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A CHARACTERIZATION OF THE INTUITIONISTIC PROPOSITIONAL LOGIC

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Dedicated to Professor Katuzi Ono on his 60th birthday

In this short note, is shown a necessary and sufficient condition for a logic to be an intermediate propositional logic in Umezawa's sense (see the reference), under such an assumption that any logic in consideration (as a subclass of $\boldsymbol{LK}$-provable propositions) contains at least the axioms of the positive propositional logic $\boldsymbol{LPS}$ (Curry's $\boldsymbol{LA}$) as its axioms and is closed with respect to the rules of detachment and substitution. Furthermore, for an expediency to our proof, we assume that in any logic negation is defined by a constant proposition $\boldsymbol{F}$ (contradiction) in the equivalence $\neg \boldsymbol{A} \equiv \boldsymbol{A} \to \boldsymbol{F}$ instead of having negation as a primitive symbol, and that axioms for negation are given in the corresponding form. For example, the intuitionistic propositional logic $\boldsymbol{LJS}$ has $\boldsymbol{F} \to \boldsymbol{a}$ as the sole axiom for negation as well as the axioms of the positive propositional logic $\boldsymbol{LPS}$. Apparently this assumption implicitly requires that any logic with negation is stronger than, or equivalent to the minimal propositional logic $\boldsymbol{LMS}$ of Johansson.[1)]

After a proof of the main theorem, are stated a number of its applications to some problems in propositional logics. Namely, it is shown, e.g., that the intuitionistic propositional logic has a kind of minimality as its significant character, and the classically minimal logic $\boldsymbol{LNS}$ a kind of maximality.

THEOREM. *A logic* $\boldsymbol{L}$ *is stronger than, or equivalent to the intuitionistic propositional logic* $\boldsymbol{LJS}$, *if and only if there is a proposition* $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$ *provable in* $\boldsymbol{L}$, *which satisfies the following condition:*

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1) A logic $\boldsymbol{L}$ is stronger than another logic $\boldsymbol{L}'$ if any proposition provable in $\boldsymbol{L}'$ is also provable in $\boldsymbol{L}$, but not the contrary. The notion of "weaker than" is defined in the similar manner.

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(i) *Let* $\boldsymbol{b}$ *be a propositional variable distinct from each* $\boldsymbol{a}_i$. ($\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$ *exhaust the propositional variables appearing in* $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$.) *The proposition* $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{b})$, *obtained from* $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$ *by substituting* $\boldsymbol{b}$ *to any occurrence of* $\boldsymbol{F}$ *in it, is not provable in the classical propositional logic* $\boldsymbol{LKS}$.

Proof.

Necessity: Let $\boldsymbol{L}$ be a logic stronger than or equivalent to $\boldsymbol{LJS}$. Then $\boldsymbol{F} \to \boldsymbol{a}$ is provable in $\boldsymbol{L}$, and in fact satisfies the condition (i).

Sufficiency: Now it will be shown that $\boldsymbol{F} \to \boldsymbol{a}$ is derived from $\boldsymbol{L}$, or more precisely from an $\boldsymbol{L}$-provable proposition satisfying the condition (i) and the axioms of $\boldsymbol{LPS}$ by the rules of detachment and substitution. In the following, we shall make use of the fact that $\boldsymbol{LKS}$ is plausible and complete with respect to the two-valued model of logic, and assume that any proposition is also referred as a function from $\{\mathbf{0}, \mathbf{1}\}^N$ to $\{\mathbf{0}, \mathbf{1}\}$ defined according to the usual truth table of propositional connectives (where $\mathbf{0}$ is the designated value).

(ii) There is a proposition $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$ provable in $\boldsymbol{L}$, which satisfies the condition (i). $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$ constitute the propositional variables occurring in $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$.

(iii) Let $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{b})$ be the proposition obtained from $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$ by substituting a propositional variable $\boldsymbol{b}$ to any occurrence of $\boldsymbol{F}$ in it, where $\boldsymbol{b}$ is different from each $\boldsymbol{a}_i$. From (i), $\boldsymbol{b}$ actually occurs in $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{b})$.

(iv) $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; 1)$ always takes the value $\mathbf{0}$ (for any assignment), because $\boldsymbol{F}$ is a constant and firmly assigned the value $\mathbf{1}$, and $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$ is provable in $\boldsymbol{LKS}$. (At this step of reasoning, we have made use of the fact that $\boldsymbol{L}$ is embraced by $\boldsymbol{LKS}$.).

