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THE BURKILL APPROXIMATELY CONTINUOUS INTEGRAL

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Abstract

This paper defines descriptive, Riemann, and constructive integrals equivalent to the approximately continuous integral of Burkill.

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1. Introduction

The simplest and most natural integral that integrates finite approximate derivatives is that of Burkill, [4]. However except for an important work of Tolstov, [25], it has not received much attention, in contrast to some fairly extensive investigations of other approximately continuous integrals; see Bullen, [3], for details and references. In this paper several alternative definitions of this Perron integral will be given; a descriptive integral, a totalization process, and a Riemann-like integral that has been suggested by Henstock, [6–8].

2. The Burkill integral and its basic properties

DEFINITION 1. (a) Let $f\colon [a, b] \to \overline{\mathbf{R}}$; then M is a major function of f, $M \in M_f^{\#}$, if and only if $M\colon [a, b] \to \mathbf{R}$ and:

(i) M is approximately continuous, $M \in C_{ap}$;

[This work was done while the author was a visiting professor at the University of Melbourne.]

(ii) $M(a) = 0$;

(iii) $lM'_{ap} > -\infty$ n.e. (except on a countable set);

(iv) $lM'_{ap} \geqslant f$ a.e.

(b) m is a minor function of f, $m \in M_{\#,f}$, if and only if $-m \in M^{\#}_{-f}$.

(c) F is P^*_{ap}-integrable, $f \in P^*_{ap}$ if and only if

$$-\infty < \sup\{t;\, t = m(b),\, m \in M_{\#,f}\} = \inf\{t;\, t = M(b),\, M \in M^{\#}_f\} < \infty,$$

when the common value will be written $\int_a^f f$.

REMARKS. (1) In case of ambiguity we will talk about P^*_{ap}-major functions on $[a, b]$, and so on.

(2) Clearly if $f \in P^*_{ap}$ then $M^{\#}_f \neq \varnothing$, $M_{\#,f} \neq \varnothing$.

LEMMA 2. (a) *If $M \in M^{\#}_f$ then M is measurable, $M \in l[ACG]$, M'_{ap} exists, finite, a.e.*

(b) *If $M \in M^{\#}_f$ and $m \in M_{\#,f}$ then $M - m$ is non-negative, increasing, continuous and differentiable a.e.*

(c) *If $M^{\#}_f \neq \varnothing$ then $f < \infty$ a.e.*

(d) *If $f \in P^*_{ap}$ then f is finite a.e.*

PROOF. (a) follows from results due to Ridder, [17, 18], while (b) follows from a result of Tolstov, [24], and O'Malley, [15], Sunouchi and Utagawa, [22]. (c), (d) are easy consequences of Definition 1.

REMARKS. (1) A function is $l[ACG]$ when $[a, b]$ is a countable union of closed sets on each of which it is lower absolutely continuous (see Ridder, [17–18]).

(2) The basic properties of the integral follow in the usual way; see, for instance, Burkill, [4]. In particular if $f \in P^*_{ap}$ then the P^*_{ap}-primitive, $F(x) = \int_a^x f$, $a \leqslant x \leqslant b$, is well-defined.

THEOREM 3. (a) *If $f \in P^*_{ap}$, $M \in M^{\#}_f$, $m \in M_{\#,f}$, $F(x) = \int_a^x f$ then $M - F$ and $F - m$ are non-negative, increasing, continuous and differentiable a.e.*

(b) *If $f \in P^*_{ap}$, $F(x) = \int_a^x f$ then $F \in [ACG]$, $F \in C_{ap}$ and $F'_{ap} = f$ a.e.*

(c) *If $f \in P^*_{ap}$ then f is measurable.*

(d) *If $F \in C_{ap}$ and* (i) *$F'_{ap}(x)$ exists, finite, $x \notin E$, $|E| = 0$,* (ii) *uF'_{ap} and lF'_{ap} are finite n.e., then if*

$$f(x) = F'_{ap}(x), \qquad x \notin E,$$
$$= 0, \qquad x \in E,$$

$f \in P^*_{ap}$.

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(e) *The P^*_{ap}- and the D-integrals are compatible.*

(f) *If $f \in P^*_{ap}[\alpha, \beta]$, for all α, β, $a < \alpha < \beta < b$ and if*

$$\lim_{\substack{\alpha \to a \\ \beta \to b}} \int_\alpha^\beta f$$

exists, with value I say, then $f \in P^*_{ap}[a, b]$ and $\int_a^b f = I$.

(g) *Let $f \in P^*_{ap}$, $F(x) = \int_a^x f$ then for all λ, $0 < \lambda < 1$, P perfect, there exists a closed portion, Q, of P, having, on $[a, b]$, closed contiguous intervals $[a_n, b_n]$, $n \in N$, such that for all $n \in N$ there exists an $E_n \subset [a_n, b_n]$, and an $M > 0$, with $|E_n| \geqslant (1-\lambda)(b_n - a_n)$ and such that for all $x_n \in E_n$, $\Sigma_n |F(x_n) - F(a_n)| < M$ and $\Sigma_n |F(b_n) - F(x_n)| < M$.*

(h) *$D - P^*_{ap} \neq \varnothing$ and $P^*_{ap} - D \neq \varnothing$.*

Proof. (b) is due to Kubota, [9]; (e) is in Kubota, [10]; (f) is a result of Grimshaw, [5]; (g) is due to Tolstov, [25]; the rest either follow easily from Lemma 2, or other parts of Theorem 3, or can be found in these references, or in Burkill, [4].

Definition 1 is not exactly that given in Burkill, [4], and the object of the next lemma is to show that the two definitions give equivalent integrals. Let Definition 1(a) be modified by replacing (iii) and (iv) by:

(iii)^1 $M'_{ap} > -\infty$;

(iv)^1 $M'_{ap} \geqslant f$;

and denote the resulting class of major functions by $M_f^{\#1}$. Clearly $M_f^{\#1} \subset M_f^{\#}$.

Lemma 4. *For all $\varepsilon > 0$, $M \in M_f^{\#}$ there exists $M^1 \in M_f^{\#1}$ such that*

$$M^1(b) \leqslant M(b) + \varepsilon. \tag{1}$$

Proof. (a) Suppose Definition 1(a) is modified by replacing (iii) by:

(iii)^2 $lM'_{ap} > -\infty$, and call the resulting class of major functions $M_f^{\#2}$. We first prove the lemma with 1 replaced by 2.

First suppose that the countable exceptional set in Definition 1(a)(iii) is the singleton $\{c\}$, $a < c < b$ (the cases $c = a$, $c = b$ can be discussed in a similar way).

