

THE FINITE-TIME RUIN PROBABILITY WITH DEPENDENT INSURANCE AND FINANCIAL RISKS

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Abstract

Consider a discrete-time insurance risk model. Within period i , the net insurance loss is denoted by a real-valued random variable X_i . The insurer makes both risk-free and risky investments, leading to an overall stochastic discount factor Y_i from time i to time $i - 1$. Assume that (X_i, Y_i) , $i \in \mathbb{N}$, form a sequence of independent and identically distributed random pairs following a common bivariate Farlie–Gumbel–Morgenstern distribution with marginal distribution functions F and G . When F is subexponential and G fulfills some constraints in order for the product convolution of F and G to be subexponential too, we derive a general asymptotic formula for the finite-time ruin probability. Then, for special cases in which F belongs to the Fréchet or Weibull maximum domain of attraction, we improve this general formula to be transparent.

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1. Introduction

Following the works of Nyrhinen (1999), (2001) and Tang and Tsitsiashvili (2003), (2004), we consider a discrete-time insurance risk model. Within period i , the net insurance loss (equal to the total claim amount minus the total premium income) is denoted by a real-valued random variable X_i , $i \in \mathbb{N}$. Suppose that the insurer makes both risk-free and risky investments, which lead to an overall stochastic discount factor Y_i from time i to time $i - 1$. See, e.g. Section 4.1 of Hashorva *et al.* (2010) for the structure of these overall stochastic discount factors. In the terminology of Tang and Tsitsiashvili (2003), (2004), we call $\{X_i, i \in \mathbb{N}\}$ insurance risks and $\{Y_i, i \in \mathbb{N}\}$ financial risks. Thus, the sum

$$S_n = \sum_{i=1}^n X_i \prod_{j=1}^i Y_j, \quad n \in \mathbb{N}, \tag{1.1}$$

represents the stochastic present value of aggregate net losses up to time n . As usual, the probability of ruin by time n is defined to be

$$\psi(x; n) = \Pr \left(\max_{1 \leq m \leq n} \sum_{i=1}^m X_i \prod_{j=1}^i Y_j > x \right), \quad n \in \mathbb{N}, \tag{1.2}$$

where $x \geq 0$ is interpreted as the initial wealth of the insurer.

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Since the work of Tang and Tsitsiashvili (2003), there has been a vast amount of literature on the asymptotic behavior of the ruin probability of this risk model in the presence of heavy-tailed insurance and/or financial risks. It has been common to assume that both $\{X_i, i \in \mathbb{N}\}$ and $\{Y_i, i \in \mathbb{N}\}$ are sequences of independent and identically distributed (i.i.d.) random variables and that they are independent of each other. Undoubtedly, this assumption of complete independence is unrealistic. A recent new approach is to introduce various dependence structures to the risk model. In this direction, we refer the reader to Goovaerts *et al.* (2005), Tang (2006a), Zhang *et al.* (2009), Weng *et al.* (2009), and Yi *et al.* (2011), among many others. However, there are few papers which take into account the dependence between insurance and financial risks, with the difficulty existing in describing the tail behavior of the product of dependent random variables.

In the present paper we assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs, and that a generic random pair (X, Y) has dependent components. We use a bivariate Farlie–Gumbel–Morgenstern (FGM) distribution to model the dependence structure of (X, Y) . Recall that a bivariate FGM distribution function is of the form

$$\Pi(x, y) = F(x)G(y)(1 + \theta\overline{F}(x)\overline{G}(y)), \quad (1.3)$$

where F and G are marginal distribution functions and θ is a real number fulfilling $|\theta| \leq 1$ in order for $\Pi(\cdot, \cdot)$ to be a proper bivariate distribution function. Trivially, if $\theta = 0$ then (1.3) describes a joint distribution function of two independent random variables. We refer the reader to Kotz *et al.* (2000) for a general account on FGM distribution functions, and to Tang and Vernic (2007) and Cossette *et al.* (2008) for applications of FGM distribution functions to risk theory.

First, under the assumption that F in (1.3) is a subexponential distribution function while G fulfills some constraints in order for the product convolution of F and G to be a subexponential distribution function too, we derive a general asymptotic formula for the ruin probability $\psi(x; n)$. Here the product convolution of F and G , denoted as $F \otimes G$, is understood as the distribution function of the product X^*Y^* , where X^* and Y^* are two independent random variables with X^* identically distributed as X and Y^* as Y .

Next, for some special cases in which F is a distribution function in the Fréchet or Weibull maximum domain of attraction, we improve the general asymptotic formula to be transparent. The obtained asymptotic formulae successfully capture the impact of the underlying dependence structure of (X, Y) .

The rest of this paper consists of three sections. In Section 2 we give some preliminaries on subexponential distributions and maximum domains of attraction. Section 3 contains the main results, and the proofs are presented in Section 4.

2. Preliminaries

Throughout the paper, all limit relationships are according to $x \rightarrow \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \leq 1$ and write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$. We also write $f(x) \asymp g(x)$ if $0 < \liminf f(x)/g(x) \leq \limsup f(x)/g(x) < \infty$.

