Canad. J. Math. Vol. **60** (3), 2008 pp. 520–531

Matrices Whose Norms Are Determined by Their Actions on Decreasing Sequences

Chang-Pao Chen, Hao-Wei Huang, and Chun-Yen Shen

Abstract. Let $A = (a_{j,k})_{j,k\geq 1}$ be a non-negative matrix. In this paper, we characterize those A for which $\|A\|_{E,F}$ are determined by their actions on decreasing sequences, where E and F are suitable normed Riesz spaces of sequences. In particular, our results can apply to the following spaces: ℓ_p, $d(w, p)$, and $\ell_p(w)$. The results established here generalize ones given by Bennett; Chen, Luor, and Ou; Jameson; and Jameson and Lashkaripour.

1 Introduction

Let $w_1 \geq w_2 \geq \cdots \geq 0$. For $1 \leq p \leq \infty$, denote by $\ell_p(w)$ the space of all sequences $x = \{x_k\}_{k=1}^{\infty}$ such that

$$\|x\|_{\ell_p(w)} := \Big(\sum_{k=1}^{\infty} |x_k|^p w_k\Big)^{1/p} < \infty.$$

The Lorentz sequence space $d(w, p)$ is the space of null sequences x for which x^* is in $\ell_p(w)$, with norm $\|x\|_{w,p} = \|x^*\|_{\ell_p(w)}$, (*cf.* [1, 7]). Here x^* is the decreasing rearrangement of $\{|x_k|\}_{k=1}^{\infty}$. When $w_k = 1$ for all k, $\ell_p(w)$ coincides with ℓ_p in the usual sense (the norm of which we denote by $\|\cdot\|_p$). We write $x \geq 0$ if $x_k \geq 0$ for all k. Similarly, $x\downarrow$ will mean that $\{x_k\}_{k=1}^{\infty}$ is decreasing, that is, $x_k \geq x_{k+1}$ for all $k \geq 1$. For $A = (a_{j,k})_{j,k\geq 1}$, we write $A \geq 0$ if $a_{j,k} \geq 0$ for all j and k. For $A \geq 0$ and two normed sequence spaces $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ in $\ell_p(w)$, let $\|A\|_{E,F}$ denote the norm of A when regarded as an operator from E to F. Clearly, $\|A\|_{E,F}$ is determined by non-negative sequences and $\|A\|_{E,F} \geq \|A\|_{E,F,\downarrow}$, where

$$\|A\|_{E,F,\downarrow} := \sup_{\|x\|_E=1, x\geq 0, x\downarrow} \|Ax\|_F.$$

Bennett [3, Problem 7.23] posed the following problem for $E = F = \ell_p$: when does the equality $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$ hold? This is of great importance in the general theory of inequalities.

Bennett established this upper bound equality for the case that $E = F = \ell_p$, $1 < p < \infty$, and A is a weighted mean matrix with decreasing weights w_n [2, p. 422],

Received by the editors September 2, 2005; revised January 19, 2007.
This work is supported by the National Science Council, Taipei, ROC, under Grant NSC 96-2115-M-364-003-MY3
AMS subject classification: Primary: 15A60, secondary: 40G05, 47A30, 47B37, 46B42.
Keywords: norms of matrices, normed Riesz spaces, weighted mean matrices, Nörlund mean matrices, summability matrices, matrices with row decreasing.

https://doi.org/10.4153/CJM-2008-025-5 Published online by Cambridge University Press

[3, p. 422]. This result was extended by Jameson [6, Theorem 2] to the case that $E = F$ is a Banach lattice of sequences with property (PS) and A satisfies the following condition:

$$\sum_{j=1}^{l}\sum_{k=1}^{r} a_{j,k} \geq \sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \qquad (l, r \geq 1; |N_l| = l, |N_r| = r) \tag{1.1}$$

(We refer the reader to §2 for the definition of (PS)). Here N_s denotes a set of positive integers having s elements and $|N_s| = s$ stands for all possibilities of N_s. In [5, Lemma 2.4], the first author extended Bennett's result to the case $\|A\|_{\ell_p,\ell_p} = \|A\|_{\ell_p,\ell_p,\downarrow}$, where $1 < p < \infty$ and A is a non-negative lower triangular matrix with rows decreasing in the sense that $a_{j,k} \geq a_{j,k+1}$ for all $j, k \geq 1$. Jameson and Lashkaripour [8, Lemma 2.1] established the equality

$$\|A\|_{d(w,p),d(w,p)} = \|A\|_{d(w,p),d(w,p),\downarrow}$$

for the same matrix and for the matrices with decreasing columns (see Corollaries 3.2 and 4.3 for details).

The purpose of this paper is to extend the aforementioned results to a more general setting. In §2 (see Theorems 2.2 and 2.3), we establish the upper bound equality for those A which obey ($*$) for some positive integer n_0:

($*$) For any $n \geq n_0$ and any $B = (b_{j,k}) \in \mathcal{R}_{A_n}$, there exists some $C = (c_{j,k}) \in \mathcal{R}_{A_n}$, depending on n and B, such that the following inequality holds:

$$\sum_{j=1}^{l}\sum_{k=1}^{r} c_{j,k} \geq \sum_{j=1}^{l}\sum_{k\in N_r} b_{j,k} \quad (1 \leq l \leq n;\ r \geq 1;\ |N_r| = r),$$

where A_n is the $n \times \infty$ matrix obtained from the first n rows of A and $\mathcal{R}_{A_n}$ is the set of all row rearrangements of A_n. The latter is defined in the following way: we say that $B = (b_{j,k})$ is a matrix obtained from A_n by row rearrangements, if there is a one-to-one mapping σ from $\{1, 2, \dots, n\}$ onto itself with $b_{j,k} = a_{\sigma(j),k}$ for all j and for all k. Obviously, (1.1) corresponds to the particular case of ($*$) with $n_0 = 1$ and $C = A_n$. On the other hand, any matrix A with rows decreasing ensures the validity of ($*$) with $n_0 = 1$ and $C = B$. Hence, Theorems 2.2 and 2.3 give a unified approach of the above upper bound problem. As proved in Theorem 2.2, the underlying spaces E and F can be any normed Riesz spaces with the properties of (PS) and (2.3), in particular, any of ℓ_p and $d(w, p)$. Hence, our results generalize [5, Lemma 2.4], [6, Theorem 2], [8, Lemma 2.1], and Bennett's result. In §3 and §4, we also give a detailed investigation of ($*$) for the matrix A. These include the investigations of the Hilbert matrix, the weighted mean matrix, the Nörlund mean matrix, summability matrices, and matrices with row decreasing. Of course, the Gamma matrix $\Gamma(\alpha)$ and the Cesàro matrix $C(\alpha)$ are also examined. We refer the reader to §2–§4 for details.