(v) From (i), $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{b})$ is not provable in $\boldsymbol{LKS}$, i.e., it takes the value $\mathbf{1}$ by some assigment $\boldsymbol{V}$. Therefore, from (iv), for some assingment $\boldsymbol{V}'$ to $(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m)$, $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \mathbf{0})$ takes the value $\mathbf{1}$.

(vi) We assume that the assignment $\boldsymbol{V}'$ gives the value $\mathbf{1}$ to $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$, and gives the value $\mathbf{0}$ to $\boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m$. This is admissible by the reason of permutation in propositional variables.

(vii) The index k is not zero, i.e., the assignment gives the value $\mathbf{1}$ at least to $\boldsymbol{a}_1$. If it were not the case, $\boldsymbol{P}(\mathbf{0}, \mathbf{0}, \cdots; \mathbf{0})$ would be $\mathbf{1}$; but such a situation could not happen in the two-valued truth table. (Note that $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{b})$ is a positive proposition).

(viii) Let $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{b})$ be the proposition obtained from $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{b})$ by substituting $\boldsymbol{a}$'s to $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$, and $(\boldsymbol{a} \rightarrow \boldsymbol{a})$'s to $\boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m$. From (v) and (vi), $\boldsymbol{Q}(\mathbf{1}, \mathbf{0})$ is $\mathbf{1}$.

(ix) $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{a} \rightarrow \boldsymbol{a})$ is equivalent to $\boldsymbol{a}$, where $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{a} \rightarrow \boldsymbol{a})$ is the proposition obtained from $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{b})$ by substituting $\boldsymbol{a} \rightarrow \boldsymbol{a}$ to $\boldsymbol{b}$. For, $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{a} \rightarrow \boldsymbol{a})$ is constructed in the sole propositional variable $\boldsymbol{a}$ and logical connectives ($\rightarrow$, $\vee$, and $\wedge$); hence, it is equivalent to $\boldsymbol{a}$ or provable in $\boldsymbol{LPS}$. However, from (viii), $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{a} \rightarrow \boldsymbol{a})$ is not provable in $\boldsymbol{LPS}$ (or $\boldsymbol{LKS}$).

(x) $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{F})$, $\boldsymbol{F} \vdash \boldsymbol{a}$ in $\boldsymbol{LPS}$, where $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{F})$ is the proposition obtained from $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{b})$ by substituting $\boldsymbol{F}$ to $\boldsymbol{b}$. For, $\boldsymbol{F} \vdash \boldsymbol{F} \equiv \boldsymbol{a} \rightarrow \boldsymbol{a}$ in $\boldsymbol{LPS}$; hence, $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{F})$, $\boldsymbol{F} \vdash \boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{a} \rightarrow \boldsymbol{a})$ in $\boldsymbol{LPS}$ by virtue of the replacement theorem. That is, $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{F})$, $\boldsymbol{F} \vdash \boldsymbol{a}$ in $\boldsymbol{LPS}$ from (ix).

(xi) According to the transformations of proposition in (iii) and (viii), $\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{F})$ is obtained from $\boldsymbol{P}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m; \boldsymbol{F})$ by substituting $\boldsymbol{a}$'s to $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$, and $(\boldsymbol{a} \rightarrow \boldsymbol{a})$'s to $\boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m$. Therefore, from (x), $\boldsymbol{F} \rightarrow \boldsymbol{a}$ is provable in $\boldsymbol{L}$.

In Umezawa's definition, an intermediate propositional logic is a logic between $\boldsymbol{LKS}$ and $\boldsymbol{LJS}$, i.e., it is stronger than, or equivalent to $\boldsymbol{LJS}$, and weaker than, or equivalent to $\boldsymbol{LKS}$. Hence, the above proved theorem gives a criterion for deciding whether a logic is an intermediate logic in this sense or not. Moreover, this criterion proves finitely performable, if a test logic is finitely axiomatized:

COROLLARY 1. *Let* $\boldsymbol{L}$ *be a finitely axiomatized propositional logic, and* $\boldsymbol{A}$ *be the conjunction of the axioms of* $\boldsymbol{L}$. $\boldsymbol{L}$ *is stronger than, or equivalent to the intuitionistic propositional logic* $\boldsymbol{LJS}$, *if and only if* $\boldsymbol{A}$ *satisfies the condition* (i) *in the theorem.*