A distribution function F on $\mathbb{R}^+ = [0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F}(x) = 1 - F(x) > 0$ for all $x \in \mathbb{R}^+$ and

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x),$$

where F^{2*} denotes the two-fold convolution of F . More generally, a distribution function F on \mathbb{R} is still said to be subexponential if the distribution function $F_+(x) = F(x) \mathbf{1}_{\{x \geq 0\}}$ is subexponential. It is well known that every subexponential distribution function F is long tailed, written as $F \in \mathcal{L}$, in the sense that the relation $\overline{F}(x + y) \sim \overline{F}(x)$ holds for some (or, equivalently, for all) $y \neq 0$; see Lemma 1.3.5(a) of Embrechts *et al.* (1997). Following the works of Konstantinides *et al.* (2002) and Tang (2006b), we say that a distribution function F on \mathbb{R} belongs to the class \mathcal{A} if $F \in \mathcal{S}$ and the relation

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(cx)}{\overline{F}(x)} < 1 \tag{2.1}$$

holds for some $c > 1$. Note that (2.1) is really a mild constraint and is fulfilled by almost all useful distribution functions with unbounded supports. Thus, as remarked in Tang (2006b), the class \mathcal{A} almost exhausts the class \mathcal{S} .

Under the assumption that the insurance risk X follows a distribution function $F \in \mathcal{S}$ or \mathcal{A} , we will derive a general asymptotic formula for the ruin probability $\psi(x; n)$. In order to further improve this general formula to be transparent, we need to introduce several concepts in extreme value theory.

A distribution function F on \mathbb{R} is said to belong to the maximum domain of attraction of an extreme value distribution function F_0 , written as $F \in \text{MDA}(F_0)$, if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - F_0(x)| = 0 \tag{2.2}$$

holds for some constants $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$. Only three choices for F_0 in (2.2) are possible: the Fréchet, Gumbel, and Weibull distributions, which we denote by Φ_α , Λ , and Ψ_α for $\alpha > 0$, respectively. A distribution function F belongs to $\text{MDA}(\Phi_\alpha)$ if and only if $\overline{F}(\cdot)$ is regularly varying at ∞ with index $-\alpha$, that is, the relation

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(cx)}{\overline{F}(x)} = c^{-\alpha}$$

holds for all $c > 0$; see Theorem 3.3.7 of Embrechts *et al.* (1997). Hence, $\text{MDA}(\Phi_\alpha) \subset \mathcal{S}$; see Lemma 1.3.1 of Embrechts *et al.* (1997). A distribution function F with an upper endpoint $\hat{x} = \sup\{x \in \mathbb{R} : F(x) < 1\} \leq \infty$ belongs to $\text{MDA}(\Lambda)$ if and only if the relation

$$\lim_{x \uparrow \hat{x}} \frac{\overline{F}(x + ca(x))}{\overline{F}(x)} = e^{-c}$$

holds for some positive auxiliary function $a(\cdot)$ and all $c \in \mathbb{R}$. Recall that the auxiliary function $a(\cdot)$ is asymptotically equivalent to the mean excess function of F , namely, $a(x) \sim E[X - x \mid X > x]$ as $x \uparrow \hat{x}$; see Theorem 3.3.27 of Embrechts *et al.* (1997). Almost all distribution functions with rapidly varying tails are included in $\text{MDA}(\Lambda)$. A distribution function F belongs to $\text{MDA}(\Psi_\alpha)$ if and only if its upper endpoint \hat{x} is finite and the relation

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(\hat{x} - c/x)}{\overline{F}(\hat{x} - 1/x)} = c^\alpha$$

holds for all $c > 0$; see Theorem 3.3.12 of Embrechts *et al.* (1997).

3. Main results

Recall the insurance risk model introduced in Section 1. In the sequel, denote by F on \mathbb{R} , G on \mathbb{R}^+ , and H on \mathbb{R}^+ the distribution functions of X , Y , and XY , respectively. Throughout the rest of the paper, F is assumed to be heavy tailed and, to avoid triviality, G is assumed to be nondegenerate at 0. For each $i \in \mathbb{N}$, denote by H_i the distribution function of $X_i \prod_{j=1}^i Y_j$. Note that $H = H_1$.

Our first main result is as follows.

Theorem 3.1. *Assume that (X_i, Y_i) , $i \in \mathbb{N}$, form a sequence of i.i.d. random pairs following a common bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Furthermore, assume that $F \in \mathcal{S}$ and that there is a function $b(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that the following statements hold simultaneously:*

- (a) $b(x) \uparrow \infty$,
- (b) $b(x)/x \downarrow 0$,
- (c) $\overline{G}(b(x)) = o(\overline{H}(x))$, and
- (d) $\overline{F}(x - b(x)) \sim \overline{F}(x)$.

Then, for each $n \in \mathbb{N}$, the distribution function of S_n belongs to the class \mathcal{S} and the ruin probability $\psi(x; n)$ satisfies

$$\psi(x; n) \sim \Pr(S_n > x) \sim \sum_{i=1}^n \overline{H}_i(x). \tag{3.1}$$

Assumptions (a)–(d) of Theorem 3.1 were first proposed in Cline and Samorodnitsky (1994) when establishing subexponentiality for the product of independent random variables; see Lemma 4.1 below. These assumptions automatically hold if Y is bounded above. Recall that $F \otimes G$ stands for the product convolution of F and G . With $\theta \in (-1, 1]$, relations (4.4) and (4.7) below show that $\overline{H}(x) \asymp \overline{F} \otimes \overline{G}(x)$. Thus, for this case, the set of assumptions (a)–(c) is equivalent to the set of assumptions (a), (b), and

$$(c^*) \overline{G}(b(x)) = o(\overline{F} \otimes \overline{G}(x)).$$

Of course, (c*) is easier to verify than (c).

