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SOME HYPERSURFACES OF SYMMETRIC SPACES

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ABSTRACT. In this paper we consider how much we can say about an irreducible symmetric space M which admits a hypersurface N with at most two distinct principal curvatures. Then we will obtain that (1) if N is locally symmetric, then M must be a sphere, a real projective space and their noncompact duals (2) if N is Einstein, then M must be rank 1.

Recently, the following problem was proposed: If we assume that an irreducible symmetric space M admits a single submanifold with a particular property, how much can we say about the ambient space? With respect to this problem, Chen and Nagano [1] obtained that the only irreducible symmetric spaces which admit totally geodesic hypersurfaces are spheres, real projective spaces and their noncompact duals. We remark that Chen & Nagano's result remains true in the case where M admits totally umbilical hypersurfaces ([2]). Also, Chen and Verstraelen [2] obtained that if M admits a hypersurface N with a constant principal curvature of multiplicity $\geq \dim N-1$, then M must be a sphere, a real projective space, a complex projective space or one of their noncompact duals.

In this paper we consider M which admits a hypersurface with at most two distinct principal curvatures and will show the following:

THEOREM A. *If M admits a (connected) locally symmetric hypersurface N* ($\dim N \geq 3$) *with at most two distinct principal curvatures, then M must be a sphere, a real projective space and their noncompact duals.*

THEOREM B. *If M admits an Einstein hypersurface N with at most two distinct principal curvatures, then M must be rank* 1.

1. **Preliminaries.** Let M be a connected Riemannian manifold and a symmetric space. As usual if G denotes the closure of the group of isometries generated by an involutive isometry for each point of M, then G acts transitively on M; hence the isotropy subgroup H, say at 0, is compact and $M=G/H$. Let $\mathfrak{g}, \mathfrak{h}$ denote the Lie algebras corresponding to G, H, respectively.

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Then we call

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m}, \quad \text{and} \quad \mathfrak{h}=[\mathfrak{m},\mathfrak{m}]$$

by the Cartan decomposition. It is well-known the space $\mathfrak{m}$ consists of the Killing vector fields X whose covariant derivative vanish at 0; in particular, the evaluation map at 0 gives a linear isomorphism of $\mathfrak{m}$ onto $T_0M: X \mapsto X(0)$. Hence we have

LEMMA 1.1. *For the curvature tensor R at 0*

$$R(X, Y)Z=-[[X, Y], Z], \quad \textit{for} \quad X, Y, Z \in \mathfrak{m}.$$

LEMMA 1.2. *A linear subspace L of the tangent space T_0M to a symmetric space M is the tangent space to some totally geodesic submanifold N of M if and only if L satisfies the condition $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset \mathfrak{n}$, where*

$$\mathfrak{n}=\mathfrak{n}^*\{X \in \mathfrak{m};\ X(0) \in L\}.$$

Next, let N be a hypersurface of an $(n+1)$-dimensional Riemannian manifold M. And let ∇ and ∇' be the covariant differentiations on N and M, respectively. Then the second fundamental form A of the immersion is given by

$$(1.1) \qquad \nabla'_X Y=\nabla_X Y+g(AX, Y)\xi,$$

$$(1.2) \qquad \nabla'_X \xi=-AX$$

for vector fields X, Y tangent to N and a vector field ξ normal to N, where g is the metric tensor of N induced by the immersion from the metric tensor of M. The equations of Gauss and Codazzi are then given respectively

$$(1.3) \qquad \begin{aligned} R'(X, Y; Z, W) &= R(X, Y; Z, W) \\ &\quad + g(AY, Z)g(AX, W)-g(AX, Z)g(AY, W), \end{aligned}$$

$$(1.4) \qquad R(X, Y; Z, \xi)=g((\nabla_X A)Y, Z)-g((\nabla_Y A)X, Z)$$

for vector fields X, Y, Z, W tangent to N and ξ normal to N, where R' and R are the curvature tensors of N and M, respectively, and $R(X, Y; Z, W)=g(R(X, Y)Z, W)$. For orthonormal vectors X, Y in M, the sectional curvature $K(X, Y)$ of the plane section spanned by X, Y is given by

$$(1.5) \qquad K(X, Y)=R(X, Y; Y, X).$$

2. **Proof of Theorem A.** Let N be a hypersurface in M with at most two distinct principal curvatures.