Let $\varepsilon > 0$, $M \in M_f^{\#}$ and let A be a set of density 1 at c on which M is continuous; choose a_1, b_1 so that $a < a_1 < c < b_1 < b$ and the oscillation of M on $A \cap [a_1, b_1]$ is less than ε. Define ω by

$$\omega(x) = \sup\{t: t = |M(y) - M(c)|, y \in A, |y - c| \leqslant |x - c|\}$$

and let χ be an increasing, differentiable function with $\chi(a) = 0$, $\chi(b) = \varepsilon$, $\chi'(c) = \infty$. Now define

$$\begin{aligned} M^2(x) &= M(x) + \chi(x), \qquad a \leqslant x \leqslant a_1, \\ &= M(x) + \chi(x) + \omega(a_1) - \omega(x), \qquad a_1 \leqslant x \leqslant c, \\ &= M(x) + \chi(c) + \omega(a_1) + \omega(x), \qquad c \leqslant x \leqslant b_1, \\ &= M(x) + \chi(x) + \omega(a_1) + \omega(b_1), \qquad b_1 \leqslant x \leqslant a; \end{aligned}$$

then $M^2 \in M_f^{\#2}$ and (1) holds.

If we let $\Delta = M^2 - M = \chi + \mu$ then the essential properties of μ are that it is increasing, continuous, $\mu(a) = 0$, $\mu(b) < 2\varepsilon$, and on a set of h having density 1 at the origin

$$M(c+h) - M(c) + \mu(c+h) - \mu(c) \geqslant 0.$$

Now suppose that the countable exceptional set in Definition 1(a)(iii) is c_n, $n \in N$, and for each c_n define a Δ_n, as Δ was defined above, but with ε replaced by $\varepsilon 2^n$; then if $M^2 = M + \Sigma_n \Delta_n$, $M \in M_f^{\#2}$ and (1) holds.

(b) From Lemma 2(a) it follows that (iv), in the definition of $M_f^{\#2}$, can be replaced by

(iv)^2 $M'_{ap} \geqslant f$, a.e.,

without affecting the definition of the integral.

(c) From (b) given $\varepsilon > 0$, $M \in \tilde{M}_f$ there exists $M^2 \in M_f^{\#2}$, satisfying (iv)^2, such that (1) holds. Now let

$$E = \left\{x;\, (M^2)'_{ap}(x) < F(x), \text{ or } (M^2)'_{ap}(x) \text{ does not exist}\right\};$$

then $|E| = 0$. If then $T \in G_\delta$, $E \subset T$, $|T| = 0$ there exists a function $g\colon [a, b] \to \mathbf{R}$ such that (i) $g \in AC$, (ii) g is increasing, (iii) g is differentiable, (iv) $g'(x) = \infty$, $x \in T$, (v) $g'(x) \neq \infty$, $x \notin T$, (vi) $g(a) = 0$, (vii) $g(b) \leqslant \varepsilon$; Zahorski, [27], Tolstov, [26]. Now if $M^1 = M^2 + g$ then $M^1 \in M_f^{\#1}$ and (1) holds.

Remarks. (1) The basic ideas for this lemma can be found in Aleksandrov, [1], Bosanquet, [2] and Grimshaw, [5].

(2) Burkill used the class $M_f^{\#2}$ to define his integral. It should also be remarked that there would be no loss in generality in assuming, in Definition 1, that f is finite, for in any case integrable functions are finite a.e. and if $f_1 = f_2$ a.e. then f_1 and f_2 are either both not integrable, or both integrable with the same integral.

Following Henstock, [6], a definition of Ward type can be given. Suppose Definition 1(a) is modified by replacing (iii) and (iv) by:

(iii)^W For all x, $a \leqslant x \leqslant b$, there exists a set E_x of density 1 at x such that $M(u) - M(v) \geqslant f(x)(u - v)$, $u \leqslant x \leqslant v$, $u, v \in E_x$, and call the resulting class of major functions $WM_f^{\#}$.

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As in Henstock it follows that the integral defined this way, the WP^*_{ap}-integral, is equivalent to the one obtained from Definition 1 in which all the exceptional sets (Definition 1(a), (iii), (iv)) are empty, and the function f finite. Hence from the above discussion this integral of Ward type is equivalent to the P^*_{ap}-integral.

A different sort of variant of Definition 1 has been given by Sunouchi and Utagawa, [22]. In Definition 1(a) replace (1), (iii) and (iv) by:

SU-(i) M is measurable;

SU-(iii) $lM'_{ap} > -\infty$ (that is, $(\text{iii})^2$);

SU-(iv) $lM'_{ap} \geqslant f$.

REMARK. The idea for this generalization is due to Saks, [20], who did the same for the classical Perron integral; he showed that the apparently more general integral was in fact equivalent to the original definition. We shall do the same in the present situation; until then we will call the integral defined this way the SU-P^*_{ap}-integral. In their work, Sunouchi and Utagawa assumed f to be measurable but this is unnecessary as this property of integrable f can be proved (Theorem 3(c)).

3. A Riemann definition

A Riemann definition of an integral equivalent to the Burkill integral is suggested in Henstock, [7, 8], but no details are given.

DEFINITION 1. (a) A collection, Δ, of closed sub-intervals of $[a, b]$ is an approximate full cover of $[a, b]$, an AFC, if and only if for all x, $a \leqslant x \leqslant b$, there exists a measurable set D_x, $x \in D_x$, of density 1 at x, such that if $\alpha \leqslant x \leqslant \beta$, $\alpha, \beta \in D_x$, then $[\alpha, \beta] \in \Delta$.

(b) If Δ is an AFC of $[a, b]$ then a Δ-partition of $[a, b]$ is a $\{a_0, \ldots, a_n; x_1, \ldots, x_n\}$, where $a = a_0 < \cdots < a_n = b$, $a_{i-1} \leqslant x_i \leqslant a_i$, $a_{i-1}, a_i \in D_{x_i}$, $1 \leqslant i \leqslant n$.

LEMMA 2. *If Δ is an AFC of $[a, b]$ and $a \leqslant \alpha < \beta \leqslant b$ then there exists a Δ-partition of $[\alpha, \beta]$.*

PROOF. This is a result of Thomson, [23].

DEFINITION 3. (a) If f: $[a, b] \to r\mathbf{R}$ then f is R^*_{ap}-integrable, $f \in R^*_{ap}$, if and only if there exists I such that for all $\varepsilon > 0$ there exists AFC, Δ, of $[a, b]$, such that for all Δ-partitions $\{a_0,\ldots,a_n;\ x_1,\ldots,x_n\}$ of $[a, b]$ we have that

$$\left| I - \sum_{i=1}^{n} f(x_i)(a_i - a_{i_1}) \right| < \varepsilon,$$

and then $\int_a^b f = I$.