2 Main Results

We have the following result.

Lemma 2.1 *Let $\{v_k\}_{k=1}^n$ and $\{u_k\}_{k=1}^n$ be two non-negative sequences such that*

$$\sum_{k=1}^{r} v_k \geq \sum_{k\in N_r} u_k \quad (r = 1, \dots, n; |N_r| = r). \tag{2.1}$$

Then

$$\sum_{k=1}^{n} v_k x_k^* \geq \sum_{k=1}^{n} u_k x_k \quad (x = \{x_k\}_{k=1}^n \geq 0).$$

Proof We have $x_k^* - x_{k+1}^* \geq 0$ for all $1 \leq k < n$. Let $\{\tilde{u}_k\}_{k=1}^n$ denote the corresponding rearrangement of $\{u_k\}_{k=1}^n$ such that $\sum_{k=1}^n u_k x_k = \sum_{k=1}^n \tilde{u}_k x_k^*$. Employing summation by parts and (2.1), we get

$$\begin{aligned}\sum_{k=1}^{n} u_k x_k = \sum_{k=1}^{n} \tilde{u}_k x_k^* &= \sum_{k=1}^{n-1} (x_k^* - x_{k+1}^*) \Big(\sum_{s=1}^{k} \tilde{u}_s\Big) + x_n^* \Big(\sum_{k=1}^{n} \tilde{u}_k\Big) \\ &\leq \sum_{k=1}^{n-1} (x_k^* - x_{k+1}^*) \Big(\sum_{s=1}^{k} v_s\Big) + x_n^* \Big(\sum_{k=1}^{n} v_k\Big) = \sum_{k=1}^{n} v_k x_k^*. \qquad \blacksquare\end{aligned}$$

Let $(F, \|\cdot\|_F)$ be a normed Riesz space of real sequences (see [9, p. 6] for definition). Following [6], we say that F has the property (PS), if for all $x \in F$, x^* exists and the following property holds:

$$y_1^* + \cdots + y_n^* \leq x_1^* + \cdots + x_n^* \quad (n \geq 1) \implies y \in F \text{ and } \|y\|_F \leq \|x\|_F. \tag{2.2}$$

Clearly, for $x \in F$, we have $\tilde{x} \in F$ and $\|\tilde{x}\|_F = \|x\|_F$, where $\tilde{x}$ is any sequence with $\tilde{x}^* = x^*$. In particular, $\tilde{x}$ can be x^* or any sequence obtained from x by reordering x_k. We have $x_1 + \cdots + x_n \leq x_1^* + \cdots + x_n^*$, so the condition in (2.2) can be replaced by $y_1^* + \cdots + y_n^* \leq x_1 + \cdots + x_n$. Applying Lemma 2.1, we get the first main result as follows.

Theorem 2.2 *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed Riesz spaces of real sequences with property (PS). In addition, the following property is satisfied:*

$$\|x\|_F = \lim_{n\to\infty} \|P_n x\|_F \quad (x \in F), \tag{2.3}$$

where $P_n x$ is the projection of x onto the first n terms. Let $A = (a_{j,k})_{j,k\geq 1}$ define an operator from E to F, given by $Ax = y$, where $a_{j,k} \geq 0$ for all j and k. If $(*)$ is true for some positive integer n_0, then $\|Ax^*\|_F \geq \|Ax\|_F$ for all $x \in E$ with $x \geq 0$. Hence, decreasing, non-negative elements x in E are sufficient to determine $\|A\|_{E,F}$.

Proof Let $x \in E$ with $x \geq 0$. Then the (PS) property of E implies $x^* \in E$. Since A sends E to F, we know that $Ax, Ax^* \in F$. We claim that $\|Ax^*\|_F \geq \|Ax\|_F$. Let $n \geq n_0$. By making row permutations, we can find $B = (b_{j,k}) \in \mathcal{R}_{A_n}$ such that $\{\sum_{k=1}^{\infty} b_{j,k}x_k\}_{j=1}^{n}$ is decreasing. Let $C = (c_{j,k}) \in \mathcal{R}_{A_n}$ be the corresponding matrix obeying $(*)$. Let l be fixed. Since $\sum_{k=1}^{r} \big(\sum_{j=1}^{l} c_{j,k} \big) \geq \sum_{k \in N_r} \big(\sum_{j=1}^{l} b_{j,k} \big)$ for all $r \geq 1$ and for all N_r, it follows from Lemma 2.1 that

$$\sum_{k=1}^{m} \Big(\sum_{j=1}^{l} c_{j,k} \Big) x_k^* \geq \sum_{k=1}^{m} \Big(\sum_{j=1}^{l} b_{j,k} \Big) x_k \quad (m \geq 1).$$

Let $m \to \infty$ and reorder the above sums. Then we obtain

$$\sum_{j=1}^{l} \Big(\sum_{k=1}^{\infty} c_{j,k}x_k^* \Big) \geq \sum_{j=1}^{l} \Big(\sum_{k=1}^{\infty} b_{j,k}x_k \Big) \quad (l = 1, \dots, n). \tag{2.4}$$