This corollary follows from the theorem thus: If $\boldsymbol{A}$ satisfies (i), $\boldsymbol{L}$ is stronger than or equivalent to $\boldsymbol{LJS}$ directly from the theorem. The converse implication can be also easily proved, but in this case taking its contraposition:[2] Let $\boldsymbol{A}$ not satisfy (i), i.e., $\boldsymbol{A}_{\boldsymbol{F}}^{\boldsymbol{b}}$ ($\boldsymbol{b}$ not occurring in $\boldsymbol{A}$) is provable in $\boldsymbol{LKS}$, where $\boldsymbol{A}_{\boldsymbol{F}}^{\boldsymbol{b}}$ comes from $\boldsymbol{A}$ through the substitution of $\boldsymbol{b}$ to $\boldsymbol{F}$. From the separation principle for $\boldsymbol{LKS}$, $\boldsymbol{A}_{\boldsymbol{F}}^{\boldsymbol{b}}$ is also provable in the positive sublogic $\boldsymbol{LQS}$ (Curry's $\boldsymbol{LC}$) of $\boldsymbol{LKS}$. ($\boldsymbol{LQS}$ is otherwise formulated from $\boldsymbol{LPS}$ by fortifying *Peirce's law*). Therefore, $\boldsymbol{A}_{\boldsymbol{F}}^{\boldsymbol{b}}$ is provable in $\boldsymbol{LNS}$ (Curry's $\boldsymbol{LE}$), which is obtained from $\boldsymbol{LQS}$ by supplying the contradiction $\boldsymbol{F}$ (and defined negation). From this, especially, $\boldsymbol{A}$ is provable in $\boldsymbol{LNS}$, that is, $\boldsymbol{L}$ is weaker than or equivalent to $\boldsymbol{LNS}$. On the other hand, $\boldsymbol{F} \to \boldsymbol{a}$ is not provable in $\boldsymbol{LNS}$. For, if it were provable in $\boldsymbol{LNS}$, $\boldsymbol{LNS}$ would have to be equivalent to $\boldsymbol{LKS}$; this is not the case. Accordingly, $\boldsymbol{F} \to \boldsymbol{a}$ is not provable in $\boldsymbol{L}$, i.e., $\boldsymbol{L}$ does not embrace $\boldsymbol{LJS}$.

In view of the fact that the decision method for $\boldsymbol{LJS}$ has been established in the form of Gentzen's cut elimination theorem, we can in addition finitely decide whether a finitely axiomatized logic $\boldsymbol{L}$ is really stronger than the intuitionistic, or equivalent to the intuitionistic, after examining that $\boldsymbol{L}$ is an intermediate logic by this criterion.

By definition, the intuitionistic logic is the weakest of the intermediate logics. However, a kind of such minimality can be established in the framework of formulation of the classical logic:

Corollary 2. *Let $\boldsymbol{L} + \boldsymbol{A}$ denote the logic obtained from $\boldsymbol{L}$ by adding $\boldsymbol{A}$ as an axiom to it. A logic $\boldsymbol{L}$ is stronger than or equivalent to the intuitionistic propositional logic $\boldsymbol{LJS}$, if and only if $\boldsymbol{L} + ((\boldsymbol{a} \to \boldsymbol{b}) \to \boldsymbol{a}) \to \boldsymbol{a}$ is equivalent to the classical propositional logic $\boldsymbol{LKS}$, or if and only if $\boldsymbol{L} + \boldsymbol{a} \vee \neg \boldsymbol{a}$ is equivalent to the classical.*

In other words, the corollory says that the intuitionistic propositional logic is the weakest of such logics that if they are fortified by *Peirce's law* (or *tertium non datur*), then the resulted logics are equivalent to the classical propositional logic. This is easily proved by making use of the main theorem similary to the proof of Corollay 1.

In another direction, although the theorem seems to say only about the properties of the intuitionistic logic, it reveals some peculiar characteristics of $\boldsymbol{LNS}$ (which is maybe called as the classically minimal propositional logic) after a few steps of reasoning from the theorem:

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[2] It would seem that a constructive or intuitionistic proof of Corollary 1 (from the main theorem) can not be obtained, while the main theorem is in fact proved constructively.