(b) If f: $[a, b] \to \mathbf{R}$ then f is VR^*_{ap}-integrable, $f \in VR^*_{ap}$, if and only if there exists F: $[a, b] \to \mathbf{R}$ such that for all $\varepsilon > 0$ there exists AFC, Δ, of $[a, b]$, and a non-decreasing ϕ: $[a, b] \to \mathbf{R}$, with $\phi(b) - \phi(a) < \varepsilon$, such that for all u, v, $u \leqslant x \leqslant v$, $u, v \in D_x$, we have

$$| F(v) - F(u) - f(x)(v - u) | \leqslant \phi(v) - \phi(u),$$

and then $\int_a^b f = F(b) - F(a)$.

REMARKS. (1) The R^*_{ap}-integral is an example of what Henstock, [6], calls a Riemann complete integral, while the VR^*_{ap}-integral is an example of what he calls a variational integral; see also Kubota, [13, 14].

(2) The basic properties of these integrals follow in the standard manner; in particular we can talk of the R^*_{ap}-primitive, and the function F in (b) above (unique by Theorem 5 below) is the VR^*_{ap}-primitive.

(3) It is also easily seen that if R^* denotes Henstock's Riemann complete integral, that is equivalent to the classical Perron integral, then $R^* \subsetneq R^*_{ap}$.

LEMMA 4. (a) *$f \in R^*_{ap}$, with primitive F, if and only if for all $\varepsilon > 0$ there exists AFC, Δ, of $[a, b]$, such that for all Δ-partitions $\{a_0,\ldots,a_n;\ x_1,\ldots,x_n\}$ of $[a, b]$ we have that*

$$\sum_{i=1}^{n} | F(a_i) - F(a_{i-1}) - f(x_i)(a_i - a_{i-1}) | < \varepsilon.$$

(b) *There is no loss in generality if, in Definition* 3(b), *it is assumed that $\phi \in C_{ap}$.*

PROOF. The proofs are similar to those for the R^*-integral; Henstock, [7; page 33, 41].

THEOREM 5. *$f \in R^*_{ap}$ if and only if $f \in VR^*_{ap}$, and then the integrals are equal.*

PROOF. The proof follows that in Henstock [7; page 40]; see also Kubota [14].

REMARK. If $E \subset [a, b]$, $| E | = 0$ and if

$$\begin{aligned} 1_E(x) &= 1, \qquad x \in E, \\ &= 0, \qquad x \notin E, \end{aligned}$$

then $1_E \in R^*$ and $\int_a^b 1_E = 0$: This can be used, in the usual way, to extend Definition 3 to functions that are finite a.e.

Let Δ be an AFC of $[a, b]$, $\pi = \{a_0, \ldots, a_n; x_1, \ldots, x_n\}$ a Δ-partition of $[a, b]$; following Pfeffer, [16], we will write

$$S(f; a, b; \pi) = \sum_{i=1}^{ns} f(x_i)(a_i - a_{i-1}),$$

$$uS(f; a, b; \Delta) = \sup_{\pi} S(f; a, b; \pi),$$

$$uS(f; a, b) = \inf_{\Delta} uS(f; a, b; \Delta),$$

with analogous definitions of $lS(f; a, b; \Delta)$ and $lS(f; a, b)$.

THEOREM 6. *$f \in R^*_{ap}$ if and only if $-\infty < lS(f; a, b) = uS(f; a, b) < \infty$.*

PROOF. The proof follows that in Pfeffer, [16].

We can now show that the P^*_{ap}- and $SU\text{-}P^*_{ap}$-integrals are equivalent, and are equivalent to the R^*_{ap}-integral.

LEMMA 7. *If $A = \inf\{t; t = M(b), M \in SU\text{-}M_f^{\#}\}$ then $A \geqslant uS(f; a, b)$.*

PROOF. Let us assume $A < uS(f; a, b)$, when there exists $M \in SU\text{-}M_f^{\#}$ such that $M(b) < uS(f; a, b)$.

Given $\varepsilon > 0$, x, $a \leqslant x \leqslant b$, set E_x of density 1 at x such that if $u, v \in E_x$ then

$$M(v) - M(u) \geqslant (f(x) - \varepsilon)(v - u).$$

This defines an AFC, Δ, of $[a, b]$; let $\pi = \{a_0, \ldots, a_n; x, \ldots, x_n\}$ be a Δ-partition of $[a, b]$ and consider

$$S(f, a, b; \pi) = \sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) \leqslant M(b) + \varepsilon(b - u);$$

or

$$uS(f; a, b) \leqslant M(b).$$

COROLLARY 8. $SU\text{-}P^*_{ap} \subset R^*_{ap}$.

PROOF. Immediate from Lemma 7 and Theorem 6.

LEMMA 9. $VR^*_{ap} \subset P^*_{ap}$.

PROOF. Let $f \in VR^*_{ap}$, F, ϕ as given in Definition 3(b), $\phi \in C_{ap}$, by Lemma 4(b); consider

$$M = F + \phi, \qquad m = F - \phi.$$

Then $M \in WM_f^{\#}$, $m \in WM_{\#,f}$ and so $f \in WP^*_{ap}$ and hence $f \in P^*_{ap}$.

COROLLARY 10. (a) $P^*_{ap} = SUP^*_{ap}$. (b) $R^*_{ap} = P^*_{ap}$.

PROOF. Immediate from Corollary 8, Lemma 9 and Theorem 5.

REMARK. The above method can be used to given an alternative proof of Sak's result for the classical Perron integral.

4. A descriptive definition

DEFINITION 1. (a) $F \in AC^*_{ap}$ on a closed set E, $F \in AC^*_{ap}(E)$, if and only if (i) $F \in AC(E)$, (ii) for all λ, $0 < \lambda < 1$, there exists, on each closed contiguous interval of E, $[a_n, b_n]$, a set E_n^λ, and an $M^\lambda > 0$, $|E_n^\lambda| > (1-\lambda)(b_n - a_n)$ such that for all $x_n \in E_n^\lambda$, $\Sigma_{n\in N} |F(x_n) - F(a_n)| < M^\lambda$, and $\Sigma_{n \in N} |F(b_n) - F(x_n)| < M^\lambda$.

(b) $F \in [ACG^*_{ap}]$ on $[a, b]$ if and only if $[a, b] = \bigcup_{n\in N} E_n$, E_n closed and $F \in AC^*_{ap}(E_n)$, $n \in N$.