For $1 \leq j \leq n$, set $y_j = \sum_{k=1}^{\infty} c_{j,k}x_k^*$ and $z_j = \sum_{k=1}^{\infty} b_{j,k}x_k$. We also set $y_j = z_j = 0$ for $j > n$. Since $\{z_j\}_{j=1}^{\infty}$ is decreasing, $z_j^* = z_j$ for all j and, consequently, (2.4) can be rewritten in the form: $\sum_{j=1}^{l} y_j \geq \sum_{j=1}^{l} z_j^*$ for all $l \geq 1$. On the other hand, $P_nAx^* \in F$ and it is of the form: $P_nAx^* = \{y_1', \dots, y_n', 0, \dots\}$. Since $C \in \mathcal{R}_{A_n}$, $y = \{y_1, \dots, y_n, \dots\}$ can be obtained from the sequence $\{y_1', \dots, y_n', 0, \dots\}$ by reordering the first n terms. The (PS) property of F implies $y \in F$, and therefore, $z = \{z_1, z_2, \cdots\} \in F$. Moreover, $\|P_nAx^*\|_F = \|y\|_F \geq \|z\|_F$. We have $B \in \mathcal{R}_{A_n}$. The same argument as above also ensures that $\|z\|_F = \|P_nAx\|_F$. Hence, $\|P_nAx^*\|_F \geq \|P_nAx\|_F$. Let $n \to \infty$. Then (2.3) implies $\|Ax^*\|_F \geq \|Ax\|_F$, which is what we want. ∎

The spaces E and F in Theorem 2.2 can be one of ℓ_p $(1 \leq p < \infty)$ or $d(w, p)$ $(1 \leq p \leq \infty)$. Hence, Theorem 2.2 generalizes [5, Lemma 2.4], [6, Theorem 2], and [8, Lemma 2.1]. We know that $\ell_2(w)$ with $w_n = 1/n^3$ fails to possess the (PS) property. Hence, the space F in Theorem 2.2 may not apply to the case $\ell_p(w)$. In the following, we show that the conclusion of Theorem 2.2 still holds for $F = \ell_p(w)$, provided that $(*)$ with $n_0 = 1$ is true.

Theorem 2.3 *Let $1 \leq p < \infty$, $A = (a_{j,k})_{j,k \geq 1} \geq 0$, and $(E, \|\cdot\|_E)$ be a normed Riesz space of real sequences with property (PS). If $(*)$ is true for $n_0 = 1$, then $\|Ax^*\|_{\ell_p(w)} \geq \|Ax\|_{\ell_p(w)}$ for all $x \in E$ with $x \geq 0$. Hence, decreasing, non-negative elements x in E are sufficient to determine $\|A\|_{E,\ell_p(w)}$.*

Proof Following the proof of Theorem 2.2, there exist $B = (b_{j,k}) \in \mathcal{R}_{A_n}$ and $C = (c_{j,k}) \in \mathcal{R}_{A_n}$ so that (2.4) holds and $\{\sum_{k=1}^{\infty} b_{j,k}x_k\}_{j=1}^{n}$ is decreasing. By (2.4) and [3, Lemma 2.8], we get

$$\sum_{j=1}^{n} \Big(\sum_{k=1}^{\infty} c_{j,k}x_k^* \Big)^p \geq \sum_{j=1}^{n} \Big(\sum_{k=1}^{\infty} b_{j,k}x_k \Big)^p. \tag{2.5}$$

Since $C, B \in \mathcal{R}_{A_n}$, (2.5) can be rewritten in the following form:

$$\sum_{j=1}^{n} \Big(\sum_{k=1}^{\infty} a_{j,k}x_k^* \Big)^p \geq \sum_{j=1}^{n} \Big(\sum_{k=1}^{\infty} a_{j,k}x_k \Big)^p.$$

We have assumed that $w_n \downarrow$. By [3, Lemma 2.1], we obtain

$$(2.6)\qquad \sum_{j=1}^{n}\Big(\sum_{k=1}^{\infty} a_{j,k}x_k^*\Big)^p w_j \geq \sum_{j=1}^{n}\Big(\sum_{k=1}^{\infty} a_{j,k}x_k\Big)^p w_j \quad (n \geq 1).$$

Let $n \to \infty$. Then (2.6) leads us to $\|Ax^*\|_{\ell_p(w)} \geq \|Ax\|_{\ell_p(w)}$, which gives the desired result. ■

3 Investigation of Condition $(*)$

We know that if $(*)$ holds for some n_0, then it is still true for any bigger n_0. The matrix $A = (a_{j,k})_{j,k\geq 1}$, with $a_{3,1} = a_{2,2} = 1$ and 0 otherwise, gives an example for which $(*)$ holds for $n_0 = 3$, but fails to hold for $n_0 = 2$. For this example, the matrix $C = (c_{j,k})$, required in $(*)$, is defined by $c_{1,1} = c_{2,2} = 1$ and 0 otherwise. In Theorems 2.2 and 2.3, we point out that $(*)$ is a sufficient condition for A to obey the equality $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$. The purpose of this section is to find those stronger conditions than $(*)$.

We know that (1.1) $\Rightarrow$ $(*)$ for $n_0 = 1$ (by letting $c_{j,k} = a_{j,k}$). Moreover, the entries of A^t satisfy (1.1) if and only if the entries of A do, where A^t denotes the transpose of A. Hence, Theorems 2.2 and 2.3 have the following consequence.

Theorem 3.1 *Theorems* 2.2 *and* 2.3 *remain true, if* $(*)$ *is replaced by* (1.1) *Moreover, the conclusions of Theorems* 2.2 *and* 2.3 *also hold for* A^t *in place of* A.

The case $E = F$ of Theorem 3.1 for A has appeared in [6, Theorem 2]. The matrix given before Theorem 3.1 shows that our results, that is, Theorems 2.2 and 2.3, can apply to a wider class than [6, Theorem 2] from the viewpoint of matrices. As proved in [6, Proposition 3], (3.1) $\Rightarrow$ (1.1), where

$$(3.1)\qquad a_{j,k} \geq a_{j+1,k} \quad (j,k \geq 1) \quad \text{and} \quad \sum_{j=1}^{l} a_{j,k} \geq \sum_{j=1}^{l} a_{j,k+1} \quad (k,l \geq 1).$$

Hence, Theorem 3.1 has the following consequence.