We suppose that there is a point x_0 at which two principal curvatures α, β are exactly distinct. Then we can choose a neighborhood U of x_0 on which $\alpha \neq \beta$. We put $T_\alpha=\{X \in TU \mid AX=\alpha X\}$ and $T_\beta=\{X \in TU \mid AX=\beta X\}$. By

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equation (1.3) of Gauss, we have

$$R'(X, Y; Z, W) = R(X, Y; Z, W) + \alpha^2\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \tag{2.1}$$

for $X, Y, Z, W \in T_\alpha$. From equation (1.4) of Codazzi we obtain

$$R(X, Y; Z, \xi) = (X\alpha)g(Y, Z) - (Y\alpha)g(X, Z) \tag{2.2}$$

for X, Y, Z in T_α.

Let T be any vector tangent to N. By taking differentiation of (2.1) with respect to T we have

$$\begin{aligned}(\nabla_T R')(X, Y; Z, W) &+ R'(\nabla_T X, Y; Z, W) + R'(X, \nabla_T Y; Z, W)\\ &+ R'(X, Y; \nabla_T Z, W) + R'(X, Y; Z, \nabla_T W)\\ &= R(\nabla'_T X, Y; Z, W) + R(X, \nabla'_T Y; Z, W)\\ &+ R(X, Y; \nabla'_T Z, W) + R(X, Y; Z, \nabla'_T W)\\ &+ (T\alpha^2)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}\\ &+ \alpha^2\nabla_T\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\},\end{aligned}$$

since M is symmetric. From (1.1), (1.3) and (2.2) we obtain

$$\begin{aligned}(\nabla_T R')(X, Y; Z, W) = (T\alpha^2)&\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}\\ &+\tfrac{1}{2}(X\alpha^2)\{g(Y, Z)g(T, W) - g(T, Z)g(Y, W)\}\\ &+\tfrac{1}{2}(Y\alpha^2)\{g(T, Z)g(X, W) - g(X, Z)g(T, W)\}\\ &+\tfrac{1}{2}(Z\alpha^2)\{g(Y, T)g(X, W) - g(X, T)g(Y, W)\}\\ &+\tfrac{1}{2}(W\alpha^2)\{g(Y, Z)g(X, T) - g(X, Z)g(Y, T)\}.\end{aligned} \tag{2.3}$$

Let $X = W$ and $Y = Z$ be orthonormal. Then (2.3) gives

$$(\nabla_T R')(X, Y; Y, X) = T\alpha^2 + (X\alpha^2)g(X, T) + (Y\alpha^2)g(Y, T).$$

Since N is locally symmetric, we have

$$T\alpha^2 + (X\alpha^2)g(X, T) + (Y\alpha^2)g(Y, T) = 0. \tag{2.4}$$

From $\dim N \geq 3$, either $\dim T_\alpha \geq 2$ or $\dim T_\beta \geq 2$. Hence we may assume $\dim T_\alpha \geq 2$ on U and take $X = T$, then we obtain

$$X\alpha^2 = 0.$$

In particular, for $T \in T_\beta$, from (2.4) we have

$$T\alpha^2 = 0.$$

Thus α is constant on U.

Now, let ω be a vector field in T_β. Then, from (1.3)

$$R'(X, Y; Z, \omega) = R(X, Y; Z, \omega) \tag{2.5}$$

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holds. By taking differentiation of (2.5) with respect to T, we may find by using $\nabla_T R = \nabla_T R' = 0$, (1.1), (1.3) and (1.4) that

$$\begin{aligned}(2.6)\qquad &\alpha\{(Y, Z)g(\nabla_T X, \omega) - g(X, Z)g(\nabla_T Y, \omega)\\ &- g(T, X)g(\nabla_Z Y, \omega) + g(T, Y)g(\nabla_Z X, \omega)\\ &- g(T, Z)g(\nabla_X Y, \omega) + g(T, Z)g(\nabla_Y X, \omega)\} = 0.\end{aligned}$$

Choosing $T = X = Z$, Y as orthonormal vectors in T_α, we find

$$\alpha g(\nabla_X Y - \nabla_Y X, \omega) = 0.$$

If we put $X = T$ and $Y = Z$ and assume X, Y are orthonormal, then (2.6) gives

$$\alpha g(\nabla_X X - \nabla_Y Y, \omega) = 0.$$

By linearization, we find

$$\alpha g(\nabla_X Y + \nabla_Y X, \omega) = 0.$$

Hence if $\alpha \neq 0$, then we obtain

$$(2.7)\qquad g(\nabla_X X, \omega) = g(\nabla_Y Y, \omega)$$

$$(2.8)\qquad g(\nabla_X Y, \omega) = 0$$

for orthonormal vectors X, Y in T_α.

