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A CHARACTERIZATION OF THE TITS' SIMPLE GROUP

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In [**6**], J. Tits has shown that the Ree group ${}^2F_4(2)$ is not simple but possesses a simple subgroup $\mathscr{T}$ of index 2. In this paper we prove the following theorem:

THEOREM. *Let G be a finite group of even order and let z be an involution contained in G. Suppose $H = C_G(z)$ has the following properties:*
(i) $J = O_2(H)$ *has order* 2^9 *and is of class at least* 3.
(ii) H/J *is isomorphic to the Frobenius group of order* 20.
(iii) *If P is a Sylow* 5*-subgroup of H, then* $C_J(P) \subseteq Z(J)$.
Then $G = H \cdot O(G)$ or $G \cong \mathscr{T}$, the simple group of Tits, as defined in [**6**].

For the remainder of the paper, G will denote a finite group which satisfies the hypotheses of the theorem as well as $G \neq H \cdot O(G)$. Thus Glauberman's theorem [**1**] can be applied to G and we have that $\langle z \rangle$ is not weakly closed in H (with respect to G). The other notation is standard (see [**2**], for example).

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1. Some properties of H**.** In the notation of the theorem we prove:

LEMMA 1. *We have that* $\mathrm{cl}(J) = 3$, $Z(J) = Z(H) = \langle z \rangle$, *and a Sylow* 2*-subgroup T of H is a Sylow* 2*-subgroup of G. Finally,* $E = J' = Z_2(J) = \Phi(J)$ *is elementary of order* 32.

Proof. Since $C_J(P) \subseteq Z(J)$ and $\mathrm{cl}(J) \geqq 3$, P cannot act trivially on J or J', so $|J : \Phi(J)| \geqq 16$ and $|J' : J' \cap Z(J)| \geqq 16$. As $|J| = 2^9$ we must have $\Phi(J) = J'$ and $|Z(J) \cap J'| = 2$ (or alternatively

$$|J : \Phi(J)| = |J' : J' \cap Z(J)| = 16).$$

Further, as $Z(J)$ is P-invariant and $\mathrm{cl}(J) \geqq 3$, $Z(J) \subseteq J'$ whence $Z(J) = \langle z \rangle$ is of order 2 and $\mathrm{cl}(J) = 3$. Put $E = J'$ and note that E is abelian (as $E' = (J')'$ and $\mathrm{cl}(J) = 3$). It follows that E is elementary abelian for $|E| = 32$ and $E = \langle z \rangle \times [P, E] = C_E(P) \times [P, E]$. We note that $E = Z_2(J)$ as $Z_2(J) \lhd H$ and $\langle z \rangle = L_3(J) = [J, J']$ as $L_3(J) \lhd H$.

If T is a Sylow 2-subgroup of H, clearly $Z(T) = \langle z \rangle$. But then $\langle z \rangle \lhd N_G(T)$ so $N_G(T) \subseteq H$. It follows immediately from Sylow's theorem that T is a Sylow 2-subgroup of G.

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Throughout this paper we need some properties of the linear group GL(5, 2).

Properties of GL(5, 2). (1) $|\mathrm{GL}(5, 2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$.

(2) GL(5, 2) is a non-abelian simple group.

(3) An involution in GL(5, 2) has centralizer of order $2^9 \cdot 3$ or $2^{10} \cdot 3 \cdot 7$. In the latter case, the centralizer is a faithful splitting extension of an elementary group of order 16 by the holomorph of an elementary group of order 8 (see [**3**]).

(4) If τ is an element of order 3 in GL(5, 2), either $C_G(\tau) \cong \langle\tau\rangle \times A_5$ or $C_G(\tau) \cong \langle\tau\rangle \times \mathrm{PSL}(2, 7)$. Further, a Sylow 3-normalizer is a faithful extension of an elementary abelian group of order 9 by D_8–the dihedral group of order 8 (see [**3**]).

From properties (1)–(4) and Sylow's theorem we also have:

(5) A Sylow 5-centralizer is cyclic of order 15 and a Sylow 5-normalizer has order $3 \cdot 4 \cdot 5$.

(6) A Sylow 7-normalizer is the direct product of a non-abelian group of order 6 and a Frobenius group of order 21.

(7) A Sylow 31-normalizer is a Frobenius group of order $5 \cdot 31$.

LEMMA 2. *We have $N_G(E) = H$ and z is conjugate* (*in* G) *to an involution in* $H - E$.

Proof. Since $C_G(E) = C_H(E) \lhd H$, $C_G(E) = E$ and therefore $N_G(E)/E$ is isomorphic to a subgroup of GL(5, 2). If t is any involution in $E - \langle z\rangle$, t has either 10 or 20 conjugates in H, whence z has 1, 11, 21, or 31 conjugates in $N_G(E)$. Under the assumption $N_G(E) \supset H$, we have $|N_G(E)/E| = 2^6 \cdot 5 \cdot 11$, $2^6 \cdot 3 \cdot 5 \cdot 7$, or $2^6 \cdot 5 \cdot 31$. As GL(5, 2) does not possess subgroups of any of these orders (this is easily seen by using properties (1)–(7) above) we therefore have $N_G(E) = H$.

Suppose z is not conjugate (in G) to any involution in $H - E$. Then $z \sim_G t$ for some involution $t \in E - \langle z\rangle$ by Glauberman's theorem. It follows that E is the normal closure of $\langle z\rangle$ in $C_G(t)$ (as $H \sim_G C_G(t)$). This clearly contradicts $N_G(E) = H$.

We now list some properties of the group J which can be derived from Lemma 1.

(a) For $j \in J - E$, $|C_E(j)| = 16$ since $L_3(J) = [J, J'] = \langle z\rangle$.

(b) If j is an involution in $J - E$, $\mho^1(\langle j, E\rangle) = \langle z\rangle$ so that not all cosets of E in J contain involutions.