As remarked in Tang (2006b), (c) requires that $b(\cdot)$ diverge to ∞ not too slowly while (d) requires that it diverge to ∞ not too fast. Very often (d) appears to be too restrictive for applications. In our second main result below we drop this annoying assumption.

Theorem 3.2. *Assume that (X_i, Y_i) , $i \in \mathbb{N}$, form a sequence of i.i.d. random pairs following a common bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Furthermore, assume that $F \in \mathcal{A}$ and that there is a function $b(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that (a)–(c) hold simultaneously. Then, for each $n \in \mathbb{N}$, the distribution function of S_n belongs to the class \mathcal{A} and the ruin probability $\psi(x; n)$ satisfies (3.1).*

Interestingly, as summarized in Lemma 4.6 below, (a)–(c) amount to the assertion that the relation $\overline{G}(cx) = o(\overline{H}(x))$ holds for all $c > 0$. This latter assertion is much easier to verify than (a)–(c).

We end this section with a corollary of Theorems 3.1 and 3.2 in which we focus on improving the asymptotic formula given in (3.1) to be completely transparent within maximum domains of attraction.

Corollary 3.1. Assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs following a common bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$.

(i) If $F \in \text{MDA}(\Phi_\alpha)$ for some $\alpha > 0$ and $E[Y^{\alpha+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then

$$\psi(x; n) \sim ((1 - \theta) E[Y^\alpha] + \theta E[(Y_1^* \vee Y_2^*)^\alpha]) \frac{1 - (E[Y^\alpha])^n}{1 - E[Y^\alpha]} \bar{F}(x), \tag{3.2}$$

where Y_1^* and Y_2^* are two independent random variables identically distributed as Y .

(ii) If $F \in \mathcal{S} \cap \text{MDA}(\Lambda)$ with an auxiliary function $a(\cdot)$ and $G \in \text{MDA}(\Psi_\alpha)$ for some $\alpha > 0$ with an upper endpoint $0 < \hat{y} < \infty$, then

$$\psi(x; n) \sim (1 + \theta) \sum_{i=1}^n \bar{F}\left(\frac{x}{\hat{y}^i}\right) \left(\Gamma(\alpha + 1) \bar{G}\left(\hat{y} - \frac{\hat{y}^{i+1}}{x} a\left(\frac{x}{\hat{y}^i}\right)\right) \right)^i. \tag{3.3}$$

Relation (3.2) suggests that if further $E[Y^\alpha] < 1$, the following simple asymptotic formula for the infinite-time ruin probability should hold:

$$\psi(x; \infty) = \Pr\left(\max_{1 \leq m < \infty} \sum_{i=1}^m X_i \prod_{j=1}^i Y_j > x\right) \sim \frac{(1 - \theta) E[Y^\alpha] + \theta E[(Y_1^* \vee Y_2^*)^\alpha]}{1 - E[Y^\alpha]} \bar{F}(x).$$

Indeed, this can be easily proven by Theorem 1 of Grey (1994). We omit the details. Also, note that, when $\hat{y} = 1$, relation (3.3) admits the following substantial simplification:

$$\psi(x; n) \sim (1 + \theta) \Gamma(\alpha + 1) \bar{F}(x) \bar{G}\left(1 - \frac{a(x)}{x}\right). \tag{3.4}$$

Since the right-hand side of (3.4) does not involve n , the same asymptotic formula should hold for the infinite-time ruin probability, though we still cannot prove it at this stage.

4. Proofs

Recall relations (1.1) and (1.2). Clearly,

$$\Pr\left(\sum_{i=1}^n X_i \prod_{j=1}^i Y_j > x\right) \leq \psi(x; n) \leq \Pr\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x\right), \tag{4.1}$$

where $X_i^+ = X_i \mathbf{1}_{\{X_i \geq 0\}}$ denotes the positive part of $X_i, i \in \mathbb{N}$. For the proof that (3.1) holds in both Theorems 3.1 and 3.2, if we can establish that the second asymptotic formula in (3.1) holds by not requiring the assumption that $F(0-) > 0$, then the same asymptotic formula should hold for the right-hand side of (4.1) as well. In this way we will have completed the proof of (3.1). Furthermore, we note the simple fact that

$$S_n = \sum_{i=1}^n X_i \prod_{j=1}^i Y_j \stackrel{D}{=} \sum_{i=1}^n X_i \prod_{j=i}^n Y_j, \quad n \in \mathbb{N}, \tag{4.2}$$

due to the i.i.d. assumption for the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$, where ‘ $\stackrel{D}{=}$ ’ denotes equality in distribution. Denote by T_n the sum on the right-hand side of (4.2). Therefore, we only need to prove the relation

$$\Pr(T_n > x) \sim \sum_{i=1}^n \bar{H}_i(x). \tag{4.3}$$

4.1. Proof of Theorem 3.1

The following lemma is a restatement of Theorem 2.1 of Cline and Samorodnitsky (1994), which is crucial for establishing our Theorem 3.1.

Lemma 4.1. *Let X and Y be two independent random variables with distribution functions F on \mathbb{R} and G on \mathbb{R}^+ , respectively. Denote by H the distribution function of their product XY . We have $H \in \mathcal{S}$ if $F \in \mathcal{S}$ and there is a function $b(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that (a)–(d), namely,*

- $b(x) \uparrow \infty$,
- $b(x)/x \downarrow 0$,
- $\bar{G}(b(x)) = o(\bar{H}(x))$, and
- $\bar{F}(x - b(x)) \sim \bar{F}(x)$

hold simultaneously.