On the other hand, if S is the Ricci tensor of M, then we have

$$S(X, \xi) = 0$$

for all X in TN, since M is Einstein [3]. Noting that, for $\omega_1, \omega_2, \omega_3 \in T_\beta$, (2.2) gives $R(\omega_1, \omega_2; \omega_3, \xi) = 0$ in both cases of $\dim T_\beta = 1$ and $\dim T_\beta \geq 2$, we have

$$\begin{aligned}0 = S(\omega, \xi) &= \sum_{i=1}^{n+1} R(\omega, X_i; X_i, \xi) = \sum_{i=1}^{p} R(\omega, X_i; X_i, \xi)\\ &= (\alpha - \beta)\sum_{i=1}^{p} g(\nabla_{X_i} X_i, \omega)\end{aligned}$$

for $\omega \in T_\beta$ and orthonormal basis $X_1, \ldots, X_{n+1}$ in $T_x M$, $x \in U$, with $X_1, \ldots, X_p \in T_\alpha$; $X_{p+1} = \omega_1, \ldots, X_n = \omega_{n-p} \in T_\beta$ and $X_{n+1} = \xi$, where p denotes the multiplicity of α. From (2.7) we obtain

$$(2.9)\qquad g(\nabla_{X_i} X_i, \omega) = 0, \qquad i = 1, 2, \ldots, p.$$

Combining (2.8) and (2.9), we have

$$g(\nabla_{X_i} X_j, \omega) = 0, \qquad i, j = 1, 2, \ldots, p.$$

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Since every two vectors X, Y in T_α are linear combinations of $X_1, \ldots, X_p$, we obtain

$$g(\nabla_X Y, \omega) = 0 \tag{2.10}$$

for all X, Y in T_α.

Assume that $\alpha \equiv 0$ on U. From (2.2) we have

$$R(X, Y; Z, \xi) = 0 \tag{2.11}$$

for $X, Y, Z \in T_\alpha = T_0$. By taking differentiation of (2.11) with respect to a vector $T \in T_0$ we obtain

$$\begin{aligned} &-g(A\nabla_Y Z, \nabla_T X) + g(A\nabla_X Z, \nabla_T Y) \\ &+ g(A\nabla_X Y, \nabla_T Z) - g(A\nabla_Y X, \nabla_T Z) = 0 \end{aligned} \tag{2.12}$$

Let $X = T$ and $Y = Z$. Then (2.12) gives

$$\begin{aligned} &-g(A\nabla_Y Y, \nabla_X X) + g(A\nabla_X Y, \nabla_X Y) \\ &+ g(A\nabla_X Y, \nabla_X Y) - g(A\nabla_Y X, \nabla_X Y) = 0. \end{aligned} \tag{2.13}$$

On the other hand, from (1.3), we have

$$R'(X, \omega; \bar{\omega}, Y) = R(X, \omega; \bar{\omega}, Y) \tag{2.14}$$

for $X, Y \in T_0$ and $\omega, \bar{\omega} \in T_\beta$.

By taking differentiation of (2.14) with respect to a vector T we obtain

$$\beta g(T, \omega) g(\nabla_Y X, \bar{\omega}) + \beta g(T, \bar{\omega}) g(\nabla_X Y, \omega) = 0. \tag{2.15}$$

Let $X = Y$ and $\omega = \bar{\omega} = T$. Then (2.15) gives

$$g(\nabla_X X, \omega) = 0 \tag{2.16}$$

for all $X \in T_0$. By linearization, we find

$$g(\nabla_X Y + \nabla_Y X, \omega) = 0 \tag{2.17}$$

for all $X, Y \in T_0$. Thus (2.13), (2.16) and (2.17) give

$$g(\nabla_X Y, \omega) = g(\nabla_Y X, \omega) = 0 \tag{2.18}$$

for all $X, Y \in T_0$. Therefore, from (2.10) and (2.18), we obtain the following

$$R(\omega, X; Y, \xi) = (\alpha - \beta) g(\omega, \nabla_X Y) = 0,$$
$$R(X, Y; \omega, \xi) = (\alpha - \beta) g(\nabla_X Y - \nabla_Y X, \omega) = 0,$$

$X, Y \in T_\alpha$, $\omega \in T_\beta$. If $\dim T_\beta = 1$, then as before we have

$$0 = S(X, \xi) = \sum_{i=1}^{n+1} R(X, X_i; X_i, \xi) = R(X, \omega; \omega, \xi)$$