(c) If $J \supset J_1 \supset J_2 \supset J_3 \supset E$ is any (maximal) chain of subgroups from J to E, then $Z(J_i), J_i' \subset E$ and $|Z(J_i)| = 2^{i+1}$ ($i = 1, 2, 3$). Further, we have $|J_1'| \geqq 8$. (This last fact may be proved by noting that we may choose $a_i \in J - E$, $i = 1, \ldots, 4$, so that $J_1 = \langle E, a_1, a_2, a_3\rangle$, $a_4 = a_1{}^p$, and $J = \langle a_i | i = 1, \ldots, 4\rangle$, where $\langle p\rangle = P$. Also $\{z, [a_i, a_j] |$ for suitable $i, j\}$ is a basis for $J' = E$. Now if $J_1' = \langle z, t\rangle$ is of order 4, $|C_{J_1}(a_i)| \geqq 2^7$, $i = 1, 2, 3$ whence $|C_J(a_4)| \geqq 2^7$. It follows that $z, t, [a_1, a_4], [a_2, a_4], [a_3, a_4]$ are not linearly independent which contradicts $J' = E$.)

(d) For $j \in J - E$, $2^5 \leqq |C_J(j)| \leqq 2^6$, while for $e \in E - \langle z \rangle$, $|C_J(e)| = 2^8$. (If $|C_J(j)| = 2^7$, $C_J(j) \cdot E$ is maximal in J and $|(C_J(j) \cdot E)'| \leqq 4$, contrary to (c) above.)

The factor group H/E. Let $x \in N_H(P)$ so that $\langle x, z \rangle$ is a Sylow 2-subgroup of $N(P)$ (recall that $C_H(P) = P \times \langle z \rangle$ and note that $x^4 = 1$ or z). Put $P = \langle p \rangle$ with $p^x = p^2$ and put $E_0 = [P, E]$ which is $N_H(P)$-invariant of order 16. The structure of H/E is uniquely determined and can be described in the following way:

Identify J/E with the additive group of GF(16); then the action of p on J/E is given by scalar multiplication by an element ζ of order 5 in the multiplicative group of GF(16) and the action of x on J/E corresponds to the Galois automorphism of GF(16).

Clearly x fixes the coset Ea corresponding to $1 = \zeta^5$ in GF(16), while x^2 fixes the cosets Eb corresponding to $\zeta + \zeta^{-1}$ and Eab which corresponds to $1 + \zeta + \zeta^{-1} = \zeta^2 + \zeta^3$ as well as Ea. Note that Ea has 5 conjugates in H while Eb has 10 conjugates.

Put $T = \langle J, x \rangle$, $A = \langle a, E \rangle$ and $B = \langle a, b, E \rangle$; then T is a Sylow 2-subgroup of H (and hence of G), $A/E = Z(T/E) = C_{J/E}(x)$, and $B/E = Z(\Omega_1(T)/E) = C_{J/E}(x^2)$.

The Centralizer of an involution in $E - \langle z \rangle$. As $N_H(P) \cdot E/\langle z \rangle \cong H/E$, the action of $N_H(P)/\langle z \rangle$ on $E_0 = [P, E]$ is exactly the same as the action of H/J on J/E. Choose $t \in E_0$ so that t has precisely 5 conjugates under the action of $N_H(P)$ and hence t has 10 conjugates in H (as $t \sim_J tz$). Thus $|C_H(t)| = |C_T(t)| = 2^{10}$. As $C_T(t)$ is maximal in T, putting $C_J(t) = D$ we must have $D/E = \Phi(T/E) \cap J/E$. Clearly $B \subset D$ and we denote $Z(B) = \langle z, t, v \rangle$ by Z. It follows that $|C_T(v)| = |C_H(v)| = 2^9$ (as $Z \lhd T$ and $Z(D) = \langle t, z \rangle$).

Further, there are precisely two classes of involutions in $E - \langle z \rangle$ in H with representatives t, v; while if $u \in E - Z$ then $C_T(u) \subset J$.

Finally, if E has basis w, u, v, t, z we describe the action of x on E by:

$$x \sim \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

The case when there are involutions in $H - J$. If k is an involution in $H - J$, by a lemma due to Suzuki (see [**2**, p. 105 and p. 328]) k inverts an element of order 5. By Sylow's theorem, k is conjugate to an involution in $N_H(P)$, and hence to an involution $y \in \langle x, z \rangle - \langle z \rangle$. Hence as $\langle x, z \rangle - \langle z \rangle$ contains two involutions (under this assumption), any involution in $H - J$ is conjugate to either y or yz in H.

Now $C_E(y) = Z$ has order 8 and $C_{J/E}(y)$ has order 4, whence $T - J$ contains precisely 32 involutions. Thus either $y \sim_H yz$ and $|C_H(y)| = |C_T(y)| = 2^6$ or $y \nsim_H yz$ and $|C_H(y)| = |C_T(y)| = 2^7$.

The notation we have used above will remain fixed for the rest of the paper.

LEMMA 3. *There are involutions in* $J - E$.

Proof. We prove the lemma by way of contradiction. Thus by Lemma 2 we may assume $z \sim_G y$, y (as above) an involution in $T - J$. Now $X = \Omega_1(C_T(y)) = \langle y \rangle \times Z$ has order 16 and so E is the only elementary abelian subgroup of order 32 in T. This implies that z is not conjugate to any involution in $E - \langle z \rangle$ in G (as $N_G(E) = H$ by Lemma 2).

As $z \sim_G y$, $C_T(y) = C_H(y)$ is not a Sylow 2-subgroup of $C_G(y)$, whence $N_G(C_T(y)) \supset N_T(C_T(y))$ by Sylow's theorem. Since X char $C_T(y)$, $N_G(X) \supset N_T(X) = N_H(X)$. From $E \subseteq N_T(X)$, it follows that y has at least 4 conjugates in $N_T(X)$ and so y has 4 or 8 conjugates in $N_G(X)$.