Corollary 3.2 *Theorems* 2.2 *and* 2.3 *remain true, if* $(*)$ *is replaced by* (3.1). *Moreover, the conclusions of Theorems* 2.2 *and* 2.3 *also hold for* A^t *in place of* A.

This corollary extends [8, Lemma 2.1] from the pair $(d(w,p), d(w,p))$ to the pair (E,F) for those A of the form (3.1). Moreover, it indicates that the condition (i) in [8, Lemma 2.1] is not needed. Clearly, the Hilbert matrix $H = (h_{j,k})_{j,k\geq 1}$, defined by $h_{j,k} = 1/(j+k-1)$, satisfies (3.1), so Corollary 3.2 can apply to this matrix. Let $A_W^{WM} = (a_{j,k}^{WM})_{j,k\geq 1}$ and $A_W^{NM} = (a_{j,k}^{NM})_{j,k\geq 1}$ denote the weighted mean matrix and the Nörlund mean matrix, respectively, which are defined by $a_{j,k}^{WM} = a_{j,k}^{NM} = 0$ for $j < k$ and

$$a_{j,k}^{WM} = w_k/(w_1 + \cdots + w_j) \quad (j \geq k),$$
$$a_{j,k}^{NM} = w_{j-k+1}/(w_1 + \cdots + w_j) \quad (j \geq k).$$

Applying Corollary 3.2 to these two kinds of matrices, we get the following result.

https://doi.org/10.4153/CJM-2008-025-5 Published online by Cambridge University Press

Corollary 3.3 *Let $w_1 > 0$ and $w_n \geq 0$ for all $n > 1$. Then Theorems* 2.2 *and* 2.3 *remain true, if* $(*)$ *is replaced by any of* (i) *or* (ii).

(i) $A = (A_W^{WM})^t$ *with* $w_n \downarrow$,

(ii) $A = (A_W^{NM})^t$, *where* $w_n \uparrow$ *and* $w_{n+1}/w_n \leq w_n/w_{n-1}$ *for all* $n \geq 2$.

Moreover, the conclusions of Theorems 2.2 *and* 2.3 *also hold for* A^t *in place of* A.

Proof Obviously, $(A_W^{WM})^t \geq 0$ and $(A_W^{NM})^t \geq 0$. Consider Case (i). Set $A = (A_W^{WM})^t = (a_{j,k})_{j,k\geq 1}$. It is easy to see that $a_{j,k} \geq a_{j+1,k}$ for all $j, k \geq 1$ if and only if $w_n \downarrow$. Moreover, we have

$$\sum_{j=1}^{l} a_{j,k} = \begin{cases} \dfrac{w_1 + \cdots + w_l}{w_1 + \cdots + w_k} & (l \leq k), \\ 1 & (l > k). \end{cases}$$

This implies $\sum_{j=1}^{l} a_{j,k} \geq \sum_{j=1}^{l} a_{j,k+1}$ for all $k, l \geq 1$. The above argument shows that (3.1) holds. By Corollary 3.2, we get (i). Next, consider (ii). Let $A = (A_W^{NM})^t = (a_{j,k})_{j,k\geq 1}$. By definition, we see that $a_{j,k} \geq a_{j+1,k}$ for all $j, k \geq 1$ if and only if $w_n \uparrow$. Moreover,

$$\sum_{j=1}^{l} a_{j,k} = \begin{cases} \dfrac{w_k + \cdots + w_{k-l+1}}{w_k + \cdots + w_1} & (l \leq k), \\ 1 & (l > k). \end{cases}$$

The hypothesis that $w_{k+1}/w_k \leq w_k/w_{k-1}$ implies $\sum_{j=1}^{l} a_{j,k+1} \leq \sum_{j=1}^{l} a_{j,k}$ for $l < k$ (see [6, Lemma 2(ii)]). The last inequality is also true for the case $l \geq k$, because $\sum_{j=1}^{l} a_{j,k+1} \leq 1 = \sum_{j=1}^{l} a_{j,k}$. Thus, (3.1) is satisfied. By Corollary 3.2, we get (ii). This completes the proof. ■

The conclusion of Corollary 3.3(i) for A^t and for $E = F = \ell_p$ was established by Bennett [2, p. 422] and [3, p. 422], where $1 < p < \infty$. For $w_n = \binom{n+\alpha-2}{n-1}$, A_W^{WM} and A_W^{NM} are denoted by $\Gamma(\alpha)$ and $C(\alpha)$, respectively. They are called the Gamma matrix and the Cesàro matrix of order α, (see [3, p. 410], [4], [10, Ch. III]). We know that $w_n \uparrow$ for $\alpha \geq 1$ and $w_n \downarrow$ for $0 < \alpha \leq 1$ [10, p. 77]. Moreover, for $\alpha \geq 1$, we have

$$\frac{w_{n+1}}{w_n} = \frac{n + \alpha - 1}{n} \leq \frac{n + \alpha - 2}{n - 1} = \frac{w_n}{w_{n-1}}.$$

Hence, by Corollary 3.3, the conclusions of Theorems 2.2 and 2.3 hold for A to be any of the matrices: $\Gamma(\alpha), \Gamma(\alpha)^t$ $(0 < \alpha \leq 1)$ and $C(\alpha), C(\alpha)^t$ $(\alpha \geq 1)$.

Following [3], we say that $A = (a_{j,k})_{j,k\geq 1}$ is a summability matrix if A is a non-negative lower triangular matrix with $\sum_{k=1}^{\infty} a_{j,k} = 1$ for all j. For such type of matrices, we have the following result.