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for $X \in T_\alpha$ and orthonormal basis $X_1, \ldots, X_{n+1}$ in T_xM, $x \in U$, with $X_1, \ldots, X_{n-1} \in T_\alpha$, $X_n = \omega$ and $X_{n+1} = \xi$. If $\dim T_\beta \geq 2$, then we obtain

$$R(X, \omega; \bar{\omega}, \xi) = (\alpha - \beta) g(X, \nabla_\omega \bar{\omega}) = 0,$$
$$R(\omega, \bar{\omega}; X, \xi) = -(\alpha - \beta) g(\nabla_\omega \bar{\omega} - \nabla_{\bar{\omega}} \omega, X) = 0,$$

$X \in T_\alpha$, ω, $\bar{\omega} \in T_\beta$. Thus we have

$$R(X, Y; Z, \xi) = 0$$

for $X, Y, Z \in TU$.

Next, if an open subset V consists of umbilical points, then we obtain a similar equation to (2.4) and know that the principal curvature is constant on V. Hence we have

$$R(X, Y; Z, \xi) = 0$$

for X, Y, $Z \in TV$.

By a continuity argument we find

$$R(X, Y; Z, \xi) = 0$$

for all X, Y, Z in TN. Therefore we have

$$R(T_xN, T_xN)T_xN \subset T_xN \tag{2.19}$$

for all $x \in N$. Since $M = G/H$ is a symmetric space and G acts on M transitively, we may assume x is the origin 0 (fixed by H). From (2.19) and Lemma 1.1, we have

$$[[T_xN, T_xN], T_xN] \subset T_xN.$$

Consequently, Lemma 1.2 implies that M admits a totally geodesic hypersurface. Theorem A then follows from the results of Chen & Nagano (See Introduction).

3. **Proof of Theorem B.** Let N be a hypersurface in M and $E_1, \ldots, E_n$ be an orthonormal basis of T_xN, $x \in N$. Then the Ricci tensor S' of N satisfies

$$\begin{aligned} S'(Y, Z) &= \sum_{i=1}^{n} R'(E_i, Y; Z, E_i) \\ &= S(Y, Z) - R(\xi, Y; Z, \xi) \\ &\quad + \text{trace } A\, g(AY, Z) - g(A^2Y, Z) \end{aligned}$$

for $Y, Z \in T_xN$, S denotes the Ricci tensor of M. Since N and M are Einstein, the scalar curvatures ρ' and ρ of N and M satisfy

$$\begin{aligned} R(\xi, Y; Z, \xi) &= \left(\frac{\rho}{n+1} - \frac{\rho'}{n}\right) g(Y, Z) \\ &\quad + \text{trace } A\, g(AY, Z) - g(A^2Y, Z). \end{aligned} \tag{3.1}$$

As in the proof of Theorem A, we take $x_0 \in N$ and U. Then (3.1) gives

$$R(\xi, Y; Z, \xi) = \left(\frac{\rho}{n+1} - \frac{\rho'}{n}\right) g(Y, Z) + (p\alpha + (n-p)\beta) g(AY, Z) - g(A^2 Y, Z), \tag{3.2}$$

where p denotes the multiplicity of α. By taking differentiation of (3.2) with respect to T, we have

$$\begin{aligned} &-g((\nabla_{AT}A)Y, Z) + g((\nabla_Y A)AT, Z) \\ &-g((\nabla_{AT}A)Z, Y) + g((\nabla_Z A)AT, Y) \\ &= (pT\alpha + (n-p)T\beta) g(AY, Z) \\ &\quad + (p\alpha + (n-p)\beta) g((\nabla_T A)Y, Z) - g((\nabla_T A^2)Y, Z). \end{aligned} \tag{3.3}$$

Let $Z = T$ and Y be orthonormal vectors in T_α, then (3.3) gives

$$X\alpha^2 = 0 \tag{3.4}$$

for all X in T_α, since we might assume that $\dim T_\alpha \geq 2$. Hence, using (2.2) and (3.2), we obtain the following

THEOREM 3.1. *Let M be a symmetric space. If M admits an Einstein hypersurface N* ($\dim N \geq 3$) *with two distinct principal curvatures of the multiplicities p* (≥ 2) *and $n-p$, respectively, then M admits a unit vector ξ and a codimension $n-p+1$ subspace V in $T_x M$ such that* (a) *the sectional curvatures of M satisfy $K(\xi, X) = K(\xi, Y)$ for any two unit vectors X, Y in V,* (b) *$R(X, Y; Z, \xi) = 0$ for X, Y, Z in V and* (c) *$T(\xi, X: Y, \xi) = 0$ for orthogonal vectors X, Y in V.*

If N is an Einstein hypersurface in M with ξ as the unit normal vector at x and two distinct principal curvatures of the multiplicities p (≥ 2) and $n-p$ ($\geqq 2$), respectively, then there exists a geodesic c through x with ξ as its tangent vector at x. Let B be a maximal flat totally geodesic submanifold of M which contains the geodesic c (and hence x). Then the rank of M is equal to the dimension of B. Thus in particular, if $\operatorname{rank} M \geq 2$, then the intersection $T_x B \cap T_x N$ contains nonzero vectors. Then for any unit vector X in $T_x B \cap T_x N$ we have $K(\xi, X) = 0$.