Recall the generic random pair (X, Y) in the insurance risk model in Section 1. Introduce the six independent random variables $X^*, X_1^*, X_2^*, Y^*, Y_1^*$, and Y_2^* , with the first three identically distributed as X and the last three identically distributed as Y . In the proof of Theorem 3.1 we also need the following lemma.

Lemma 4.2. *Let (X, Y) follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Under (a)–(c), it holds that*

$$\bar{H}(x) \sim (1 - \theta) \Pr(X^*Y^* > x) + \theta \Pr(X^*(Y_1^* \vee Y_2^*) > x). \tag{4.4}$$

Proof. Note the decomposition

$$\Pi = (1 + \theta)FG - \theta F^2G - \theta FG^2 + \theta F^2G^2. \tag{4.5}$$

Clearly, $X_1^* \vee X_2^*$ is distributed by F^2 and $Y_1^* \vee Y_2^*$ is distributed by G^2 . Thus, it follows from (4.5) that

$$\begin{aligned} \bar{H}(x) &= \Pr(XY > x) \\ &= (1 + \theta) \Pr(X^*Y^* > x) - \theta \Pr((X_1^* \vee X_2^*)Y^* > x) \\ &\quad - \theta \Pr(X^*(Y_1^* \vee Y_2^*) > x) + \theta \Pr((X_1^* \vee X_2^*)(Y_1^* \vee Y_2^*) > x). \end{aligned} \tag{4.6}$$

In terms of the function $b(\cdot)$ in (a)–(c), we have

$$\begin{aligned} \Pr((X_1^* \vee X_2^*)Y^* > x) &= \int_0^{b(x)} \Pr\left(X_1^* \vee X_2^* > \frac{x}{y}\right) dG(y) + O(\bar{G}(b(x))) \\ &= (2 + o(1)) \int_0^{b(x)} \Pr\left(X^* > \frac{x}{y}\right) dG(y) + O(\bar{G}(b(x))) \\ &= (2 + o(1)) \Pr(X^*Y^* > x) + o(\bar{H}(x)). \end{aligned}$$

Similarly as above,

$$\Pr((X_1^* \vee X_2^*)(Y_1^* \vee Y_2^*) > x) = (2 + o(1)) \Pr(X^*(Y_1^* \vee Y_2^*) > x) + o(\overline{H}(x)).$$

Substituting these estimates into (4.6) yields

$$\overline{H}(x) = (1 - \theta + o(1)) \Pr(X^*Y^* > x) + (\theta + o(1)) \Pr(X^*(Y_1^* \vee Y_2^*) > x) + o(\overline{H}(x)),$$

which implies relation (4.4).

When $\theta \in (-1, 1]$, we apply the two-sided inequality

$$\Pr(X^*Y^* > x) \leq \Pr(X^*(Y_1^* \vee Y_2^*) > x) \leq 2 \Pr(X^*Y^* > x)$$

to obtain an asymptotic estimate for the right-hand side of (4.4) as

$$(1 - \theta) \Pr(X^*Y^* > x) + \theta \Pr(X^*(Y_1^* \vee Y_2^*) > x) \asymp \Pr(X^*Y^* > x). \tag{4.7}$$

Therefore, there is no problem with the implication in the last step of the proof of Lemma 4.2. When $\theta = -1$, however, we need to be aware that the right-hand side of (4.4) could reduce to $o(\Pr(X^*Y^* > x))$, e.g. when F is rapidly varying tailed and G does not assign a mass at the upper endpoint $0 < \hat{y} \leq \infty$. For this case, (4.4) may fail to give an exact asymptotic relationship. This explains why we have to exclude $\theta = -1$ in Lemma 4.2 and hereby in Theorems 3.1 and 3.2.

Lemma 4.2 offers the following insight. For $\theta \in (-1, 1]$, introduce a function $G_\theta(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ as

$$G_\theta(y) = (1 - \theta)G(y) + \theta G^2(y), \tag{4.8}$$

which clearly defines a proper distribution function on \mathbb{R}^+ . Furthermore, introduce a nonnegative random variable Z_θ distributed according to G_θ and independent of (X, Y) . Then relation (4.4) can be rewritten as

$$\Pr(XY > x) \sim \Pr(XZ_\theta > x). \tag{4.9}$$

In this way, the dependence structure of (X, Y) is dissolved.

By Lemma 4.1, it is easy to establish the subexponentiality of the independent product XZ_θ in (4.9). Also, recall that the class \mathcal{S} is closed under asymptotic equivalence; see, e.g. Theorem 3 of Teugels (1975). Therefore, by relation (4.9) we readily arrive at the following result.

Lemma 4.3. *Let (X, Y) follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. If $F \in \mathcal{S}$ and (a)–(d) hold then $H \in \mathcal{S}$.*

For a distribution function F on \mathbb{R} , under (a), (b), and (d), it is straightforward to verify that the relation

$$\overline{F}(x - cb(x)) \sim \overline{F}(x) \tag{4.10}$$

holds for all $c > 0$. The following lemma is inspired by the work of Zhou *et al.* (2011).