REMARK. It follows from Solomon's lemma, [1], that Definition 1(b) can be rephrased as:

> $F \in [ACG^*_{ap}]$ on $[a, b]$ if and only if for all λ, $0 < \lambda < 1$, P perfect, there exists a closed portion Q of P, having on $[a, b]$ closed contiguous intervals $[a_n, b_n]$, $n \in N$, such that for all $n \in N$ there exists $E_n^\lambda \subset [a_n, b_n]$, $M^\lambda > 0$, $|E_n^\lambda| > (1-\lambda)(b_n - a_n)$ and such that for all $x_n \in E_n^\lambda$, $\Sigma_{n\in N} |F(x_n) - F(a_n)| < M^\lambda$ and $\Sigma_{n\in N} |F(b_n) - F(\lambda_n)| < M^\lambda$.

We will first obtain some alternative forms of Definition 1(a). Let us define for F: $[a, b] \to \mathbf{R}$ and $A \in [a, b]$

$$\omega(F; A) = \sup\{t;\, t = |F(x) - F(y)|\,,\, x, y \in A\}.$$

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LEMMA 2. *If E is a bounded closed set, with extremities a, b, $a < b$, and closed contiguous intervals in $[a, b]$, $[a_n, b_n]$, $n > 1$, then if $E_n \subset [a_n, b_n]$, $a_n, b_n \in E_n$, $n \geqslant 1$, $E_0 = E \cup \bigcup_{n\geqslant 1} E_n$,*

$$\omega(F; E_0) \leqslant V(F; E) + 2 \sum_{n\geqslant 1} \omega(F; E_n),$$

where $V(F; E)$ is the variation of F on E.

(This is a slight generalization of a result in Saks, [1; page 231].)

LEMMA 3. *If $f \in C_{ap}[a, b]$ then for all λ, $0 < \lambda < 1$, there exists $E^\lambda \subset [a, b]$, $a, b \in E^\lambda$ such that $|E^\lambda| > (1 - \lambda)(b - a)$ and $\omega(F; E^\lambda) < \infty$.*

PROOF. Given $\varepsilon > 0$, $x \in [a, b]$, λ, $0 < \lambda < 1$, there exists $\delta > 0$, $E_x \subset]x - \delta, x + \delta[$ such that if $0 < h < \delta$, $|E_x \cap [x - h, x + h]| > 2(1 - \lambda)h$ and if $u \leqslant x \leqslant v$, $u, v \in E_x$, then $|F(v) - F(u)| < \varepsilon$.

The set of such E_x, $a \leqslant x \leqslant b$, defines an AFC, Δ, of $[a, b]$; let $\{a_0, \ldots, a_p; x_1, \ldots, x_p\}$ be a Δ-partition of $[a, b]$: and define

$$E^\lambda = \bigcup_{k=1}^{p} E_{x_k}.$$

Then $|E^\lambda| > (1 - \lambda)(b - a)$ and if $u, v \in E^\lambda$, $u \in [a_{m-1}, a_m]$, $v \in [a_{n-1}, a_n]$, say,

$$|F(v) - F(u)| \leqslant \sum_{k=m+1}^{n-1} |F(a_k) - F(a_{k-1})| \\ + |F(a_m) - F(u)| + |F(v) - F(a_{n-1})| \leqslant \varepsilon p,$$

which is sufficient to prove the lemma.

THEOREM 4. *$F \in AC^*_{ap}(E)$ if and only if* (a) $F \in AC(E)$, (b) *for all λ, $0 < \lambda < 1$, there exists, on each closed contiguous interval $[a_n, b_n]$ of E, a set E_n^λ, $a_n, b_n \in E_n^\lambda$, $|E_n^\lambda| > (1 - \lambda)(b_n - a_n)$ and $\Sigma_{n\in N}\omega(F; E_n^\lambda) < \infty$.*

PROOF. Let $F \in AC^*_{ap}(E)$, $\tilde{E}_n^\lambda = E_n^\lambda \cup \{a_n, b_n\}$, where E_n^λ are the sets of Definition 1(a)(ii); let $x_n, y_n \in \tilde{E}_n^\lambda$, $n \in N$. Then

$$|F(y_n) - F(x_n)| \leqslant |F(x_n) - F(a_n)| + |F(y_n) - F(b_n)| + |F(a_n) - F(b_n)|;$$

since $F \in AC(E)$, $\Sigma_{n\in N} |F(a_n) - F(b_n)| < \infty$ and the result follows from Definition 1(a)(ii). The converse is immediate.

THEOREM 5. *$F \in AC^*_{ap}(E)$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\alpha_1 < \beta_1 < \cdots < \beta_p$, points of E, if $\sum_{k=1}^p(\beta_k - \alpha_k) < \delta$ then for all λ, $0 < \lambda < 1$, there exists $E_k^\lambda \subset [\alpha_k, \beta_k]$, α_k, $\beta_k \in E_k^\lambda$, $|E_k^\lambda| > (1-\lambda)(\beta_k - \alpha_k)$, $1 \leqslant k \leqslant p$, and $\sum_{k=1}^p \omega(F; E_k^\lambda) < \varepsilon$.*

PROOF. (i) Let $F \in AC^*_{ap}(E)$; then $F \in AC(E)$ and so given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\alpha_1 < \beta_1 < \cdots < \beta_p$, points of E, if $\sum_{k=1}^p(\beta_k - \alpha_k) < \delta$ then $\sum_{k=1}^p V(F; E_n[\alpha_k, \beta_k]) < \varepsilon$. Further, by Theorem 4, and with its notation, there exists n_0 such that $\sum_{n > n_0} \omega(F; E_n^\lambda) < \varepsilon$.

Let $\delta_0 = \min\{\delta; b_n - a_n, n \leqslant n_0\}$ and let $\alpha_1 < \beta_1 < \cdots < \beta_p$, points of E, be such that $\sum_{k=1}^p \beta_k - \alpha_k < \delta_0$. Define

$$\tilde{E}_k^\lambda = E \cap [\alpha_k, \beta_k] \cup \bigcup_{n \in N_k} E_n^\lambda$$

where

$$n_k = \{n; [a_n, b_n] \subset [\alpha_k, \beta_k]\};$$

clearly if $n \in N_k$, then $n > n_0$. By Lemma 2,

$$\omega(F; \tilde{E}_k) \leqslant V(F; E_n[\alpha_k, \beta_k]) + 2 \sum_{n \in N_k} \omega(F; E_n^\lambda).$$

Hence

$$\sum_{k=1}^p \omega(F; \tilde{E}_k) \leqslant 3\varepsilon.$$

(ii) To prove the converse first note that the condition given implies that $F \in AC(E)$. Using the notation of Definition 1(a)(ii) let N_0 be such that if $n > n_0$ then $\sum_{n > n_0}(b_n - a_n) < \delta$: then from the condition given $E_n^\lambda \subset [a_n, b_n]$, a_n, $b_n \in E_n^\lambda$, $|E_n^\lambda| > (1-\lambda)(b_n - a_n)$ and $\sum_{n > n_0} \omega(F; E_n^\lambda) < \varepsilon$.