In the latter case z has 9 conjugates in $N_G(X)$ (as z is not conjugate to any involution in $E - \langle z \rangle$). This implies that $\{e : e \in Z - \langle z \rangle\} \lhd N_G(X)$. Hence $Z = \langle \{e : e \in Z - \langle z \rangle\} \rangle \lhd N_G(X)$ and $\langle z \rangle \lhd N_G(X)$, which is a contradiction. Therefore z has 5 conjugates in $N_G(X)$; i.e., $|N_G(X) : N_T(X)| = 5$. Because y has only 4 conjugates in $N_T(X)$, $N_T(X) = \langle B, x \rangle$ and $|N_T(X) : C_T(X)| = 8$ for $x \in N_T(X) - C_T(X)$. However, this yields $|N_G(X) : C_G(X)| = 5 \cdot 8 = 2^3 \cdot 5$ which contradicts the structure of $A_8 \cong \mathrm{GL}(4, 2)$. The lemma is proved.

From the remarks above, either the coset Ea or the coset Eb contains involutions, but not both.

LEMMA 4. *There are involutions in the coset* Ea (*or, alternatively,* Eb *does not contain involutions*).

Proof. Suppose Eb contains involutions. We use the same notation as above; that is, J/E is identified with the additive group of GF(16) and ζ is an element of order 5 in the multiplicative group of GF(16). Then $Eb \leftrightarrow \zeta + \zeta^{-1}$ so that the cosets $Ea_i \leftrightarrow 1 + \zeta^i$ ($i = 1, 2, 3, 4$) also contain involutions. (The conjugates of Eb in H are the cosets which correspond to $\zeta^l + \zeta^j$ ($0 \leqq l < j \leqq 4$).) Further, $Ea_ia_j \leftrightarrow \zeta^i + \zeta^j$ so the coset Ea_ia_j also contains an involution. By (b), $E\langle a_i, a_j \rangle / \langle z \rangle$ is elementary; that is, $[a_i, a_j] \in \langle z \rangle$. Thus we may choose $\bar{a}_j \in Ea_j$ ($j = 1, 2, 3, 4$) such that $[\bar{a}_j, a_i] = 1$ for any fixed i ($i = 1, 2, 3$, or 4). This is clearly a contradiction as $a_i \in C_J\langle \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4 \rangle = C_J(J) = Z(J) = \langle z \rangle$.

2. The fusion of involutions in G. As Ea contains involutions, we take a to be an involution. Put $F = \langle a \rangle \times C_E(a)$ so that F is elementary of order 32. Clearly $F \lhd T$ (as $A \lhd T$) and $C_G(F) = C_T(F) = F$. We show by way of contradiction that $N_G(F) \supset T$.

Suppose $N_G(F) = T$; we will show that z is not conjugate to any involution in $T - \langle z \rangle$ in G which will contradict Glauberman's Theorem. First consider the case that z is not conjugate to any involution in $J - \langle z \rangle$ in G. Then there are involutions in $T - J$ and we may assume $z \sim_G y$. Put $X = \Omega_1(C_T(y))$ and $W = O_2(N_G(X))$. If $X = Z \times \langle y \rangle$, then we get a contradiction in exactly the same way as in Lemma 3. Hence X is elementary of order 32 and y must have 8 or 16 conjugates in $N_T(X)$ (as $yaE \sim_T yE$ and X covers A/E). Note that $N_G(X) \supset N_T(X)$ as $z \sim_G y$ and $N_G(X)/C_G(X)$ is isomorphic to a subgroup of GL(5, 2). From the structure of T, it follows that $C_T(X) = X$ or $|C_T(X) : X| = 2$ (in this latter case, $C_T(X)$ covers B/E). If y has 16 conjugates in $N_T(X)$, z has 17 conjugates in $N_G(X)$ which contradicts the order of GL(5, 2). Thus z has 9 conjugates in $N_G(X)$, whence $|N_G(X) : N_T(X)| = 9$ and $|N_G(X)| = 2^{10} \cdot 3^2$. Now from property (3) (of GL(5, 2)) it follows that $|W : C_G(X)| = 2^4$, so $|W| = 2^9$ or 2^{10}. As $W \subseteq T$, $z \in Z(W)$. However from the structure of T we see that $|Z(W)| \leqq 4$. This implies that z has at most 3 conjugates in $N_G(X)$, which is a contradiction.

Next consider the case when z is conjugate to an involution in $J - E$, but not conjugate to an involution in $E - \langle z \rangle$. Without loss of generality we may suppose $z \sim_G a$. Put $S = C_T(a) = C_H(a)$, and note that $N_G(S) \supset N_T(S) = N_H(S)$ and that a has at most 16 conjugates in T (i.e., $|C_T(a)| \geqq 2^7$, so by (d), $2^7 \leqq |C_T(a)| \leqq 2^8$).

We claim that $Z(S) = \langle z, t, a \rangle$. If S covers T/J this follows immediately from $C_E(x) = \langle z, t \rangle$. (Note that as S covers $\langle J, y \rangle / J$ in any case and $|C_J(a)| \leqq 2^7$, $Z(S) \supseteq \langle z, t, a \rangle$.) In the other possibility we must have $F \subset J \cap S$, and hence $z \in S'$. By assumption z is not conjugate to any involution in $E - \langle z \rangle$ which implies $S/S \cap E$ is non-abelian. This forces $Z(S) = \langle z, t, a \rangle$.

It now follows immediately that $N_G(Z(S))/C_G(Z(S)) \cong S_3$, the symmetric group on 3 letters. Clearly $S = C_G(Z(S))$ and $E \cdot S$ is a Sylow 2-subgroup of $N_G(S)$. As $3 \| N_G(\Omega_1(S))|$, F cannot be maximal in $\Omega_1(S)$, and hence $|\Omega_1(S) : F| = 4$. Thus $|S| = 2^8$ and in particular S covers T/J. A simple computation shows that $S' = \langle z, t, a, v \rangle \subset \Omega_1(S)$. By another computation we see that for $w \in E - F$, $[w, \Omega_1(S)] \subseteq S'$.

By Suzuki's lemma, w inverts a Sylow 3-subgroup Q of $N_G(S)$. It follows immediately that Q stabilizes the chain $S \supset \Omega_1(S) \supset S'$ (as $|S : \Omega_1(S)| = 2$). Hence Q centralizes S and in particular $Q \subseteq C_G(z) = H$ which is impossible.