Lemma 4.4. *Let (X, Y) follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Under (a)–(d), it holds for all $c > 0$ that*

$$\overline{H}(x - cb(x)) \sim \overline{H}(x). \tag{4.11}$$

Proof. We only need to prove that (4.11) holds with $c = 1$. For arbitrarily fixed $0 < \varepsilon, \delta < 1$, by (4.9) we have

$$\begin{aligned} \bar{H}(x - b(x)) &\sim \Pr(XZ_\theta > x - b(x)) \\ &= \left(\int_0^\varepsilon + \int_\varepsilon^1 + \int_1^{b(x)} + \int_{b(x)}^\infty \right) \bar{F}\left(\frac{x - b(x)}{y}\right) dG_\theta(y) \\ &= \sum_{i=1}^4 I_i(x). \end{aligned} \tag{4.12}$$

We deal with the four terms on the right-hand side of (4.12) one by one. By (a) and relation (4.10) with $c = 1/\varepsilon$, it holds for all large x that

$$\begin{aligned} I_1(x) &\leq \bar{F}\left(\frac{x - b(x)}{\varepsilon}\right) G_\theta((0, \varepsilon]) \\ &\leq \bar{F}\left(\frac{x}{\varepsilon} - \frac{1}{\varepsilon}b\left(\frac{x}{\varepsilon}\right)\right) G_\theta((0, \varepsilon]) \\ &\leq (1 + \delta) G_\theta((0, \varepsilon]) \bar{F}\left(\frac{x}{\varepsilon}\right) \\ &= (1 + \delta) \frac{G_\theta((0, \varepsilon])}{\bar{G}_\theta(\varepsilon)} \Pr\left(X > \frac{x}{\varepsilon}, Z_\theta > \varepsilon\right) \\ &\leq (1 + \delta) \frac{G_\theta((0, \varepsilon])}{\bar{G}_\theta(\varepsilon)} \Pr(XZ_\theta > x). \end{aligned}$$

Similarly, it holds for all large x that

$$\begin{aligned} I_2(x) &\leq \int_\varepsilon^1 \bar{F}\left(\frac{x}{y} - \frac{b(x)}{\varepsilon}\right) dG_\theta(y) \\ &\leq \int_\varepsilon^1 \bar{F}\left(\frac{x}{y} - \frac{1}{\varepsilon}b\left(\frac{x}{y}\right)\right) dG_\theta(y) \\ &\leq (1 + \delta) \int_\varepsilon^1 \bar{F}\left(\frac{x}{y}\right) dG_\theta(y). \end{aligned}$$

By (b) and (d), it holds for all large x that

$$I_3(x) \leq \int_1^{b(x)} \bar{F}\left(\frac{x}{y} - b\left(\frac{x}{y}\right)\right) dG_\theta(y) \leq (1 + \delta) \int_1^{b(x)} \bar{F}\left(\frac{x}{y}\right) dG_\theta(y).$$

Finally, by (4.8) and (c),

$$I_4(x) \leq \bar{G}_\theta(b(x)) \asymp \bar{G}(b(x)) = o(\bar{H}(x)).$$

Substituting these estimates into (4.12) yields

$$\begin{aligned} \bar{H}(x - b(x)) &\lesssim (1 + \delta) \frac{G_\theta((0, \varepsilon])}{\bar{G}_\theta(\varepsilon)} \Pr(XZ_\theta > x) + (1 + \delta) \int_\varepsilon^{b(x)} \bar{F}\left(\frac{x}{y}\right) dG_\theta(y) + o(\bar{H}(x)) \end{aligned}$$

$$\begin{aligned} &\lesssim (1 + \delta) \left(\frac{G_\theta((0, \varepsilon])}{\bar{G}_\theta(\varepsilon)} + 1 \right) \Pr(XZ_\theta > x) + o(\bar{H}(x)) \\ &\sim (1 + \delta) \left(\frac{G_\theta((0, \varepsilon])}{\bar{G}_\theta(\varepsilon)} + 1 \right) \bar{H}(x), \end{aligned}$$

where in the last step we used (4.9) again. Letting both $\varepsilon \downarrow 0$ and $\delta \downarrow 0$ on the right-hand side above leads to $\bar{H}(x - b(x)) \lesssim \bar{H}(x)$, which is equivalent to (4.11) with $c = 1$.

The following lemma is well known and can be found in Embrechts and Goldie (1980), Cline (1986, Corollary 1), and Tang and Tsitsiashvili (2003, Lemma 3.2).

Lemma 4.5. *Let F_1 and F_2 be two distribution functions on \mathbb{R} , and let $F = F_1 * F_2$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$, and $\bar{F}_2(x) = O(\bar{F}_1(x))$, then $F \in \mathcal{S}$ and $\bar{F}(x) \sim \bar{F}_1(x) + \bar{F}_2(x)$.*

Proof of Theorem 3.1. As mentioned in the beginning of this section, we only need to prove that relation (4.3) holds and that the distribution function of T_n belongs to the class \mathcal{S} . We proceed by induction on n . Note that

- (1:1) $\bar{G}(b(x)) = o(\bar{H}(x))$ and $\bar{H}(x - b(x)) \sim \bar{H}(x)$ (by Lemma 4.4),
- (1:2) relation (4.3) trivially holds for $n = 1$,
- (1:3) $T_1 = X_1 Y_1$ follows a subexponential distribution function (by Lemma 4.3).

Now assume that

- (n:1) $\bar{G}(b(x)) = o(\bar{H}_n(x))$ and $\bar{H}_n(x - b(x)) \sim \bar{H}_n(x)$,
- (n:2) relation (4.3) holds for n ,
- (n:3) both $X_n \prod_{j=1}^n Y_j$ and T_n follow subexponential distribution functions.