If $n \leqslant n_0$ divide each $[a_n, b_n]$ into a finite number of intervals each of length less than δ, and we easily see that there exists $E_n^\lambda \subset [a_n, b_n]$, $|E_n^\lambda| > (1-\lambda)(b_n - a_n)$ and $\omega(F; E_n^\lambda) < \infty$. From this it follows that $F \in AC^*_{ap}(E)$.

DEFINITION 6. Let E be a closed set, with closed contiguous integrals $[a_n, b_n]$, $n \in N$; let $x \in E'$, E_x a set of unit density at x such that there exists $\varepsilon > 0$ with a_n, $b_n \in E_x$ if $[a_n, b_n] \subset]x - \frac{1}{2}\varepsilon, x + \frac{1}{2}\varepsilon[$, say if $n \in N_x$ for short; we will write for $F\colon [a, b] \to \mathbf{R}$, a, b the extremities of E,

$$\omega_{n,ap}(F) = \sup_{\alpha, \beta \in E_x \cap [a_n, b_n]} |F(\beta) - F(\alpha)|.$$

THEOREM 7. *If $F \in AC^*_{ap}(E)$ then* (a) $F \in AC(E)$, (b) *for all $x \in E'$, $\sum_{n \in N_x} \omega_{n,ap}(F) < \infty$.*

PROOF. It suffices to prove (b). Since $F \in AC^*_{ap}(E)$, by Theorem 4, for all λ, $0 < \lambda < 1$, there exists n_λ such that

$$\sum_{n > n_\lambda} \omega\left(F; E_n^\lambda\right) \leqslant \frac{1}{2^\lambda}.$$

Let $\varepsilon_\lambda = \min_{n \leqslant n_\lambda}(b_n - a_n)$, $N_{x,\lambda} = \{n; [a_n, b_n] \subset]x - \frac{1}{2}\varepsilon_\lambda, x + \frac{1}{2}\varepsilon_\lambda[\}$ when $\sum_{n \in N_{x,\lambda}} \cdot \omega(F, E_n^\lambda) < 1/2^\lambda$; put $E_x^\lambda = \bigcup_{\lambda \in N_{x,\lambda}} E_n^\lambda$. Now define $E_x^0 = \bigcup_{n \geqslant 1} E_x^{1/2}$, $\varepsilon_0 = \sup_{n \geqslant 1} \varepsilon_{1/n}$, when $E_x^0 \subset]x - \frac{1}{2}\varepsilon_0, x + \frac{1}{2}\varepsilon_0[$; let $N_x = \{n; [a_n, b_n] \subset]x - \frac{1}{2}\varepsilon_0, x + \frac{1}{2}\varepsilon_0[\}$ and finally $E_x = E_x^0 \cup E \cap]x - \frac{1}{2}\varepsilon_0, x + \frac{1}{2}\varepsilon_0[$.

Then E_x has density 1 at x and if $x_n, y_n \in E_x \cap [a_n, b_n]$, $n \in N_x$,

$$\sum_{n \in N_x} |F(x_n) - F(y_n)| = \sum_{m \geqslant 1} \sum_{n \in N_{x,1/m}} |F(x_n) - F(y_n)| \leqslant 1,$$

which completes the proof.

THEOREM 8. *If E is a closed set with extremities a, b, $a < b$, F: $[a, b] \to \mathbf{R}$ and if* (a) $F \in C_{ap}[a, b]$, (b) $F \in AC(E)$, (c) *for all $x \in E'$, $\sum_{n \in N_x} \omega_{n,ap}(F) < \infty$, then $F \in AC^*_{ap}(E)$.*

PROOF. If $x \in E'$ consider $E_x \cap [a_n, b_n]$, $n \in N_x$, then for all λ, $0 < \lambda < 1$, there exists $E_n^\lambda \subset [a_n, b_n]$, a_n, $b_n \in E_n^\lambda$, such that $|E_n^\lambda| > (1 - \lambda)(b_n - a_n)$ and clearly $\omega(F; E_n^\lambda) < \omega_{n,ap}(F)$. The family of $]x - \frac{1}{2}\varepsilon, x + \frac{1}{2}\varepsilon[$ covers E' and so a finite sub-family of these intervals also covers E'. Hence there exists a finite set of integers N_0 such that $\sum_{n > N_0} \omega(F; E_n^\lambda) < \infty$; since $F \in C_{ap}[a, b]$, the intervals $[a_n, b_n]$, $n \in N_0$, can be handled using Lemma 3.

DEFINITION 9. If E is a closed set with extremities a, b, F: $[a, b] \to \mathbf{R}$, then F is lAC^*_{ap} on E, $F \in lAC^*_{ap}(E)$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\alpha_1 < \beta_1 < \cdots < \beta_p$, points of E, if $\sum_{k=1}^p (\beta_k - \alpha_k) < \delta$, then for all λ, $0 < \lambda < 1$, there exists $E_k^\lambda \subset [\alpha_k, \beta_k]$, α_k, $\beta_k \in E_k^\lambda$, $|E_k^\lambda| > (1 - \lambda)(\beta_k - \alpha_k)$ such that for all $x_k \in E_k^\lambda$, $1 \leqslant k \leqslant p$,

$$\sum_{k=1}^p F(x_k) - F(\alpha_k) > -\varepsilon,$$

$$\sum_{k=1}^p F(\beta_k) - F(x_k) > -\varepsilon.$$

REMARKS. (1) An analogous definition can be made for $F \in uAc^*_{ap}(E)$ and, from Theorem 5, $F \in AC^*_{ap}(E)$ if and only if $F \in AC^*_{ap}(E) \cap lAC^*_{ap}(E)$.

(2) Further, as in Definition 1(b), we can now define the classes $u[ACG^*_{ap}]$ and $l[ACG^*_{ap}]$.

THEOREM 10. *If* F: $[a, b] \to \mathbf{R}$, $F \in C_{ap}[a, b]$, $lF'_{ap} > -\infty$ *n.e. then* $F \in l[ACG^*_{ap}]$.