COROLLARY 3. *There is no propositional logic between* $\boldsymbol{LKS}$ *and* $\boldsymbol{LNS}$. *In other words, if* $\boldsymbol{LNS}$ *is fortified by adding as axiom a proposition* $\boldsymbol{A}$, *which is provable in* $\boldsymbol{LKS}$, *but not so in* $\boldsymbol{LNS}$, *then the resulted logic* $\boldsymbol{LNS}+\boldsymbol{A}$ *is equivalent to* $\boldsymbol{LKS}$.

This is assured from the theorem as follows: Let $\boldsymbol{A}$ be provable in $\boldsymbol{LKS}$, but not so in $\boldsymbol{LNS}$. Then $\boldsymbol{A}$ satisfies the condition (i) in the theorem. For, if $\boldsymbol{A}_F^b$ (the same notation above) were provable in $\boldsymbol{LKS}$, then $\boldsymbol{A}_F^b$ would be provable in $\boldsymbol{LQS}$. Accordingly, $\boldsymbol{A}$ would be provable in $\boldsymbol{LNS}$; this is a contradiction. Therefore, by the theorem, $\boldsymbol{F}\rightarrow\boldsymbol{a}$ is provable in $\boldsymbol{LNS}+\boldsymbol{A}$, that is $\boldsymbol{LNS}+\boldsymbol{A}$ is equivalent to $\boldsymbol{LKS}$.

This fact seems to present a kind of peculiarity of the classically minimal propositional logic in comparison with the intuitionistic and the classical. The property can be restated in another form as in the next corollary, which apparently shows maximality of $\boldsymbol{LNS}$ in the framework of $\boldsymbol{LKS}$:

COROLLARY 4. *If* $\boldsymbol{F}\rightarrow\boldsymbol{a}$ *is not provable in* $\boldsymbol{L}$, *then* $\boldsymbol{L}$ *is weaker than or equivalent to* $\boldsymbol{LNS}$.

COROLLARY 5. $\boldsymbol{LN}$ ($\boldsymbol{LNS}$) *can not be formulated by fortifying* $\boldsymbol{LM}$ ($\boldsymbol{LMS}$) *by any axiom schema of just one* (*syntacitcal*) *variable* (*any axiom of just one propositional variable*). (*On the other hand,* $\boldsymbol{LK}$ ($\boldsymbol{LKS}$), $\boldsymbol{LJ}$ ($\boldsymbol{LJS}$), *and* $\boldsymbol{LD}$ ($\boldsymbol{LDS}$) *can be obtained in such a format.*)

These corollaries are derived from the theorem by applying it quite similarly to the case of Corollary 3.

CONCLUSION. By virtue of the theorem, it may be safe to say that only intermediate logics are the logics which have *proper* negation in the sense that the contradiction $\boldsymbol{F}$ is not merely a symbol, but has its own property (as *contradiction*) which characterizes negation-notion. In other words, it is the axiom $\boldsymbol{F}\rightarrow\boldsymbol{a}$ (*Donus Scotus' principle*) that assures us to unify all contradictory propositions to the *contradiction*. Furthermore, the theorem says that the intuitionistic logic is the weakest among the class of logics with proper negation. This fact is a remarkable feature of the intuitionistic logic at least from the view-point of propositional logics. For, the model-theoretic characterization of the intuitionistic propositional logic would seem somewhat tedious and not transparent in employing topological notion, but the property revealed by substitution of a propositional variable to the contradiction is as simple as it proves an effective test to decide whether a logic in question is the intuitionistic or not, by referring to the two-valued truth table (the simplest model of logic).

Remark. For predicate logics, we can give the following weak criterion, which is rather apparent in view of the main theorem:

A predicate logic $\boldsymbol{L}$ (which is stronger than or equivalent to $\boldsymbol{LM}$,) is stronger than or equivalent to $\boldsymbol{LJ}$, if and only if there is a $\boldsymbol{L}$-provable *proposition* $\boldsymbol{A}$ such that $\boldsymbol{A}_F^b$ is not provable in $\boldsymbol{LK}$.

However, it remains unsettled whether a *proposition* can be replaced by a *predicate* in the above assertion. (Strong criterion for predicate logics.)

Reference

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