We will prove that these assertions hold for $n + 1$, namely,

- (n+1:1) $\bar{G}(b(x)) = o(\bar{H}_{n+1}(x))$ and $\bar{H}_{n+1}(x - b(x)) \sim \bar{H}_{n+1}(x)$,
- (n+1:2) relation (4.3) holds for $n + 1$,
- (n+1:3) both $X_{n+1} \prod_{j=1}^{n+1} Y_j$ and T_{n+1} follow subexponential distribution functions.

First, note that T_n and X_{n+1} are independent of each other, that, by (n:3), both follow subexponential distribution functions (hence are long tailed), and that, by (n:2), if $\bar{G}(1) > 0$ then $\Pr(X_{n+1} > x) = O(\Pr(T_n > x))$, while if $\bar{G}(1) = 0$ then $\Pr(T_n > x) = O(\Pr(X_{n+1} > x))$. Thus, by Lemma 4.5,

$$\Pr(T_n + X_{n+1} > x) \sim \Pr(T_n > x) + \Pr(X_{n+1} > x) \sim \sum_{i=1}^n \bar{H}_i(x) + \bar{F}(x), \tag{4.13}$$

where in the last step we used (n:2).

Consider (n+1:1). If $\bar{G}(1) > 0$ then

$$\bar{H}_{n+1}(x) \geq \Pr\left(X_{n+1} \prod_{j=1}^{n+1} Y_j > x, Y_1 > 1, \dots, Y_n > 1\right) \geq \bar{H}(x) \bar{G}(1)^n.$$

Thus, $\bar{G}(b(x)) = o(\bar{H}(x)) = o(\bar{H}_{n+1}(x))$. If $\bar{G}(1) = 0$ then the relation $\bar{G}(b(x)) = o(\bar{H}_{n+1}(x))$ trivially holds since $\bar{G}(b(x)) = 0$ for all large x . Moreover, note that, since

$\overline{H}_n(x - b(x)) \sim \overline{H}_n(x)$ by (n:1), the proof of $\overline{H}_{n+1}(x - b(x)) \sim \overline{H}_{n+1}(x)$ is similar to but easier than the proof of Lemma 4.4.

Next, consider (n+1:2). Introduce the six independent random variables $X^*, X_1^*, X_2^*, Y^*, Y_1^*$, and Y_2^* , as for Lemma 4.2, and let them be independent of T_n . Similarly as done in (4.6),

$$\begin{aligned} \Pr(T_{n+1} > x) &= \Pr((T_n + X_{n+1})Y_{n+1} > x) \\ &= (1 + \theta) \Pr((T_n + X^*)Y^* > x) - \theta \Pr((T_n + X_1^* \vee X_2^*)Y^* > x) \\ &\quad - \theta \Pr((T_n + X^*)(Y_1^* \vee Y_2^*) > x) + \theta \Pr((T_n + X_1^* \vee X_2^*)(Y_1^* \vee Y_2^*) > x) \\ &= (1 + \theta)J_1(x) - \theta J_2(x) - \theta J_3(x) + \theta J_4(x). \end{aligned} \tag{4.14}$$

We only deal with the tail probability $J_1(x)$ in detail. By (4.13), for $x > 0$,

$$\begin{aligned} J_1(x) &= \left(\int_0^{b(x)} + \int_{b(x)}^\infty \right) \Pr\left(T_n + X^* > \frac{x}{y}\right) dG(y) \\ &= (1 + o(1)) \int_0^{b(x)} \left(\sum_{i=1}^n \overline{H}_i\left(\frac{x}{y}\right) + \overline{F}\left(\frac{x}{y}\right) \right) dG(y) + O(\overline{G}(b(x))) \\ &\sim \int_0^\infty \left(\sum_{i=1}^n \overline{H}_i\left(\frac{x}{y}\right) + \overline{F}\left(\frac{x}{y}\right) \right) dG(y), \end{aligned}$$

where we have used $\overline{G}(b(x)) = o(1) \int_0^\infty \overline{F}(x/y) dG(y)$ due to (c) and relations (4.4) and (4.7). In a similar way, with some obvious modifications, we have

$$\begin{aligned} J_2(x) &\sim \int_0^\infty \left(\sum_{i=1}^n \overline{H}_i\left(\frac{x}{y}\right) + 2\overline{F}\left(\frac{x}{y}\right) \right) dG(y), \\ J_3(x) &\sim \int_0^\infty \left(\sum_{i=1}^n \overline{H}_i\left(\frac{x}{y}\right) + \overline{F}\left(\frac{x}{y}\right) \right) dG^2(y), \\ J_4(x) &\sim \int_0^\infty \left(\sum_{i=1}^n \overline{H}_i\left(\frac{x}{y}\right) + 2\overline{F}\left(\frac{x}{y}\right) \right) dG^2(y). \end{aligned}$$

Substituting these estimates into (4.14), and then using (4.8) and (4.9), we obtain

$$\begin{aligned} \Pr(T_{n+1} > x) &\sim \int_0^\infty \sum_{i=1}^n \overline{H}_i\left(\frac{x}{y}\right) dG(y) + (1 - \theta) \int_0^\infty \overline{F}\left(\frac{x}{y}\right) dG(y) \\ &\quad + \theta \int_0^\infty \overline{F}\left(\frac{x}{y}\right) dG^2(y) \\ &= \sum_{i=2}^{n+1} \overline{H}_i(x) + \int_0^\infty \overline{F}\left(\frac{x}{y}\right) dG_\theta(y) \\ &\sim \sum_{i=1}^{n+1} \overline{H}_i(x), \end{aligned} \tag{4.15}$$

showing that relation (4.3) still holds for $n + 1$.