Corollary 3.4 *Let $A = (a_{j,k})_{j,k\geq 1}$ be a summability matrix. Then* (3.2) $\Rightarrow$ (1.1), *where*

$$a_{j,k} \geq \max(a_{j+1,k}, a_{j+1,k+1}) \quad (j \geq k \geq 1). \tag{3.2}$$

Hence, Theorems 2.2 *and* 2.3 *remain true if* $(*)$ *is replaced by* (3.2). *Moreover, the conclusions of Theorems* 2.2 *and* 2.3 *also hold for* A^t *in place of* A.

https://doi.org/10.4153/CJM-2008-025-5 Published online by Cambridge University Press

Proof The second part follows from Theorem 3.1. We claim that (3.2) $\Rightarrow$ (1.1). Divide the proof into three cases. For the case $l \leq r$, we have

$$\sum_{j=1}^{l}\sum_{k=1}^{r} a_{j,k} \geq \sum_{j=1}^{l}\sum_{k=1}^{l} a_{j,k} = l. \tag{3.3}$$

On the other hand, we know that A is a summability matrix. Thus, $\sum_{k=1}^{\infty} a_{j,k} = 1$ for all j. This implies

$$\sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \leq \sum_{j\in N_l}\Bigl(\sum_{k=1}^{\infty} a_{j,k}\Bigr) = |N_l| = l. \tag{3.4}$$

Putting (3.3) and (3.4) together yields (1.1).

Next, consider the case that $l > r$ and $N_r = \{1, 2, \dots, r\}$. Write $N_l = \{j_1, \dots, j_l\}$ in alphabetical order. Then

$$\sum_{s=1}^{r}\sum_{k\in N_r} a_{j_s,k} \leq \sum_{s=1}^{r}\Bigl(\sum_{k=1}^{\infty} a_{j_s,k}\Bigr) = r = \sum_{s=1}^{r}\sum_{k=1}^{r} a_{s,k}. \tag{3.5}$$

On the other hand, for $r < s \leq l$ and $k \in N_r$, by (3.2), we get $a_{s,k} \geq a_{j_s,k}$, and so $\sum_{k\in N_r} a_{j_s,k} \leq \sum_{k=1}^{r} a_{s,k}$. Sum up both sides over $s \in \{r+1, \dots, l\}$. Then

$$\sum_{s=r+1}^{l}\sum_{k\in N_r} a_{j_s,k} \leq \sum_{s=r+1}^{l}\sum_{k=1}^{r} a_{s,k}. \tag{3.6}$$

Putting (3.5) and (3.6) together yields $\sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \leq \sum_{s=1}^{l}\sum_{k=1}^{r} a_{s,k}$. This is (1.1).

It remains to prove the case that $l > r$ and N_r is any set of positive integers with $|N_r| = r$. Obviously, this case will follow from the previous case if we can find two index sets N_l^* and N_r^* such that $|N_l^*| = l$, $N_r^* = \{1, 2, \dots, r\}$, and $\sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \leq \sum_{j\in N_l^*}\sum_{k\in N_r^*} a_{j,k}$. To get these two sets, we first find $N_l' = \{j_1', \dots, j_l'\}$ and $N_r' = \{k_1', \dots, k_r'\}$ such that $j_1' \geq k_1'$, $j_l' \geq k_r'$, and $\sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \leq \sum_{j\in N_l'}\sum_{k\in N_r'} a_{j,k}$. Write $N_r = \{k_1, \dots, k_r\}$ in the alphabetical order. If $j_l \geq k_r$, let $N_r' = N_r$ and skip the consequent discussion. If $j_l < k_r$, then choose $k' \in \{1, 2, \dots, j_l - 1\} \setminus N_r$ and let $N_r' = \{k_1, \dots, k_{r-1}, k'\}$. Such k' exists, because $j_l \geq l > r$. We have $a_{j,k_r} = 0$ for all $j \in N_l$ (since A is a lower triangular matrix). Therefore, $\sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \leq \sum_{j\in N_l}\sum_{k\in N_r'} a_{j,k}$. If j_l is still less than k_r', then replace k_r' by another k' and let N_r' be the new corresponding index set. Do this procedure several times. Eventually, we can assume $j_l \geq k_r'$. Similarly, for $j_1 \geq k_1'$, let $N_l' = N_l$. If $j_1 < k_1'$, then choose $j' > j_l$ and let $N_l' = \{j_2, \cdots, j_l, j'\}$. Then

$$\sum_{j\in N_l}\sum_{k\in N_r} a_{j,k} \leq \sum_{j\in N_l}\sum_{k\in N_r'} a_{j,k} \leq \sum_{j\in N_l'}\sum_{k\in N_r'} a_{j,k}. \tag{3.7}$$

Without loss of generality, we can assume $j_1' \geq k_1'$. This finishes the proof of the existence of N_l' and N_r'. Next, we come back to find N_l^* and N_r^*. By (3.2), we obtain

$$\sum_{j\in N_l'} \sum_{k\in N_r'} a_{j,k} \leq \sum_{j\in N_l^1} \sum_{k\in N_r^1} a_{j,k}, \tag{3.8}$$

where N_l^1 and N_r^1 are of the form

$$N_l^1 = \{ j - k_1' + 1 : j \in N_l' \} := \{ j_1^1, j_2^1, \dots, j_l^1 \},$$

$$N_r^1 = \{ k - k_1' + 1 : k \in N_r' \} := \{ 1, 2, \dots, t^*, k_{t^*+1}^1, \dots, k_r^1 \}.$$