Under the assumption $N_G(F) = T$, we must have either $z \sim_G t$ or $z \sim_G v$. In the first case, put $C = C_T(t)$ and note that C covers T/J. Thus we have $E \cap F \subseteq C'$ while $(C/E)' = B/E$ whence $E \cap F \subseteq \Omega_1(C') \subseteq A$.

From $Z(C) = \langle t, z \rangle$ and $t \sim_G z$ it follows immediately that $N_G(\langle t, z \rangle)/C \cong S_3$ ((clearly $C = C_G(\langle t, z \rangle)$). If $\Omega_1(C') \subset A$, we must have $\Omega_1(C') = E \cap F$. But $C_G(E \cap F) = A$ so in any case $A \lhd N_G(\langle t, z \rangle)$. However,

$$\langle z \rangle = A' \text{ char } A \lhd N_G(\langle t, z \rangle)$$

which immediately gives a contradiction.

Finally we suppose $z \sim_G v$ and put $V = C_T(v)$. From $Z(V) = \langle z, v\rangle$ we have as above, $N_G(V)/V \cong S_3$. A computation shows $(V/E)' = \langle h, E\rangle/E$ for some $h \in B - A$. If V' is non-abelian, $(V')' = \langle z\rangle$ which is impossible. However from V' is abelian it follows that $\mho^1(V') = \langle k\rangle \lhd N_G(V)$ for some involution k, which is also impossible. We have proved:

LEMMA 5. *The normalizer $N = N_G(F)$ of F in G properly contains T.*

Put $K = O_2(N)$ and recall that $C_G(F) = F$. We have N/F is isomorphic to a subgroup of $GL(5, 2)$ and $|N/F| = 2^6 \cdot n$, where $1 < n \leqq 31$, n odd.

Using properties (1)–(7) above, Sylow's theorem, and the transfer theorem, we see that $|O_2(N/F)| \geqq 2^4$ unless $n = 3 \cdot 7$, in which case $|O_2(N(F))| \geqq 2^3$. In the latter case, if $|K| = 2^8$, $|Z(K)| \leqq 8$. Clearly $z \in Z(K)$ so z has at most 7 conjugates in N, contradicting $n = 21 = |N : T| = |N : C_N(z)|$.

Therefore we have $|K| \geqq 2^9$, whence $|Z(K)| \leqq 4$. As $z \in Z(K)$, z has at most 3 conjugates in N. It follows immediately that $n = 3$, $|Z(K)| = 4$, and $N/K \cong S_3$. Further, the structure of T shows that $Z(K) = \langle t, z\rangle$; i.e., $K = C_T(t)$. If there are no involutions in $T - J$, $\Omega_1(K) = A$ which implies $\langle z\rangle \lhd N$. Thus $T - J$ contains involutions. The structure of T shows $\Omega_1(K) = \langle B, y\rangle$ (of index 4 in K) and so $Z(\Omega_1(K)) = Z = \langle z, t, v\rangle$. It follows immediately that a Sylow 3-subgroup Q of N centralizes v (note that all involutions in $Z - \langle t, z\rangle$ are conjugate in K). We have:

LEMMA 6. *If $N = N_G(F)$ and $K = O_2(N)$, then $N/K \cong S_3$. Further, $T - J$ possesses involutions, $z \sim_G t$, but $z \nsim_G v$ as $3||C_G(v)|$.*

Put $U = \Omega_1(K) = C_G(Z)$ and note that $C_Z(Q) = \langle v\rangle$. We claim that $C_F(Q) = \langle v\rangle$. If not, $|C_F(Q)| = 8$, and so there exists an involution $e \in E$ with $e \in C_F(Q) - \langle v\rangle$. As $C_K(e) \subseteq C_J(e)$, it follows immediately that $C_U(e) = A$. However $C_U(e)$ is Q-invariant whence Q normalizes $A' = \langle z\rangle$, which is a contradiction.

Since N/U is isomorphic to a subgroup of $GL(3, 2)$, we have $N/U \cong S_4$. An easy computation shows that $K' = \langle b, F\rangle$ (where $b \in B - A$ and b is of order 4). From $|K' : F| = 2$ we have without loss that $b \in C_U(Q)$. Thus $b^2 = v$ and $C_K(Q) = \langle b\rangle$ is cyclic of order 4.

The involution $u \in (F \cap E) - Z$ has centralizer $C_N(u)$ of order 2^8 $(C_N(u) = C_J(u) = C_T(u))$ whence u has 24 conjugates in N. All involutions in $F - Z$ are therefore conjugate to v in G (for $E \cap F - Z$ must contain involutions conjugate to v in H). This means that all involutions in $J - E$, and all involutions in $F - \langle t, z\rangle$, are conjugate to v in G.

In U, F has precisely three (non identity) cosets which contain involutions: Fw, Fy, and $Fywb$ where $w \in E - F$ (as above). Clearly Q permutes these three cosets. (Remark. $Fywb$ contains involutions rather than Fyb because $C_T(y)$ covers $\langle F, b\rangle/F$ as $bf \in C_T(w)$ (for some $f \in F$); thus $v \in \mho^1(\langle yb, F\rangle)$ whereas $\mho^1(\langle w, F\rangle) = \langle z\rangle$, $\mho^1(\langle y, F\rangle) = \langle t\rangle$ and $\langle t, z\rangle \lhd N$.) The coset Fw

contains 16 involutions, 8 of which are conjugate to z in G (as they are conjugate to t in H) and 8 of which are conjugate to v (in H). It follows therefore that $y \nsim_G yz$. Without loss we take $z \sim_G y$ (whence $v \sim_G yz$) and note that $|C_T(y)| = 2^7$.