PROOF. Ridder, [19], proves under these conditions that $F \in l[ACG]$; the rest follows from Tolstov's proof of Theorem 1.3(g), Tolstov, [25].

REMARK. The basic lemma in Tolstov, [25], can be used to shorten Ridder's result since it shows that certain sets in Ridder's proof are closed.

COROLLARY 12. *If* F: $[a, b] \to \mathbf{R}$, $F \in C_{ap}[a, b]$, $-\infty < lF'_{ap} \leqslant uF'_{ap} < \infty$, *n.e. then* $F \in [ACG^*_{ap}]$.

We can now define a descriptive integral that will be equivalent to the P^*_{ap}-integral.

DEFINITION 13. *If* f: $[a, b] \to \overline{\mathbf{R}}$ *then* $f \in D^*_{ap}$, f *is* D^*_{ap}*-integrable, if and only if there exists* $F \in C_{ap}[a, b]$, $F \in [ACG^*_{ap}]$ *and* $F'_{ap} = f$ *a.e.; then* $\int_a^x f = F(x) - F(a)$.

REMARK. The basic properties of the class of approximately continuous $-[ACG]$ functions, of which the approximately continuous $-[ACG^*_{ap}]$ functions is a sub-class, Ridder, [18, 19], Kubota, [9], show that this definition is meaningful.

THEOREM 14. *If* $f \in P^*_{ap}$ *then* $f \in D^*_{ap}$, *with integrals equal.*

PROOF. This follows from Theorems 1.3(b), (g), and the remark following Definition 1.

To prove the converse of Theorem 14 we will use the R^*_{ap}-integral and for this need to show that this integral has what are usually called Cauchy and Harnack properties. That the R^*_{ap}-integral has the Cauchy property follows from the fact that the equivalent P^*_{ap}-integral does, Theorem 1.3(f), but we will give an independent proof.

THEOREM 15. *If* $f \in R^*_{ap}[\alpha, \beta]$, *for all* β, β, $a < \alpha < \beta < b$ *and if*

$$\lim_{\substack{\alpha \to a \\ \beta \to b}} {}_{ap} \int_\alpha^\beta f$$

exists, with value I *say, then* $f \in R^*_{ap}[a, b]$, *and* $\int_a^b f = I$.

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PROOF. It is sufficient to consider the case where for all β, $a<\beta<b$, $f \in R^*[a, \beta]$ and $\lim_{\beta \to b} {}_{ap} \int_a^\beta f = I$. Let $a=\beta_0<\beta_1<\cdots$, $\lim_{n\to\infty}\beta_n = b$, $\varepsilon>0$, then since $k \geqslant 1$, $f \in R^*_{ap}[\beta_k, \beta_{k-1}]$, there exists AFC, Δ_k, of $[\beta_k, \beta_{k-1}]$ such that for all Δ_k-partitions of $[\beta_k, \beta_{k-1}]$, $\{a_0^k, \ldots,; x_1^k \cdots\}$,

$$\left| \int_{\beta_{k-1}}^{\beta_k} f - \sum \cdot f(x_i^k)(a_i^k - a_{i-1}^k) \right| < \frac{\varepsilon}{2^k}.$$

Since $\lim_{\beta \to b} {}_{ap} \int_a^\beta f = I$, given $\varepsilon > 0$ there exists $\delta > 0$ and a set A of density 1 at b, $A \subset [b-\delta, b]$, such that if $x \in A$, $|I - \int_a^x f| < \varepsilon$, and $|(b-x)f(b)| < \varepsilon$. $\Delta = \bigcup_{x\in A}[x, b] \cup \bigcup_{k\geqslant 1}\Delta_k$ is an AFC of $[a, b]$ and consider the Δ-partition $\{a_0, \ldots, a_n; x_1, \ldots, x_n\}$ of $[a, b]$:

$$\begin{aligned}\left| I - \sum_{i=1}^n f(x_i)(a_i - a_{i-1}) \right| &\leqslant \left| \int_a^{a_{n-1}} f - \sum_{i=1}^{n-1} f(x_i)(a_i - a_{i-1}) \right| \\ &\quad + \left| I - \int_a^{a_{n-1}} f \right| + |f(b)(b - a_{n-1})| \\ &< 3\varepsilon,\end{aligned}$$

and so $\int_a^b f$ exists, with value I.

THEOREM 16. *Let E be a perfect set, end points a, b, with closed contiguous intervals in $[a, b]$, $[a_n, b_n]$, $n \in N$; suppose that $f 1_E \in R^*_{ap}[a, b]$ and that for all $n \in N$, $f \in R^*_{ap}[a_n, b_n]$; suppose further that for all $x \in E$ there exists a set E_x of unit density at x, $\delta > 0$, with $a_n, b_n \in E_x$ if $[a_n, b_n] \subset]x - \frac{1}{2}\delta, x + \frac{1}{2}\delta[$, $n \in N_x$, for short and $\sum_{n\in N_x}\{\sup_{\alpha,\beta \in E_x \cap [a_n, b_n]} |\int_\alpha^\beta f|\} < \infty$; then $f \in R^*_{ap}[a, b]$ and*

$$\int_a^b f = \int_a^b f 1_E + \sum_{n\in N} \int_{a_n}^{b_n} f. \tag{1}$$

PROOF. It is sufficient to prove that $f(1 - 1_E) \in R^*_{ap}[a, b]$. Note that the above conditions imply that for all $\varepsilon > 0$ there exists n_0 such that

$$\sum_{n > n_0} \left\{ \sup_{\alpha, \beta \in E_x \cap [a_n, b_n]} \left| \int_\alpha^\beta f \right| \right\} < \varepsilon,$$

and so, in particular, the right-hand side of (1) is defined.

For each $n \in N$ there exists AFC Δ_n of $[a_n, b_n]$ such that for all Δ_n-partitions of $[a_n, b_n]$, $\{a_0^n, \ldots; x_1^n \cdots\}$,

$$\left| \int_{a_n}^{b_n} f - \sum f(x_i^n)(a_i^n - a_{n-1}^n) \right| < \frac{\varepsilon}{2^n}.$$

At each $x \in E$ there exists $\tilde{E}_x \subset E_x$, of density 1 at x, containing all a_n, b_n, $n > n_0$, and $[a_n, b_n] \subset]x - \frac{1}{2}\delta, x + \frac{1}{2}\delta[$: let $\tilde{E}_x^* = \{[u, v]; u \leqslant x \leqslant v, u_1 v \in \tilde{E}_x\}$.