Finally, consider (n+1:3). Note that $H_n \in \mathcal{S}$ by (n:3) and that $\overline{G}(b(x)) = o(\overline{H}_n(x))$ and $\overline{H}_n(x - b(x)) \sim \overline{H}_n(x)$ by (n:1). Thus, applying Lemma 4.1 we obtain $H_{n+1} \in \mathcal{S}$. In other

words, $H_i \in \mathcal{S} \subset \mathcal{L}$ for all $i = 1, \dots, n + 1$. Hence, by (4.15), T_{n+1} is long tailed. Moreover, by (4.15) again, if $\overline{G}(1) > 0$ then $\Pr(T_{n+1} > x) \asymp \overline{H}_{n+1}(x)$, while if $\overline{G}(1) = 0$ then $\Pr(T_{n+1} > x) \asymp \overline{H}(x)$. Therefore, for both cases, applying Theorem 2.1(a) of Klüppelberg (1988) we obtain the subexponentiality of T_{n+1} . This completes the proof of Theorem 3.1.

4.2. Proof of Theorem 3.2

The following two lemmas are restatements of Lemma 3.2 and Theorem 2.1 of Tang (2006b), respectively.

Lemma 4.6. *For two distribution functions G and H with $\overline{G}(x) > 0$ and $\overline{H}(x) > 0$ for all $x \in \mathbb{R}^+$, the relation $\overline{G}(cx) = o(\overline{H}(x))$ holds for all $c > 0$ if and only if there is a function $b(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that (a)–(c), namely,*

- $b(x) \uparrow \infty$,
- $b(x)/x \downarrow 0$, and
- $\overline{G}(b(x)) = o(\overline{H}(x))$

hold.

Lemma 4.7. *Let X and Y be two independent random variables with distribution functions F on \mathbb{R} and G on \mathbb{R}^+ , respectively. Denote by H the distribution function of their product XY . If $F \in \mathcal{A}$ and $\overline{G}(cx) = o(\overline{H}(x))$ for all $c > 0$ then $H \in \mathcal{A}$.*

Recall relations (4.8) and (4.9) under the conditions of Lemma 4.2. Thus, applying Lemma 4.7, we can easily obtain an analogue of Lemma 4.3 as follows.

Lemma 4.8. *Let (X, Y) follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. If $F \in \mathcal{A}$ and (a)–(c) hold, then $H \in \mathcal{A}$.*

Proof of Theorem 3.2. As mentioned before, we only need to prove that relation (4.3) holds and that the distribution function of T_n belongs to the class \mathcal{A} . We still proceed by induction on n . Note that

$$(1:1) \quad \overline{G}(b(x)) = o(\overline{H}(x)),$$

$$(1:2) \quad \text{relation (4.3) trivially holds for } n = 1,$$

$$(1:3) \quad T_1 = X_1 Y_1 \text{ follows a distribution function in the class } \mathcal{A} \text{ (by Lemma 4.8).}$$

Now assume that

$$(n:1) \quad \overline{G}(b(x)) = o(\overline{H}_n(x)),$$

$$(n:2) \quad \text{relation (4.3) holds for } n,$$

$$(n:3) \quad \text{both } X_n \prod_{j=1}^n Y_j \text{ and } T_n \text{ follow distribution functions in the class } \mathcal{A}.$$

We will prove that these assertions hold for $n + 1$, namely,

$$(n+1:1) \quad \overline{G}(b(x)) = o(\overline{H}_{n+1}(x)),$$

$$(n+1:2) \quad \text{relation (4.3) holds for } n + 1,$$

$$(n+1:3) \quad \text{both } X_{n+1} \prod_{j=1}^{n+1} Y_j \text{ and } T_{n+1} \text{ follow distribution functions in the class } \mathcal{A}.$$

The same as in the proof of Theorem 3.1, the relations in (4.13) hold. For (n+1:1), the verification of $\overline{G}(b(x)) = o(\overline{H}_{n+1}(x))$ is the same as in the proof of Theorem 3.1. For (n+1:2), recalling Lemma 4.6, by relations (4.3) and (4.13) for n , the derivations of (4.14)–(4.15) are still valid, showing that relation (4.3) holds for $n + 1$.

Finally, consider (n+1:3). Note that $H_n \in \mathcal{A}$ by (n:3) and that $\overline{G}(b(x)) = o(\overline{H}_n(x))$ by (n:1). Thus, applying Lemmas 4.6 and 4.7, we obtain $H_{n+1} \in \mathcal{A}$. In other words, $H_i \in \mathcal{A} \subset \mathcal{B}$ for all $i = 1, \dots, n + 1$. Then, similarly as in the proof of Theorem 3.1, we obtain the subexponentiality of T_{n+1} . Furthermore, by definition, $H_i \in \mathcal{A}$ implies that there is some constant $c_i > 1$ such that

$$\limsup_{x \rightarrow \infty} \frac{\overline{H}_i(c_i x)}{\overline{H}_i(x)} < 1.$$

With $c = \max\{c_1, c_2, \dots, c_{n+1}\} > 1$, by (4.15) we have

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{\Pr(T_{n+1} > cx)}{\Pr(T_{n+1} > x)} &\leq \limsup_{x \rightarrow \infty} \frac{\sum_{i=1}^{n+1} \overline{H}_i(c_i x)}{\sum_{i=1}^{n+1} \overline{H}_i(x)} \\ &\leq \max_{1 \leq i \leq n+1} \limsup_{x \rightarrow \infty} \frac{\overline{H}_i(c_i x)}{\overline{H}_i(x)} \\ &< 1. \end{aligned}$$

Therefore, the distribution function of T_{n+1} belongs to the class \mathcal{A} . This completes the proof of Theorem 3.2.