LEMMA 8. *The group G has precisely two conjugate classes of involutions with representatives z and v.*

3. Generators and relations for N and H. We recall that $E = \langle z, t, v, u, w\rangle$, $F = \langle z, t, v, u, a\rangle$, $T = \langle x, J\rangle$, and $\langle x\rangle \times \langle z\rangle$ is a Sylow 2-subgroup of $N_H(P)$. Further

(1) $$x^4 = [x, t] = 1, [x, v] = t, [x, u] = v, [x, w] = u.$$

From these relations we derive $[y, v] = 1$, $[y, u] = t$ and $[y, w] = v$. Without loss we take $[b, w] = 1$ so

(2) $$[b, w] = 1, [a, w] = [b, u] = z.$$

As $u \sim_N a$ it follows that $|C_K(a)| = |C_T(a)| = 2^7$. Also $C_J(a) = F$ and so $C_T(a)$ covers T/J. As $C_K(b)$ is Q-invariant and $w \in C_K(b)$, $C_K(b)$ covers U/F; but $x \notin C_T(b)$ so $C_K(b) = C_U(b)$. Let d be an involution in $J - D$ (i.e., $d \in T - K$); by Suzuki's result, d inverts an element of odd order in N so we may assume $d \in N_N(Q)$ by Sylow's theorem. However, as $d \sim_H a$, $C_J(d) = \langle d\rangle \times C_E(d)$ whence $d \notin C_N(b)$. As $d \in N_N(\langle b\rangle) = N_N(C_K(Q))$, we have

(3) $$[d, b] = b^2 = v, [d, t] = z.$$

Note that $|C_T(b)| = 2^6$ so that bE possesses two classes of elements of order 4 in T with representatives b and bw. As $x \sim_N c^*$ for some $c^* \in D - B$, $|C_E(x)| = 8$ whence $C_T(x)$ covers A/E and $|C_T(x)| = 2^5$. Further, as $(c^*)^2 \sim_H v$,

(4) $$x^2 = yz.$$

Since $\langle z, [a, b]\rangle = \langle b, F\rangle' \operatorname{char}\langle b, F\rangle \lhd N$, $[a, b]$ must be either t or tz. Replacing a by au if necessary, we take

(5) $$[a, b] = t.$$

Choose $q \in Q$ so that $z^q = t$. Now $[u, b] = z$ and $[a, b] = t$ so $u^q = ae$ for some $e \in Z$. Put $x^{q^{-1}} = c^*$; then as $[x, Z] = \langle t\rangle$, $[c^*, Z] = \langle z\rangle$. We see that $c^* \in D - B$. Thus $[c^*, u] \in \langle z\rangle$ yields $[x, ae] \in \langle t\rangle$ so $[x, a] = 1$ or t. In either case

(6) $$[y, a] = 1.$$

Next we choose w, u more exactly. Namely, replacing w by wt if necessary,

(7) $$[d, w] = 1$$

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and replacing u by ut and hence w by wv (so that (1) still holds),

(8) $$[d, u] = 1.$$

We now choose $c \in D - B$ so that

(9) $$[c, u] = [c, w] = 1$$

and clearly

(10) $$[c, t] = 1, [c, v] = z.$$

Using (1) we see that $[c, x] \in ab\,E$, $[d, x] \in abc\,E$, and the cosets aE, dE, $abcdE$, cbE, and dbE contain involutions. Further, the conjugates of t in H are found easily by noticing that $C_E(d)$ contains precisely three involutions conjugate to z in G. The conjugates of t in H are: t, tz, wt, wtz, wut, $wutz$, wvt, $wvtz$, wuv, $wuvz$.

Since $\langle a, d\rangle$ is dihedral of order 8, $[a, d] \in \langle u, v, z\rangle = C_E(\langle a, d\rangle)$. From (2) and (3), replacing a by at if necessary, either

(11) $$[a, d] = u$$

or

(11′) $$[a, d] = uv.$$

Case 1: *Relation* (11) *holds*. Using (11), (1), $[a, x] \in \langle t\rangle$, $[d, x] \in abc\,E$, and (5), we deduce that $[a, c]$ lies in the coset $vt\langle z\rangle$. We can replace c by cw if need be to get

(12) $$[a, c] = vt.$$

As $c^2 \in E - F$ and $c^2 \sim_H v$, $c^2 \in w\langle u, z\rangle$. From $[c, y] \in aE$, (1), and (6), it follows that $c^2 \in wu\langle z\rangle$. We choose

(13) $$c^2 = wu$$

since we may replace c by cv if necessary. Now cdE contains involutions so we assume $(cd)^2 = 1$ as c so far is chosen only up to a factor in $\langle u, t, z\rangle$ and $[d, t] = z$. We have

(14) $$[c, d] = wu.$$

A simple calculation yields $[b, c] = uv$ or uvz, so by our remark above and as $[b, u] = z$ we may choose

(15) $$[b, c] = uv$$

Case 2: *relation* (11′) *holds*. As above, we may choose c in the appropriate way to get:

(12′) $$[a, c] = v$$

(13′) $$c^2 = w$$

(14′) $$[c, d] = w$$

and

(15′) $$[b, c] = ut.$$

In Case 2 we now replace j by j' for any $j \in J$ and write J' for J. Then the isomorphism $\sigma : J \rightarrow J'$ given by

$$\sigma(a) = a', \quad \sigma(b) = a'b't', \quad \sigma(c) = c', \quad \sigma(d) = c'd'u'$$

shows that Cases 1 and 2 give isomorphic groups for J. From now on we assume we are in Case 1; i.e., relations (1)–(15) hold. Next we choose

(16) $$[a, x] = 1;$$

for, if $[a, x] = t$ (we know $[a, x] \in \langle t \rangle$) and if we put $a' = av$ and $c' = cw$, we see that a', b, c', d satisfy (1)–(15) and $[a', x] = 1$.