Clearly, $j_l^1 \geq k_r^1$. If $k_{t^*+1}^1 = t^* + 1 = r$, then $N_l^* = N_l^1$ and $N_r^* = N_r^1$ are what we want. If $k_{t^*+1}^1 > t^* + 1$, we claim that there exist $N_l^2 = \{ j_1^2, j_2^2, \dots, j_l^2 \}$ and $N_r^2 = \{ 1, 2, \dots, t^*, k_{t^*+1}^2, \dots, k_r^2 \}$ such that $j_l^2 \geq k_r^2$, $k_{t^*+1}^2 = k_{t^*+1}^1 - 1$, and $\sum_{j\in N_l^1} \sum_{k\in N_r^1} a_{j,k} \leq \sum_{j\in N_l^2} \sum_{k\in N_r^2} a_{j,k}$. If so, we can assume $k_{t^*+1}^2 = t^* + 1$ after applying this process $k_{t^*+1}^1 - (t^* + 1)$ times. With the help of (3.7) and (3.8), this leads us to the choices of N_l^* and N_r^* after we apply the same argument to the other k_t^2's. It remains to prove the existence of N_l^2 and N_r^2. Set $j_0^1 = 0$ and let s^* be the smallest positive integer with $j_{s^*}^1 \geq k_{t^*+1}^1$. This s^* exists, because $j_l^1 \geq k_r^1 \geq k_{t^*+1}^1$. We have $j_{s^*-1}^1 < k_{t^*+1}^1$, so $a_{j_s^1, k_t^1} = 0$ for all $1 \leq s < s^*$ and $t^* < t \leq r$. Here we use the fact that A is a lower triangular matrix. If $j_{s^*}^1 > j_{s^*-1}^1 + 1$, let $N_l^2 = \{ j_s^1 : 1 \leq s < s^* \} \cup \{ j_s^1 - 1 : s^* \leq s \leq l \}$ and $N_r^2 = \{ 1, 2, \cdots, t^* \} \cup \{ k_t^1 - 1 : t^* < t \leq r \}$. It follows from (3.2) that

$$\begin{aligned} \sum_{j\in N_l^1} \sum_{k\in N_r^1} a_{j,k} &\leq \sum_{1\leq s<s^*} \sum_{t=1}^{t^*} a_{j_s^1,t} + \sum_{s=s^*}^{l} \sum_{t=1}^{t^*} a_{j_s^1-1,t} + \sum_{s=s^*}^{l} \sum_{t=t^*+1}^{r} a_{j_s^1-1,k_t^1-1} \\ &\leq \sum_{j\in N_l^2} \sum_{k\in N_r^2} a_{j,k}. \end{aligned}$$

Clearly, N_l^2 and N_r^2 satisfy the desired properties. Next, consider the case $j_{s^*}^1 = j_{s^*-1}^1 + 1$. Suppose that s^{**} is the smallest positive integer in $[s^*, l]$ such that $j_{s^{**}}^1 > j_{s^{**}-1}^1 + 1$. Such s^{**} may not exist. In this case, we set $s^{**} = l+1$. We have $j_s^1 = j_{s-1}^1 + 1$ for all $s^* \leq s < s^{**}$. Let $N_l^2 = \{ j_s^1 : 1 \leq s < s^{**} \} \cup \{ j_s^1 - 1 : s^{**} \leq s \leq l \}$ and $N_r^2 = \{ 1, 2, \dots, t^* \} \cup \{ k_t^1 - 1 : t^* < t \leq r \}$. By (3.2), we get

$$\begin{aligned} \sum_{j\in N_l^1} \sum_{k\in N_r^1} a_{j,k} \leq &\sum_{1\leq s<s^{**}} \sum_{t=1}^{t^*} a_{j_s^1,t} + \sum_{s=s^{**}}^{l} \sum_{t=1}^{t^*} a_{j_s^1-1,t} \\ &+ \sum_{s=s^*}^{s^{**}-1} \sum_{t=t^*+1}^{r} a_{j_s^1-1,k_t^1-1} + \sum_{s=s^{**}}^{l} \sum_{t=t^*+1}^{r} a_{j_s^1-1,k_t^1-1}. \end{aligned} \tag{3.9}$$

We have

$$\sum_{s=s^*}^{s^{**}-1} \sum_{t=t^*+1}^{r} a_{j_s^1-1,k_t^1-1} = \sum_{s=s^*-1}^{s^{**}-2} \sum_{t=t^*+1}^{r} a_{j_s^1,k_t^1-1} \leq \sum_{1\leq s<s^{**}} \sum_{t=t^*+1}^{r} a_{j_s^1,k_t^1-1}. \tag{3.10}$$

https://doi.org/10.4153/CJM-2008-025-5 Published online by Cambridge University Press

Putting (3.9) and (3.10) together yields $\sum_{j\in N_l^1}\sum_{k\in N_r^1} a_{j,k} \le \sum_{j\in N_l^2}\sum_{k\in N_r^2} a_{j,k}$. This leads us to the conclusion. ∎

Corollary 3.4 allows us to deal with the case $A = A_W^{NM}$ with $w_n\downarrow$.

Corollary 3.5 *Let $w_1 > 0$ and $w_n \ge 0$ for all $n > 1$. Then Theorems 2.2 and 2.3 remain true, if $(*)$ is replaced by $A = A_W^{NM}$ with $w_n\downarrow$. Moreover, the conclusions of Theorems 2.2 and 2.3 also hold for $(A_W^{NM})^t$ in place of A.*

Proof We know that $A_W^{NM} = (a_{j,k}^{NW})_{j,k\ge 1}$ is a summability matrix. The hypothesis that $w_n \ge 0$ and $w_n\downarrow$ implies

$$\frac{w_1+\cdots+w_{j+1}}{w_1+\cdots+w_j} \ge 1 \ge \frac{w_{j-k+2}}{w_{j-k+1}} \qquad (j \ge k \ge 1).$$

This leads us to (3.2) for $a_{j,k} = a_{j,k}^{NW}$. By Corollary 3.4, we get the desired result. ∎

For $w_n = \binom{n+\alpha-2}{n-1}$, $A_W^{NM} = C(\alpha)$. Moreover, $w_n\downarrow \Leftrightarrow 0 < \alpha \le 1$. Hence, by Corollary 3.5, the conclusions of Theorems 2.2 and 2.3 hold for A to be one of the matrices $C(\alpha), C(\alpha)^t$ $(0 < \alpha \le 1)$.