Consider $\Delta = \bigcup_{n \in N} \Delta_n \cup \bigcup_{x \in E} \tilde{E}_x$, an AFC on $[a, b]$, and $\{a_0, \ldots, a_p; x_1, \ldots, x_p\}$ any Δ-partition of $[a, b]$:

$$\sum_{n \in N} \left| \int_{a_n}^{b_n} f - \sum_{i=1}^{p} f(1 - 1_E)(x_i)(a_i - a_{i-1}) \right|$$

$$\leqslant \sum_{n > n_0} \left| \int f \right| + \left| \sum_{n \leqslant n_0} \int_{a_n}^{b_n} f - \sum f(x_i^n)(a_i^n - a_{i-1}^n) \right|$$

$$< 2\varepsilon,$$

which completes the proof.

THEOREM 17. *If* $f \in D^*_{ap}$ *then* $f \in R^*_{ap}$ *and the integrals are equal.*

PROOF. Let $f \in D^*_{ap}$, $E = \{x;\, a \leqslant x \leqslant b$ and f is not R^*_{ap}-integrable in some neighbourhood of $x\}$; assume $E \neq \varnothing$. From Theorem 15, E is perfect and if $[a_n, b_n]$, $n \in N$ are the closed continuous intervals of E, in $[a, b]$, then $f \in R^*_{ap}[a_n, b_n]$, $n \in N$. If $F(x) = D^*_{ap} - \int_a^x f$ then $f \in [ACG^*_{ap}]$ and so E contains a portion E_0 on which F is AC^*_{ap}; let α, β be the extremities of E_0. Since $F \in AC(E_0)$, $F'_{ap} = f$ a.e. on E_0 and f is L-integrable there, and so $f 1_E \in R^*_{ap}[a, b]$. Further since $F \in AC^*_{ap}(E_0)$, by Theorem 8, all the conditions of Theorem 16 are satisfied on $[\alpha, \beta]$, and so $f \in R^*_{ap}[\alpha, \beta]$. This proves that $E = \varnothing$, and so $f \in R^*_{ap}[a, b]$.

COROLLARY 18. $P^*_{ap} = R^*_{ap} = D^*_{ap}$.

5. An approximate total

The approximate-total* of f, f: $[a, b] \to \mathbf{R}$, $T^*_{ap} - \int_a^b f$, is constructed by the transfinite induction as indicated below; if the construction is possible we say that $f \in T^*_{ap}$.

The process uses four operations:

(1) if $a \leqslant \alpha \leqslant \beta \leqslant b$, $f \in L[\alpha, \beta]$ then $T^*_{ap} - \int_\alpha^\beta f = L - \int_\alpha^\beta f$;

(2) if for all α', β', $a \leqslant \alpha < \alpha' < \beta' < \beta \leqslant b$ we have evaluated $T^*_{ap} - \int_{\alpha'}^{\beta'} f$ and if

$$\lim_{\substack{\alpha' \to \alpha \\ \beta' \to \beta}} {}_{ap}\, T^*_{ap} - \int_{\alpha'}^{\beta'} f$$

exists, then $T^*_{ap} - \int_\alpha^\beta f$ is defined to be this limit;

(3) if $T^* - \int_\alpha^\beta f$ and $T^* = \int_\beta^\delta f$, $a \leqslant \alpha < \beta < \beta \leqslant b$, have been evaluated then $T^*_{ap} - \int_\alpha^\delta f$ is defined to be their sum;

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(4) if $P \subset [a, b]$ is perfect, with extremities α, β, and if $f 1_P \in L[\alpha, \beta]$, and if $f \in T^*_{ap}[\alpha_n, \beta_n]$, $[\alpha_n, \beta_n]$ being the closed contiguous intervals of P in $[\alpha, \beta]$, $n \in N$, and if further for all $x \in P$ there exists a set E_x of density 1 at x, $\delta > 0$ with $a_n, b_n \in E_x$, if $[a_n, b_n] \subset]x - \frac{1}{2}\delta, x + \frac{1}{2}\delta[$, $n \in N_x$, for short, and $\Sigma_{n \in N_x}\{\sup_{\alpha'_n, \beta'_n E_x \cap [\alpha_n, \beta_n]} | T^*_{ap} - \int_{\alpha'_n}^{\beta'_n} f|\} < \infty$, then $T^*_{ap} - \int_\alpha^\beta f$ is evaluated as $L - \int_\alpha^\beta f 1_P + \Sigma_{n \in N} T^*_{ap} - \int_{\alpha_n}^{\beta_n} f$.

Remark. This operation is related to that used in an integral defined by Kubota, [11, 12], in the same way as the corresponding operation in the special Denjoy integral is related to that in the general Denjoy integral; Saks, [20; page 255].

The construction of $T^*_{ap} - \int_a^b f$ can now be described as follows.

Stage 1: *Step* 1. Let $E = \{x; a \leqslant x \leqslant b, f$ is not summable at $x\}$. If E is not nowhere dense, $f \notin T^*_{ap}$, if E is nowhere dense proceed to

Step 2. For all $[\alpha, \beta]$, $[\alpha, \beta] \cap E = \varnothing$ compute $T^*_{ap} - \int_\alpha^\beta f$ by operation (1).

Step 3. If $[\alpha, \beta]$ is a closed contiguous interval of E see if

$$\lim_{\substack{\alpha' \to \alpha \\ \beta' \to \beta}}{}_{ap}\, T^*_{ap} - \int_\alpha^\beta f$$

exists; if not $f \notin T^*_{ap}$, if so compute $T^*_{ap} - \int_\alpha^\beta f$ by operation (2).

Step 4. For all $[\alpha, \beta]$, $[\alpha, \beta] \cap E' = \varnothing$ compute $T^*_{ap} - \int_\alpha^\beta f$ by operation (3).

Step 5. Applying step 3 to the contiguous intervals of E', then by a transfinite process using steps 4 and 3, we either find that $f \notin T^*_{ap}$, or will have computed $T^*_{ap} - \int_\alpha^\beta f$ for all $[\alpha, \beta]$, closed contiguous intervals of the perfect kernel P of E; if $P = \varnothing$ we have completed the calculation, if not proceed to

Stage 2: *Step* 1. Let $\tilde{E} = \{x; x \in P$ and $f 1_p$ is not summable at $x\}$. If $\tilde{E}$ is not nowhere dense in P, $f \notin T^*_{ap}$; if $\tilde{E}$ is nowhere dense on P, proceed to

Step 2. For all $[\alpha, \beta]$, $[\alpha, \beta] \cap \tilde{E} = \varnothing$ compute $T^*_{ap} - \int_\alpha^\beta f$ as described in stage 3 below. If this is not possible $f \notin T^*_{ap}$, if it is use steps 3, 4 of stage 1 to compute, if possible $T^*_{ap} - \int_\alpha^\beta f$ for all $[\alpha, \beta]$, closed contiguous intervals of the perfect kernel of $\tilde{E}$.