Consequently, from (3.1) and Theorem 3.1 we obtain the following

THEOREM 3.2. *Let M, N and p be as in Theorem* 3.1. *If $n-p \geq 2$ and rank of M is ≥ 2, then the Ricci tensor S of M satisfies one of*

(1) $S(\xi, \xi) = (n-p)K(\xi, \omega)$ for a unit vector $\omega \in T_\beta$,

(2) $S(\xi, \xi) = pK(\xi, Y)$ for a unit vector $Y \in T_\alpha$,

(3) $S(\xi, \xi) = pK(\xi, X_\alpha/\|X_\alpha\|) + (n-p)K(\xi, X_\beta/\|X_\beta\|)$ *and* $R(\xi, X_\alpha; X_\alpha, \xi) + R(\xi, X_\beta; X_\beta, \xi) = 0$,

where X_α, X_β and $\|\cdot\|$ denote the components of a unit vector X to T_α, T_β and the length of vectors, respectively.

If M is an irreducible symmetric space of dimension ≤ 4, then M is one of spheres, real projective spaces, complex projective spaces and their noncompact duals. So we may assume that the dimension of M is ≥ 5. In the case of M admitting an Einstein hypersurface N with two distinct principal curvatures of the multiplicities p (≥ 2) and $n-p$ (≥ 2), respectively, from Lemma 1.1 and Theorem 3.2, we see that the rank of M is 1.

From a continuity argument, the case of M ($\dim M \geq 5$) admitting an Einstein hypersurface N with two distinct principal curvatures of the multiplicities $n-1$ and 1 remains. Chen and Verstraelen ([2], Theorem 9.1) showed the following.

THEOREM 3.3. *If N is an Einstein quasiumbilical hypersurface in M* ($\dim M \geq 4$), *then M is either a sphere, a real projective space or one of their noncompact duals.*

However, the proof of the above Theorem is not precise. Here we will give the precise proof.

Now, let the dimensions of T_α and T_β are $n-1$ and 1, respectively. By the result of Chen and Verstraelen (See Introduction), we may assume that $\alpha \neq 0$ on N. Then, from (3.4), we have

$$X\alpha^2 = 0$$

for all $X \in T_\alpha$. Let T and Y be in T_α and $Z = \omega \in T_\beta$. Then (3.3) gives

$$(\alpha-\beta)\alpha\{ng(\nabla_T Y, \omega) - g(\nabla_Y T, \omega)\} = \alpha(\omega\alpha)g(T, Y)$$

from which we get

$$\alpha(\alpha-\beta)(n-1)g(\nabla_T T, \omega) = \alpha(\omega\alpha) \tag{3.5}$$

for unit vector $T \in T_\alpha$, and

$$\alpha\{g(\nabla_T Y, \omega) - g(\nabla_Y T, \omega)\} = 0 \tag{3.6}$$

for $T, Y \in T_\alpha$. From (3.5) we find

$$\alpha g(\nabla_T T, \omega) = \alpha g(\nabla_Y Y, \omega)$$

for unit vectors $T, Y \in T_\alpha$. By linearization, we find

$$\alpha\{g(\nabla_T Y, \omega) + g(\nabla_Y T, \omega)\} = 0$$

for orthonormal vectors $T, Y \in T_\alpha$. Since $\alpha \neq 0$, we obtain

$$g(\nabla_T T, \omega) = g(\nabla_Y Y, \omega)$$
$$g(\nabla_T Y, \omega) = 0$$

for orthonormal vectors $T, Y \in T_\alpha$. Using (2.9) of the proof of Theorem A, we obtain

$$g(\nabla_T Y, \omega) = 0$$

for all X, Y in T_α. Thus we have

$$R(X, Y; Z, \xi) = 0$$

for $X, Y, Z \in TN$. By a similar argument to the last part of the proof of Theorem A we see that M admits a totally geodesic hypersurface. Hence we obtain Theorem 3.3.

This completes the proof of Theorem B.

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