The matrix A_W^{NM} involved in Corollary 3.5 is row increasing in the triangular sense, that is, $a_{j,k} \le a_{j,k+1}$ for all $j > k$. This fact does not imply that Corollary 3.5 can be extended to any summability matrix with rows increasing in the triangular sense. A counterexample is given below:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For this example, we have

$$\begin{aligned} \|A\|_{\ell_p,\ell_p} &= \sup_{\|x\|_p=1, x\ge 0} \left(x_1^p + x_2^p + \left(\frac{x_2+x_3}{2}\right)^p + x_4^p + \cdots \right)^{1/p} \\ &= \sup_{\|x\|_p=1, x\ge 0} \left(1 + \left(\frac{x_2+x_3}{2}\right)^p - x_3^p \right)^{1/p} \quad (1 \le p < \infty). \end{aligned}$$

Such a supremum is not attained by non-negative decreasing sequences.

4 An Equivalent Form of $(*)$ and Its Consequences

For an $n\times\infty$ matrix $B = (b_{j,k})$, we write $B \in \mathcal{R}_{A_n}^{\downarrow}$ if $B \in \mathcal{R}_{A_n}$ and (i)–(iii) are satisfied by some pair (γ,λ) with $0 \le \gamma \le \lambda \le n$, where

(i) $b_{j,k} \ge b_{j+1,k}$ for $j \le \gamma$ and for $j \ge \lambda$.

https://doi.org/10.4153/CJM-2008-025-5 Published online by Cambridge University Press

(ii) $b_{r_1,k} \geq b_{j,k} \geq b_{r_2,k}$ for $r_1 \leq \gamma < j < \lambda \leq r_2$.
(iii) No α with $\gamma < \alpha < \lambda$ possesses the property $b_{\alpha,k} \geq b_{j,k}$ for all $\gamma < j < \lambda$ or $b_{\alpha,k} \leq b_{j,k}$ for all $\gamma < j < \lambda$.

In other words, B is of the form:

$$B = \begin{pmatrix} \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ * & * & * & * & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \ddots \end{pmatrix}. \tag{4.1}$$

In (4.1), the top $\geq$ the middle $\geq$ the bottom (ii), the top and the bottom are decreasing (i), and the middle has no decreasing property (iii). Moreover, the top (respectively middle, bottom) disappears for the case $\gamma = 0$ (respectively $\gamma = \lambda$, $\lambda = n$). By taking row permutations, we can easily prove $\mathcal{R}_{A_n}^{\downarrow} \neq \varnothing$ for all $n \geq 1$. It is clear that $(*) \Rightarrow (**)$, where

> for any $n \geq n_0$ and any $B = (b_{j,k}) \in \mathcal{R}_{A_n}^{\downarrow}$, there exists some $C = (c_{j,k}) \in \mathcal{R}_{A_n}$, depending on n and B, such that the following inequality holds:

$$\sum_{j=1}^{l} \sum_{k=1}^{r} c_{j,k} \geq \sum_{j=1}^{l} \sum_{k \in N_r} b_{j,k} \quad (1 \leq l \leq n;\ r \geq 1;\ |N_r| = r). \tag{$**$}$$

In the following, we show that the converse is also true. This enables us to rewrite Theorems 2.2 and 2.3 in a more general form.

Theorem 4.1 *For $A \geq 0$, we have $(**) \Leftrightarrow (*)$. Hence, Theorems 2.2 and 2.3 remain true if $(*)$ is replaced by $(**)$.*

Proof It suffices to prove $(**) \Rightarrow (*)$. Let $n \geq n_0$ and $B = (b_{j,k}) \in \mathcal{R}_{A_n}$. By taking suitable row permutations, we can find a matrix $\tilde{B} = (\tilde{b}_{j,k}) \in \mathcal{R}_{A_n}^{\downarrow}$ such that

$$\sum_{j=1}^{l} \sum_{k \in N_r} b_{j,k} \leq \sum_{j=1}^{l} \sum_{k \in N_r} \tilde{b}_{j,k} \quad (1 \leq l \leq n, r \geq 1; |N_r| = r). \tag{4.2}$$

After replacing $b_{j,k}$ in $(**)$ by $\tilde{b}_{j,k}$, we conclude that the desired result follows from the combination of $(**)$ and (4.2). ∎

In general, $\mathcal{R}_{A_n}^{\downarrow}$ is much smaller than $\mathcal{R}_{A_n}$, so the investigation of $(**)$ is easier than that for $(*)$ in some cases. We shall see this advantage below. Consider the following condition for $n \geq n_0$ and $B = (b_{j,k}) \in \mathcal{R}_{A_n}^{\downarrow}$:

$$\Big\{ \sum_{j=1}^{l} b_{j,k} \Big\}_{k=1}^{\infty} \text{ is a decreasing sequence } (1 \leq l \leq n). \tag{4.3}$$

By considering $N_r = \{1, \ldots, r-1, r+1\}$, we see that (4.3) is equivalent to the particular case $c_{j,k} = b_{j,k}$ of $(**)$. Applying Theorem 4.1, we obtain the following consequence.

https://doi.org/10.4153/CJM-2008-025-5 Published online by Cambridge University Press

Corollary 4.2 *Theorems* 2.2 *and* 2.3 *remain true, if* $(*)$ *is replaced by* (4.3).

In some cases, (4.3) is satisfied by the entries of $B \in \mathcal{R}^{\downarrow}_{A_n}$, while it may be false for some $B \in \mathcal{R}_{A_n}$ (see the example given after Corollary 4.3).

For $B = (b_{j,k}) \in \mathcal{R}^{\downarrow}_{A_n}$, we have $\sum_{j=1}^{l} b_{j,k} = \sum_{j \in N_l} a_{j,k}$ for some index set N_l with $|N_l| = l$. This enables us to prove that $(* * *) \Rightarrow$ (4.3) with $n_0 = 1$:

$(* * *)$ A is row decreasing, that is, $a_{j,k} \geq a_{j,k+1}$ for all $j, k \geq 1$.