Step 3. A transfinite process using the above steps then either finds $f \notin T^*_{ap}$ or computes $T^*_{ap} - \int_\alpha^\beta f$ on the closed contiguous intervals of $E_1 = E$, $E_2 = \tilde{E}$, $E_3, \ldots, E_\lambda, \ldots$, where if λ has a predecessor E_λ is nowhere dense in the perfect kernel, $P_{\lambda-1}$ of $E_{\lambda-1}$ and $E_\lambda = \{x; x \in P_{\lambda-1}$ and $f 1_{P_{\lambda-1}}$ is not summable at $x\}$, while if λ has no predecessor $E_\lambda = \bigcap_{\mu<\lambda} E_\mu$. For some $\nu < \Omega$, $E_\nu = \varnothing$, $E_{\nu-1} \neq \varnothing$, that is, either $P_{\nu-1} = \varnothing$ or $f 1_{P_{\nu-1}} \in L[a, b]$; in either case stages 1–3 applied to $E_{\nu-1}$ completes the computation.

Stage 3: (From step 2 of stage 2 we have to compute $T^*_{ap} - \int_\alpha^\beta f$ where $[\alpha, \beta]$ defines a closed portion of a perfect set P, Q say, with $f 1_Q$ summable, and on the closed contiguous intervals in $[\alpha, \beta]$ of Q, $[\alpha_n, \beta_n]$, $T^*_{ap} - \int_{\alpha_n}^{\beta_n} f$ has already been computed, $n \in N$.)

Step 1. Let x be a regular point of Q if there exists a set E_x, of density 1 at x, $\delta > 0$, with $\alpha_n, \beta_n \in E_x$ if $[\alpha_n, \beta_n] \subset]x - \frac{1}{2}\delta, x + \frac{1}{2}\delta[$, $n \in N_x$ for short, and $\sum_{n \in N_x s}\{\sup_{\alpha'_n, \beta'_n \in E_x \cap [\alpha_n, \beta_n]} | T^*_{ap} - \int_{\alpha'}^{\beta'} f |\} < \infty$; let E be the set of non-regular points of Q. If E is not nowhere dense in Q, $f \notin T^*_{ap}$, if it is proceed to

Step 2. For all $[\alpha', \beta'], [\alpha', \beta'] \cap E = \varnothing$ compute $T^*_{ap} - \int_{\alpha'}^{\beta'} f$ by operation (4).

Step 3. Proceed as in stage 1 to obtain $T^*_{ap} - \int_{\alpha'}^{\beta'} f$ on all $[\alpha', \beta']$ closed contiguous intervals of the perfect kernel of E; then proceed to stage 2 again.

To facilitate the discussion of the T^*_{ap}-integral we define for all α, $0 \leqslant \alpha \leqslant \Omega$, on $[a, b]$ an integral $L^{*,\alpha}_{ap}$; this follows the ideas of Saks, [20], and Kubota, [11, 12].

(a) $L^{*,0}_{ap} = L$.

(b) If for all $\alpha < \beta \leqslant \Omega$ we have defined $L^{*,\alpha}_{ap}$ in such a way that the integrals are compatible and if $\alpha < \alpha' < \beta$ then $L^{*,\alpha}_{ap} \subset l^{*,\alpha'}_{ap}$ then I_1^β is the integral defined by

$$I_1^\beta = \bigcup_{\alpha < \beta} L^{*,\alpha}_{ap}, \quad I_1^\beta - \int_a^b f = L^{*,\alpha_0}_{ap} - \int_a^b f,$$

where

$$\alpha_0 = \min\{\alpha; f \in L^{*,\alpha}_{ap}\}.$$

(c) (i) If $\beta < \Omega$ then I_2^β is the integral $(I_1^\beta)^C_{ap}$, see Definition 1(a) below; and

$$L^{*,\beta}_{ap} = \left(I_2^\beta\right)^{H^*}_{ap},$$

see Definition 1(b) below;

(ii) if $\beta = \Omega$,

$$L^{*,\Omega}_{ap} = I_1^\Omega.$$

DEFINITION 1. If I is an integral let $S_f = S = \{x; f \text{ is not } I\text{-integrable at } x\}$; then:

(a) the approximate Cauchy extension of I, I^C_{ap}, is defined as follows: $f \in I^C_{ap}$ if and only if there exists $F \in C_{ap}$ such that if $[a', b'] \cap S = \phi$ then $I - \int_{a'}^{b'} f = F(b') - F(a')$ then $I^C_{ap} - \int_a^b f = F(b) - F(a)$.

(b) the approximate Harnack* extension of I, $I^{H^*}_{ap}$, is defined as follows: $f \in I^{H^*}_{ap}$ if and only if (i) $f 1_S \in L$, (ii) if $[a_n, b_n]$, $n \in N$ are the closed contiguous intervals of S in $[a, b]$ then f is I-integrable on each, and if x is a limit point of the

$[a_n, b_n]$ there exists a set E_x of unit density, $\delta > 0$, with $a_n, b_n \in E_x$ if $[a_n, b_n] \subset]x - \frac{1}{2}\delta, x + \frac{1}{2}\delta[$, $x \in N_x$ for short, and $\Sigma_{n\in N_x}\{\sup_{a'_n, b'_n \in E_x \cap [a_n, b_n]} |I - \int_{a'_n}^{b'_n} f|\} < \infty$, then

$$I_{ap}^{H^*} - \int_a^b f = L - \int_a^b f 1_S + \Sigma_{n\in N} I - \int_a^b f.$$

The following theorem is then easily deduced, using the methods of Saks, [20], and Kubota, [11, 12].

THEOREM 2. (a) $((L_{ap}^{*,\Omega})_{ap}^C)_{ap}^{H^*} = L_{ap}^{*,\Omega}$.
(b) $L_{ap}^{*,\Omega} = T_{ap}^*$.
(c) $L_{ap}^{*,\Omega} = D_{ap}^*$.
(d) If I is an approximately continuous integral such that (i) $L \subset I$, (ii) $(I_{ap}^C)_{ap}^* = I$, then $D_{ap}^* \subset I$.

COROLLARY 3. $P_{ap}^* = R_{ap}^* = D_{ap}^* = T_{ap}^* = L_{ap}^{*,\Omega}$.

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