Taking $q \in Q$ as above (i.e., $z^q = t$), we have $w^q = yf$ and $y^q = ybwf'$, $f, f' \in F$ (because $[w, F] = z$ and $[y, F] = t$). It now follows (from $b \in C_N(q)$, (1), (2), (4), and (5)) that $[y, b] = 1$. Hence $(ybw)^2 = 1$ and $[y, d] = bwf_1$ for some $f_1 \in \langle v, z \rangle$ by (3). Replacing y by yt (and then x by xv) if necessary, we may assume $f_1 \in \langle v \rangle$. Suppose $f_1 = v$; then putting $d' = dw$ and $a' = at$, a', b, c, d' satisfy (1)–(16) and $[y, d'] = 1$. We may choose $f_1 = 1$:

(17) $$[y, d] = bw.$$

Next we see that $[b, x] \in a\langle t, z \rangle$ because of (1) and $[y, b] = 1$. We may replace x by xu (and then y by yv) if necessary, to choose $[b, x] \in a\langle t \rangle$, and then repeat the argument used above to choose

(18) $$[b, x] = a \quad (\text{with } [b, y] = 1).$$

A computation, using the relations above, yields $[c, y] = atz^\delta$, $\delta = 0, 1$. However, $y^{dxd} = yatz \not\sim_G yat$ so $[c, y] = atz$. Two further computations enable us to determine $[x, c]$ uniquely and $[x, d]$ up to a factor of z. Thus replacing x by xt if necessary, we have

(19) $$[y, c] = atz, \quad [x, c] = abuv, \quad [x, d] = abcuv.$$

We have now given all relations between the generators x, a, b, c, d of T.

Generators and relations for H. If X is a subset of G, put $I(X)$ equal to the subset of all involutions of X. An easy computation yields

$$|\{j^i | i \in I(T - J)\}| = 8$$

for any $j \in I(J - A)$.

Let $r \in I(H - T)$ with $r \sim_H y$ and $(aE)^r = dE$. Note that $H = \langle T, r \rangle$. We have $C_E(r) = \langle z, u, wvt \rangle$, $C_{J/E}(r) = \langle dc, ad \rangle E/E$, and $(ry)^5 = 1$. Let T_1 denote the Sylow 2-subgroup of H which contains r. If $(yr)^2 = \sigma$, then $T^\sigma = T_1$, $F^\sigma = F_1 = \langle cd, tv, w, u, z \rangle$, and of course $y^\sigma = r$. Thus $t^r = wuv$ or $wuvz$, so replacing r by an appropriate involution in $rcdE$ if necessary,

(20) $$r^2 = (ry)^5 = 1, \quad t^r = wuvz.$$

Further, $v^r = uv$ or uvz so we may choose r (as r can be replaced by an involution in $racE$ if need be) to get

(21) $$v^r = uv,$$

and so

(22) $$u^r = u, w^r = vtz.$$

By the remark above, $|\{a^i|i \in I(T_1 - J)\}| = 8$ and so

$$\{a^i|i \in I(T_1 - J)\} = \{df \mid f \in C_{F_1}(d) = \langle w, u, z\rangle\} \text{ or } = \{dvf \mid f \in C_{F_1}(d)\}.$$

These two possibilities yield isomorphic groups $H = \langle T, r\rangle$ (in fact under the correspondence in § 6, this isomorphism is given by the outer automorphism induced by the element $u_5 \in {}^2F_4(2)$-$\mathscr{T}$ in Tits' notation [**6**].)

We take the first possibility; i.e., $a^r = df$ for some $f \in \langle w, u, z\rangle$. Put $(ab)^\sigma = adh$ where $h \in \langle z, v, u, wt\rangle$ so that $adh \in C_J(r)$ (because $y^\sigma = r$). This forces $f \in \langle u, z\rangle$ and $h \in v\langle z, u, wvt\rangle$; i.e., $a^r = du^\alpha z^\beta$ ($\alpha, \beta = 0, 1$). However, we may replace r by $rvtw$ if necessary, to have $a^r = du^\alpha$ ($\alpha = 0, 1$) and thus $d^r = au^\alpha$. It follows immediately that $c^r = cdau^\alpha$ and a computation shows $b^r = dcb(vt)^\alpha w^{\alpha+1}z^\gamma$ ($\alpha, \gamma = 0, 1$). Replacing r by ru if need be, we choose $\gamma = 0$. Thus we have the following possible two sets of relations between r and J:

(23) If $\alpha = 1$, then $a^r = du$, $d^r = au$, $c^r = cdau$, $b^r = dcbvt$;

(23′) if $\alpha = 0$, then $a^r = d$, $d^r = a$, $c^r = cda$, $b^r = dcbv$.

If the elements of $\langle J, r\rangle$ satisfy (23′) put $J = J'$, $r = r'$, and $j = j'$ for any $j \in J$. Then $\lambda : \langle J, r\rangle \rightarrow \langle J', r'\rangle$ given by

$$\lambda(a) = a'v't'z', \quad \lambda(b) = b't'z', \quad \lambda(c) = c'w'z', \quad \lambda(d) = d'u'w', \quad \lambda(r) = r'$$

is an isomorphism. As usual we suppose (23) holds from now on.

Finally, a simple but tedious computation shows

$$(ryrx^{-1}rx)j(x^{-1}rxryr) = j$$

for each $j \in \{a, b, c, d\}$. As $J = \langle a, b, c, d\rangle$ and $C_G(J) = \langle z\rangle$, this implies $ryrx^{-1}rx \in \langle z\rangle$. Thus $ryr = x^{-1}rxz$ or $x^{-1}rx$. However, $y \sim_H r$ but $y \nsim_H yz$ so $ryr = x^{-1}rx$, or

(24) $$rxr = (yr)^2x.$$

Generators and relations for N. Let $s \in I(N - T)$ with $(yF)^s = wF$. Note that $N = \langle T, s\rangle$, and put $T_2 = \langle s, K\rangle$. Since $|\{y^i|i \in I(T - K)\}| = 8$, it follows that all involutions in wF conjugate to y in G are conjugate to y under $I(T_2 - K)$. Without loss therefore we take $y^s = wuvz$. Hence $t^s = [y, F]^s = [wuvz, F] = z$ and $(vt)^s = [y, wuvz]^s = vt$; i.e., $v^s = vtz$.

