big(o(u)\big) = p - 1$$

so, in both cases, $p \leqslant n$ as claimed in (3). We note for future reference that much the same argument as in the last sentence, together with Lemma 5, shows that $o(b) \leqslant 2n^2$.

Suppose that K has a unique maximal normal subgroup, K_0 say; $|K : K_0|$ is prime since K is soluble. K is not nilpotent because it is not cyclic. Minimality means that $|K_0|$ and $|K : K_0|$ are both prime and each, by Corollary 2, is at most n, so $|K| \leqslant n^2$. Now $\mathbb{F}_pK$, the regular K-module over the field of p elements, contains a section isomorphic to U, so $|G| \leqslant p^{n^2} \cdot n^2 \leqslant n^{n^2+2}$, using (3).

If, on the other hand, K has different maximal normal subgroups K_1, K_2 then

$$K = K_1K_2, K' = [K_1, K_2] \leqslant K_1 \cap K_2 \leqslant Z(K)$$

and K is nilpotent of class 2. By minimality K is a q-group for some prime $q \neq p$; and $q \leqslant n$ by Corollary 2. Also $Z(K)$ is cyclic and no non-identity element of it centralises a non-identity element of U as U is faithful and irreducible for the conjugation action of K. Because G is not Frobenius there exists $a \in K \setminus \{1\}$ and $u \in U \setminus \{1\}$ for which $au = ua$; and, as no non-identity power of a is central, we may suppose that $o(a) = q$. As above, for some $b \in K$, $\langle a, b\rangle$ is not Abelian and therefore it is K. However $1 = [a^q, b] = [a, b]^q$ meaning that K' has order q. The sentence ending the penultimate paragraph shows that $o(b) \leqslant 2n^2$. Consequently $|K/K'| \leqslant 2n^2 \cdot q \leqslant 2n^3$ so $|K| \leqslant 2n^3 \cdot q = 2n^4$; and then, as in the last paragraph, $|G|$ is functionally bounded.

The last two paragraphs show that there is a uniform bound on the orders of the groups in $\mathfrak{P}_n^*$. This contradicts (2) and with it the assumption that there are in $\mathfrak{X}^{[n]} \setminus \mathfrak{X}$ groups of unbounded order. The proof of Theorem 6 is therefore complete. ∎

PROOF OF COROLLARY 9: Since $\mathfrak{A} \subseteq \mathfrak{N}$ it follows from Corollary 7 that a group G in $\mathfrak{A}^{[n]}$ of order greater then $f_{nilp}(n)$ is nilpotent and therefore the direct product of its Sylow subgroups, one of which, a q-subgroup Q, say, is non-Abelian. Let N be a normal subgroup of G maximal with respect to not containing Q'. Then G/N is non-Abelian so, by Lemma 3, $|N| \leqslant n$. Also $Q'N/N$ is the unique minimal normal subgroup of $H := G/N$ which therefore has class 2 and cyclic centre. Moreover $H \in \mathfrak{A}^{[n]} \setminus \mathfrak{A}$.

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It suffices, therefore, to show that groups H of class 2 and with cyclic centre in $\mathfrak{A}^{[n]}$ have bounded order. Now H is a central product of non-Abelian two-generator groups $H_i = \langle a_i, b_i\rangle$ ($1 \leqslant i \leqslant m$) of class 2 with cyclic centre and, possibly, a cyclic group (see [**1**, Theorem 2.1]). By Lemma 1 an element h with $\phi(o(h)) \geqslant n$ is central so each H_i has bounded exponent and therefore bounded order. Moreover with $b := b_1 b_2 \dots b_m$ and $A := \{a_1, a_2, \dots, a_m\}$, $|A| = |Ab| = m$ so, if $m \geqslant n$ then, for some j, k, $1 = [a_j, a_k b] = [a_j, b_j]$ a contradiction. It follows that H is a central product of a group H_0 of bounded order and a cyclic group $C = \langle c\rangle$. If the result claimed is false then there are such groups H in $\mathfrak{A}^{[n]}$ with arbitrarily large C. If $o(c) > o(a_1)$ then $o(a_1 c) = o(c)$ so $\phi(o(c)) > n$ would mean, by Lemma 1, that $a_1 c$, and therefore a_1, were central, a contradiction. Hence, by Lemma 5, $|H|$ is bounded. □

4. Final comments

Our definition of $\mathfrak{Y}^{[n]}$ is a one-parameter version of a two-parameter definition noted by Neumann [**11**]. He defines a class we might write as $\mathfrak{Y}^{[m,n]}$ consisting of those groups in which, whenever X, Y are subsets of cardinalities m, n respectively, there is an $x \in X$ and a $y \in Y$ such that $\langle x, y\rangle \in \mathfrak{Y}$.

Direct proofs may be given for Corollaries 7 and 9 independently of our main theorem and not depending on the classification of finite simple groups. By way of example a sketch of a direct proof of Corollary 9 goes like this. If the result is false then there is a smallest $n \geqslant 2$ for which $\mathfrak{A}^{[n]} \setminus \mathfrak{A}$ contains a non-Abelian group G of order greater than $f_{ab}(n-1)$. There are subsets X, Y of cardinality $n-1$ in G for which no $x \in X$ commutes with a $y \in Y$. Observe that $o(x), o(y)$ are bounded by $2n^2$ for $x \in X$ and $y \in Y$ by Lemmas 1 and 5. Either every $g \in G \setminus Y$ commutes with some element of X, in which case

$$G = \bigcup_{y \in Y} \langle y\rangle \cup \bigcup_{x \in X} C_G(x);$$

or, for some $y' \in G \setminus Y$, y' commutes with no element of X, and then

$$G = \bigcup_{x \in X} \langle x\rangle \cup \bigcup_{y \in Y} C_G(y) \cup C_G(y').$$

In the first of these unions not every $\langle y\rangle$, and in the second not every $\langle x\rangle$, is omissible. That is G is an irredundant union of at most $2n-1$ subgroups whose intersection has order at most $2n^2$. The theorem of Neumann [**9**] then shows that $|G|$ is bounded by a function of n.

A proof of Corollary 7 in this style is somewhat more complicated.

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