4.3. Proof of Corollary 3.1

Copied below is the well-known Breiman theorem; see Breiman (1965) and Cline and Samorodnitsky (1994).

Lemma 4.9. *Let X and Y be two independent random variables with distribution functions F on \mathbb{R} and G on \mathbb{R}^+ , respectively. Denote by H the distribution function of their product XY . If $F \in \text{MDA}(\Phi_\alpha)$ for some $\alpha > 0$ and $E[Y^{\alpha+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then*

$$\lim_{x \rightarrow \infty} \frac{\overline{H}(x)}{\overline{F}(x)} = E[Y^\alpha].$$

The following lemma forms the main ingredient of the proof of Corollary 3.1.

Lemma 4.10. *The following statements hold.*

(i) *Under the conditions of Corollary 3.1(i), it holds for each $i \in \mathbb{N}$ that*

$$\overline{H}_i(x) \sim ((1 - \theta) E[Y^\alpha] + \theta E[(Y_1^* \vee Y_2^*)^\alpha]) (E[Y^\alpha])^{i-1} \overline{F}(x),$$

where Y_1^* and Y_2^* are two independent random variables identically distributed as Y .

(ii) *Under the conditions of Corollary 3.1(ii), it holds for each $i \in \mathbb{N}$ that*

$$\overline{H}_i(x) \sim (1 + \theta) \overline{F}\left(\frac{x}{\hat{y}^i}\right) \left(\Gamma(\alpha + 1) \overline{G}\left(\hat{y} - \frac{\hat{y}^{i+1}}{x} a\left(\frac{x}{\hat{y}^i}\right)\right) \right)^i.$$

Proof. (i) Apply Lemma 4.9 to relation (4.4) to obtain

$$\lim_{x \rightarrow \infty} \frac{\overline{H}(x)}{\overline{F}(x)} = (1 - \theta) E[Y^\alpha] + \theta E[(Y_1^* \vee Y_2^*)^\alpha]; \tag{4.16}$$

see also Theorem 2.1(i) of Jiang and Tang (2011) for a result more general than (4.16). Thus, the distribution function H of $X_i Y_i$ belongs to the class $MDA(\Phi_\alpha)$. Then, for $i = 2, 3, \dots$, by Lemmas 4.9 and relation (4.16), in turn,

$$\begin{aligned} \overline{H}_i(x) &= \Pr\left(X_i Y_i \prod_{j=1}^{i-1} Y_j > x\right) \\ &\sim (E[Y^\alpha])^{i-1} \overline{H}(x) \\ &\sim ((1 - \theta) E[Y^\alpha] + \theta E[(Y_1^* \vee Y_2^*)^\alpha]) (E[Y^\alpha])^{i-1} \overline{F}(x). \end{aligned}$$

(ii) Note that every distribution function $F \in MDA(\Lambda)$ with an upper endpoint $\hat{x} = \infty$ is rapidly varying tailed. We apply Theorem 2.1(ii) of Jiang and Tang (2011), which, in our situation, implies that

$$\lim_{x \rightarrow \infty} \frac{\Pr(XY > x)}{\Pr(X^*Y^* > x)} = 1 + \theta, \tag{4.17}$$

where X^* and Y^* are two independent random variables with X^* identically distributed as X and Y^* as Y , and are independent of $\{(X_n, Y_n), n \in \mathbb{N}\}$. Then, by (4.17) and Theorem 3.1(i) of Hashorva *et al.* (2010), for $i = 1, 2, \dots$,

$$\begin{aligned} \overline{H}_i(x) &= E\left[\Pr\left(X_i Y_i \prod_{j=1}^{i-1} Y_j > x \mid Y_1, \dots, Y_{i-1}\right)\right] \\ &\sim (1 + \theta) \Pr\left(X^* Y^* \prod_{j=1}^{i-1} Y_j > x\right) \\ &= (1 + \theta) \Pr\left(X^* \prod_{j=1}^i \frac{Y_j}{\hat{y}} > \frac{x}{\hat{y}^i}\right) \\ &= (1 + \theta) \overline{F}\left(\frac{x}{\hat{y}^i}\right) \left(\Gamma(\alpha + 1) \overline{G}\left(\hat{y} - \frac{\hat{y}^{i+1}}{x} a\left(\frac{x}{\hat{y}^i}\right)\right)\right)^i. \end{aligned}$$

This completes the proof of Lemma 4.10.

Proof of Corollary 3.1. For statement (i), by (4.16), it is easy to verify that (a)–(c) hold with $b(x) = x^{1-\delta}$ for some $0 < \delta < \varepsilon/(\alpha + \varepsilon)$; see also Lemma 3.7 of Tang and Tsitsiashvili (2003). For statement (ii), (a)–(c) automatically hold because Y is bounded above. Hence, Theorem 3.2 is applicable for both statements (i) and (ii). Then we apply Lemma 4.10 to (3.1) to complete the proof of Corollary 3.1.

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