As $x^2 = yz$, $x^s = cae'$ for some $e' \in C_E(ca) = \langle w, u, t, z\rangle$. From (18) and (1) we have $y \sim_N yaz$ so $(yaz)^s \in wt\langle v, z\rangle$, whence $a^s \in u\langle v, z\rangle$. If $x^s = cae'$ with $e' \in \langle u, t, z\rangle$, then $[x^s, a] = vt = [x, a^s]$ implies $a^s = uv$ or uvz. But $1 = [x, a]^s = [cae', uv] = z$, which is a contradiction. Therefore $x^s = cawe$, $e \in \langle u, t, z\rangle$, and $a^s = u$ or uz.

If m is an element of order 4 in $K - U$, then m^2 has precisely 16 roots in K, 8 in each of the cosets mF and $m^{-1}F$. Further, these roots are precisely the set $\{m^i, (m^{-1})^i, (mk)^i, (m^{-1}k)^i | i \in I(N - K) \cap C_N(m^2)$ and $\langle k\rangle = Z(\langle i, K\rangle)\}$. This means we may choose s more exactly so that $x^s = caw$ or $cawtz$ (note that above we showed $x^s \in c^{-1}F = cwF \neq cF$).

In the former case, $x^s = caw$, using (18), (19), and the possibilities above, $a^s = uz$ and $b^s = bauvt$. However, a further computation shows $(bauvt)^s = btz \neq b$, which is clearly a contradiction. Thus $x^s = cawtz$, and from (18) and (19) we have:

(25) $$t^s = z, \quad v^s = vtz, \quad y^s = wuvz, \quad w^s = yavz,$$
$$a^s = u \quad b^s = bauvz, \quad c^s = xyauvt, \quad x^s = cawtz.$$

Finally we compute $x^{(sd)^3}$ to be xt whence $(sd)^3 = vz$. Our final relation is then

(26) $$(sdvz)^3 = 1.$$

4. Determination of the second centralizer.

LEMMA 8. *We have $C_G(v) = C_N(v)$, an extension of a 2-group of order 2^8 by S_3.*

Proof. Put $C = C_G(v)$, $V = C_N(v)$, and $U = O_2(V)$ and recall that $U = \langle F, y, w, b\rangle$ and $V = U \cdot Q\langle d\rangle$ with $V/U \cong S_3$.

We note that as $\langle z, v\rangle = Z(T \cap V)$, it follows that $T \cap V = \langle U, d\rangle$ is a Sylow 2-subgroup of C. Also if $\langle z, t\rangle$ normalizes a subgroup O of odd order of C then $O = 1$. For, $z \in O_2(C_G(t)) \cap O_2(C_G(tz))$ (by the structure of H), whence $\langle C_O(t), C_O(tz)\rangle \subseteq C_O(z) = 1$, which immediately implies $O = 1$.

We have $C_V(d) = S$ is elementary of order 32 and $S \sim_H F$ so that $N_C(S) = N_V(S)$. On the other hand for any involution $l \in U$, $|C_V(l)| \geqq 2^7$. Thus if $d \sim_C l$, a Sylow 2-subgroup of $C_C(d)$ would contain S properly, whence $N_C(S) \supset N_V(S)$. This is not the case so d is not conjugate to any involution in U. Thompson's transfer lemma [**5**, p. 411] yields that C has a subgroup M of index 2, and in fact we may assume $d \notin M$.

Clearly U is then a Sylow 2-subgroup of M and we claim $Z = Z(U)$ is weakly closed in U with respect to M. For, if $z \sim_M l$ with $l \in U - Z$, then $2^8 | |C_M(l)|$. However, $C_U(l)' = \langle z\rangle, \langle t\rangle$, or $\langle tz\rangle$ and obviously $C_M(z) = C_M(t) = C_M(tz) = U$. Therefore a Sylow 2-subgroup of $C_M(l)$ has order 2^6 which means $z \nsim_M l$. Thus, as $\langle v\rangle \lhd M$, Z is weakly closed in U; i.e., M is 2-normal. Grün's transfer theorem [**2**, p. 256] implies that M has a subgroup X of index 2 with

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$b \notin X$; i.e., $X \cap U = \langle F, bw, by\rangle$. Hence $Y = X \cap U$ is a Sylow 2-subgroup of X, $\Omega_1(Y) = F$, and $C_Y(Q) = \langle v\rangle$.

Next we make two remarks:

(i) $\langle z, t\rangle$ is characteristic in any 2-subgroup of X containing it (in Y, z has precisely 3 G-conjugates), and $N_X(\langle z, t\rangle) = YQ$;

(ii) if $k \in F - Z$, then $N_Y(\langle k, v\rangle) = C_Y(k) = F$, while if $k \in Y - F$, then $N_Y(\langle k, v\rangle) = C_Y(k) = \langle k\rangle Z$.

Put $\bar{X} = X/\langle v\rangle$ and use the bar convention. From (i) and (ii) we have $C_{\bar{Y}}(\bar{k})$ is an abelian Sylow 2-subgroup of $C_{\bar{X}}(\bar{k})$, $k \in Y - Z$. As a Sylow 2-subgroup of $C_{\bar{X}}(\bar{k})/\langle\bar{k}\rangle$ has order at most 8 and as $\bar{z}$, $\bar{t}$, $\bar{t}\bar{z}$ lie in distinct conjugate classes in $C_{\bar{X}}(\bar{k})$, the transfer theorem implies $C_{\bar{X}}(\bar{k})$ has a normal 2-complement $\bar{O}$. If O denotes the inverse image of $\bar{O}$ in X, and $O = \langle v\rangle \times O_1$, then $\langle z, t\rangle \subseteq N_X(O_1) \subseteq N_C(O_1)$. By the remark at the beginning of the proof, $O_1 = 1$; i.e., $C_{\bar{X}}(\bar{k}) = C_{\bar{Y}}(\bar{k})$. It follows immediately that $\bar{Y}$ contains the centralizer of each of its involutions, and hence $\bar{Y}$ is disjoint from its conjugates.