As a consequence of Corollary 4.2, we get the following result.

Corollary 4.3 *Theorems* 2.2 *and* 2.3 *remain true, if* $(*)$ *is replaced by* $(* * *)$.

This corollary extends [5, Lemma 2.4] (respectively [8, Lemma 2.1]) from the pair (ℓ_p, ℓ_p) (respectively $(d(w,p), d(w,p))$) to the pair (E, F) for those A of the form $(* * *)$. Moreover, it indicates that the condition (i) in [8, Lemma 2.1] is redundant. The matrix $A = (a_{j,k})_{j,k \geq 1}$, with $a_{2,1} = a_{2,2} = a_{3,1} = 1$ and 0 otherwise, shows that $(* * *)$ is not a special case of (1.1). Next, consider the matrix A, defined by

$$A = \begin{pmatrix} 4 & 3 & 2 & 2 & 0 & \cdots \\ 1 & 1.5 & 2 & 2 & 0 & \cdots \\ 1.5 & 1.5 & 1.8 & 1.8 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 4 & 3 & 2 & 2 & 0 & \cdots \\ 1.5 & 1.5 & 1.8 & 1.8 & 0 & \cdots \\ 1 & 1.5 & 2 & 2 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this example, $\mathcal{R}^{\downarrow}_{A_2} = \{A_2\}$ and $\mathcal{R}^{\downarrow}_{A_n} = \{A_n, \tilde{A}_n\}$ for $n \geq 3$. It is easy to see that (4.3) with $n_0 = 2$ is satisfied by such A, but $(* * *)$ fails to hold for this matrix. This example differs (4.3) from $(* * *)$.

The entries of the Hilbert matrix H satisfy $(* * *)$. Hence, Corollary 4.3 can apply to $A = H$. On the other hand, by applying Corollary 4.3 to the Nörlund mean matrix A^{NW}_{W}, we get the following consequence.

Corollary 4.4 *Let* $w_1 > 0$ *and* $w_n \geq 0$ *for all* $n > 1$. *Then Theorems* 2.2 *and* 2.3 *remain true if* $(*)$ *is replaced by* $A = A^{NM}_{W}$ *with* $w_n \uparrow$.

Corollary 4.4 is a generalization of Corollary 3.3(ii) for the Nörlund mean matrix A^{NM}_{W}. For this matrix, the condition $w_{n+1}/w_n \leq w_n/w_{n-1}$, required in Corollary 3.3(ii), is not necessary. However, we do not know whether this condition can be removed from there for the transpose $(A^{NM}_{W})^t$. For the case $w_n = \binom{n+\alpha-2}{n-1}$, it does, (see the statement given after the proof of Corollary 3.3). It is still open for general w_n.

Remark. It is clear that (4.4) is a weak form of (1.1):

$$\sum_{j=1}^{l} \sum_{k=1}^{r} a_{j,k} \geq \sum_{j=1}^{l} \sum_{k \in N_r} a_{j,k} \qquad (l, r \geq 1; \quad |N_r| = r). \tag{4.4}$$

By considering $N_r = \{1, \dots, r-1, r+1\}$, we infer that (4.4) is equivalent to the second inequality in (3.1). The matrix $A = (a_{j,k})_{j,k\geq 1}$, with $a_{1,1} = a_{2,2} = a_{2,3} = 1$ and 0 otherwise, satisfies (4.4) and obeys the inequality: $\|A\|_{\ell_2,\ell_2} > \|A\|_{\ell_2,\ell_2,\downarrow}$. The last fact follows from the observation that

$$\|A\|_{\ell_2,\ell_2} = \sup_{\|x\|_2=1, x\geq 0} (x_1^2 + x_2^2 + x_3^2 + 2x_2x_3)^{1/2}$$

is attained only at

$$x = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \cdots\right),$$

which is not a decreasing sequence. This example shows that the first inequality in (3.1) cannot be removed from the hypotheses of Corollary 3.2 and the condition (1.1) in Theorem 3.1 cannot be relaxed to (4.4).

Acknowledgement We would like to express our gratitude to the referee for his valuable comments in developing the final version of this article.

References

[1] M. A. Ariño and B. Muckenhoupt, *A characterization of the dual of the classical Lorentz sequence space $d(w, q)$*. Proc. Amer. Math. Soc. **112**(1991), no. 1, 87–89.
[2] G. Bennett, *Some elementary inequalities.* Quart. J. Math. Oxford **38**(1987), no. 152, 401–425.
[3] ———, *Inequalities complimentary to Hardy.* Quart. J. Math. Oxford **49**(1998), no. 196, 395–432.
[4] J. Boos, *Classical and Modern Methods in Summability.* Oxford University Press, New York, 2000.
[5] C.-P. Chen, D.-C. Luor, and Z.-Y. Ou, *Extensions of Hardy inequality.* J. Math. Anal. Appl. **273**(2002), no. 1, 160–171.
[6] G. J. O. Jameson, *Norms and lower bounds of operators on the Lorentz sequence space $d(w, 1)$*. Illinois J. Math. **43**(1999), no. 1, 79–99.
[7] G. J. O. Jameson and R. Lashkaripour, *Lower bounds of operators on weighted ℓ_p spaces and Lorentz sequence spaces.* Glasg. Math. J. **42**(2000), no. 2, 211–223.
[8] ———, *Norms of certain operators on weighted ℓ_p spaces and Lorentz sequence spaces.* J. Ineq. Pure Appl. Math. **3**(2002), no. 1, 17 pp. (electronic)
[9] P. Meyer-Nieberg, *Banach Lattices.* Springer-Verlag, New York, 1991.
[10] A. Zygmund, *Trigonometric Series.* Vol. I, Second edition, Cambridge University Press, London, 1968.

Department of Applied Mathematics, Hsuan Chuang University, Hsinchu 300, Taiwan, Republic of China
e-mail: cpchen@wmail.hcu.edu.tw

Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.
e-mail: huang39@indiana.edu
e-mail: shenc@indiana.edu