A standard argument (see [**2**, p. 302], for example) yields that $\bar{X}$ has precisely one class of involutions or $\bar{Y} \lhd \bar{X}$. The first possibility implies $z\langle v\rangle \sim_X u\langle v\rangle$, which is a contradiction. From $\bar{Y} \lhd \bar{X}$ follows $Y \lhd X$ and then $Z = Z(Y) \lhd X$ so $X \subseteq N_G(Z) \subset N$. It follows immediately that $C \subset N$ and so $C = V$. The lemma is proved.

5. Generators and relations for $\mathscr{T}$. In [**6**], Tits gives generators and relations for the group ${}^2F_4(2)$. Using these generators and relations for ${}^2F_4(2)$ and the method of Reidemeister-Schreier (see [**4**, p. 86–95]), one can derive the following presentation for the group $\mathscr{T}$:

Generators for $\mathscr{T}$: $r_1, r_8, s_i, i = 1, \ldots, 8$.
(In the notation of [**6**], we have:

$$s_i = u_i \qquad i \text{ even},$$
$$s_i = u_1u_i \qquad i = 1, 3, 5,$$
$$s_7 = u_7u_1^{-1}, \text{ and } r_1, r_8 \text{ are as in [6].)}$$

Relations.

(I) $r_1^2 = r_8^2 = s_1^2 = s_2^2 = s_4^2 = s_6^2 = s_8^2 = 1$, $s_3^4 = s_5^4 = s_7^4 = 1$.
Put $r_3 = s_1s_2s_3^2$, $r_5 = s_1s_5^2$ and $r_7 = r_3 \cdot s_3s_5s_7$.

(II) $[s_1, s_2] = [s_1, s_3] = [s_1, s_5] = 1$.

(III) $[s_1, s_6] = r_3$; $[s_1, s_7] = s_2r_3r_5$; $[s_1, s_8] = s_7^2r_3s_1$.

(IV) $[s_2, s_4] = [s_2, s_6] = 1$; $[s_2, s_8] = s_4s_6$; $[s_7, s_2] = s_4r_5$.

(V) $[s_7, s_4] = r_3r_5$; $[s_3, s_5] = s_2r_3s_4$; $[s_5, s_4] = r_3$.

(VI) (i) $(r_1r_8)^8 = 1$, (ii) $(s_1r_1)^5 = 1$, (iii) $(r_8s_8)^3 = 1$.

(VII) $r_1s_2r_1 = s_8$; $r_1s_4r_1 = s_6$; $r_1s_5r_1 = (s_1r_1)^2s_5$;
$r_1s_3r_1 = (s_1r_1)^2r_7$; $r_1r_3r_1 = s_7r_7$.

(VIII) $r_8s_2r_8 = s_6$; $r_8s_4r_8 = s_4$; $r_8s_1r_8 = s_7r_7$; $r_8s_7r_8 = s_7^{-1}$;
$r_8s_3r_8 = s_7s_5$; $r_8r_3r_8 = r_5$.

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6. Identification of G with $\mathscr{T}$. We consider the following correspondence:

$$\begin{array}{rr} y \leftrightarrow s_1 & x \leftrightarrow s_5 \\ avz \leftrightarrow s_2 & uvz \leftrightarrow s_6 \\ bxv \leftrightarrow s_3 & x^{-1}cawtz \leftrightarrow s_7 \\ vz \leftrightarrow s_4 & dvz \leftrightarrow s_8 \\ r \leftrightarrow r_1 & s \leftrightarrow r_8 \end{array}$$

(It follows that

$$t \leftrightarrow r_3, \quad z \leftrightarrow r_5, \quad xbcwt \leftrightarrow r_7, \quad wuvz \leftrightarrow s_7r_7).$$

Under this correspondence, using the fact that E and F are elementary and relations (1)–(26) of § 4, we see that all the relations of § 5 are satisfied with the possible exception of VI (i).

Verification of VI (i) (*i.e., we prove* $(rs)^8 = 1$). By the choice of r and s, $r \sim_G z$ while $s \sim_G v$ which means rs has even order. Further, $(rs)^4y(sr)^4 = ryr \neq y$ shows rs has order at least six. Now the structures of H and C imply that either $(rs)^8 = 1$ or $(rs)^{10} = 1$. Suppose $(rs)^{10} = 1$; then $(rs)^5 = i \sim_G z$. A simple computation yields $dvz \in C_G(r(sr)^3)$, whence

$$s \cdot dvz \cdot s \in C_G(s \cdot r(sr)^3 \cdot s) = C_G((sr)^4s) = C_G(ir).$$

Clearly $r \in C_G(ir)$ as $(rs)^5 = i$, and so $(sdvzs)r \in C_G(ir)$. Using relation (26) (i.e., $sdvzs = dvzsdvz$), a computation gives

$$(*) \qquad (sdvzsr)^4 = advz(sr)^4zvda.$$

But $r \in O_2(C_G(ir))$ as $r \sim_G z$ and $ir \sim_G v$ (i.e., $C - U$ only contains involutions conjugate to v), which implies $(sdvzs)r$ lies in a Sylow 2-subgroup of $C_G(ir)$. This implies by equation (*) that $(sr)^4$ is also a 2-element. However, as $(sr)^{10} = (rs)^{10} = 1$, $(sr)^4$ is of order 5. This contradiction shows $(rs)^{10} \neq 1$; we have proved $(rs)^8 = 1$.

With the verification of relation VI (i), we have proved that G possesses a subgroup $G_0 = \langle H, N\rangle$ isomorphic to a factor group of $\mathscr{T}$, so $G_0 \cong \mathscr{T}$ as $\mathscr{T}$ is simple. It therefore remains to show that $G = G_0$.

At this stage, Thompson's order formula [**3**, p. 279] may be applied to determine $|G|$. However, the actual computation of $|G|$ is not necessary, for the formula, along with §§ 2, 3, and 4 show that $|G|$ is unique. Since $\mathscr{T}$ satisfies the assumptions of the theorem, it follows therefore that $|G| = |\mathscr{T}|$. Thus $|G| = |G_0|$ and as $G_0 \subseteq G$, we have shown $G = G_0$, as required.

This completes the proof of the theorem.

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