Open questions on Tantrix graphs

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Tantrix™ tiles are black hexagons imprinted with three coloured paths [1] joining pairs of edges. There are three different kinds of path. One is a straight line going from an edge to the opposite edge, one a circular arc joining adjacent edges and one an arc of larger radius joining alternate edges (or two apart). Tiles can be rotated but, since they are opaque, they cannot be turned over. A careful enumeration would indicate that, identifying tiles under rotation but not under reflection, there are 16 such tiles. However, the two tiles consisting of three straight lines (meeting at the centre of the hexagon) are not part of the set, so actually there are only 14 different tiles. The game is played by matching tiles to connect paths of the same colour; the goal is to create loops or long paths of a single colour This easy to learn yet hard to master game has inspired research on strategy (e.g. [2]) and complexity (e.g. [3]).

The authors of this paper chose to study combinatorial relationships between game pieces. Inspired by the permutohedron [4], we initially worked to match some idealised set of Tantrix tiles to the vertices of a polytope. While this 'Tantrix-tope' remains elusive, our search uncovered the large family of graphs described here. To the best of our knowledge, these graphs are largely unstudied. In this article we explore two members of this family; avenues for further research are offered as an exercise to the reader.

1. *Tantrix tiles*

Each Tantrix tile is a hexagon imprinted with three coloured paths joining pairs of edges [1]. We describe the tiles by listing the colours of the path ends as they appear clockwise around the tile; see Figure 1. Under this system the same tile may have several different descriptions, such as BRBRYY and YYBRBR. By choosing the alphabetically earliest member of the set of possible descriptions we assign a unique identifier to each tile.

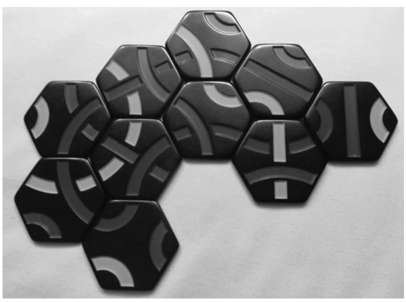

FIGURE 1: The cover image, reproduced here in black and white, shows a loop in red. The lowest tile in this figure is designated BBYYRR and the rightmost is BRRBYY

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This naming system arose from our work on the permutohedron (Figure 2) and is based on notation describing permutations of the numbers 1 to n. These permutations are identified with the vertices of the permutohedron and its edges correspond to transpositions of adjacent numbers [5, p. 17].

2. *Tantrix graph*

Imitating the construction of the permutohedron, we assign Tantrix tiles to vertices in a graph. Figure 3 depicts the graph arising from the commercially available Tantrix tiles printed with blue, red and yellow paths; in particular we do not include tiles BRYBRY and BYRBYR or tiles printed with green paths. The edges of this graph correspond to transpositions of adjacent path endings — BBYYRR is adjacent to BYBYRR, BBYRYR, and BRBYYR. Although swapping the two blue path ends sends BBYYRR to itself, we do not allow loops in our graph; if we did we would have no chance of finding a Tantrix-tope!

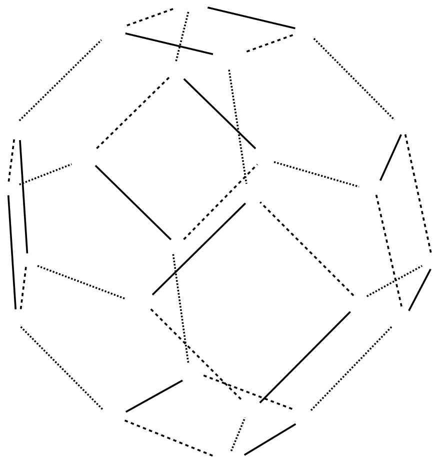

FIGURE 2: The 4-permutohedron

Perhaps the biggest difference between graphs described using this system and the edge graph of the permutohedron is that the arrangement of path ends on a Tantrix tile is cyclic — BBYYRR is adjacent to RBYYRB = BRBYYR in our Tantrix graph, while 3 4 1 2 is not adjacent to 2 4 1 3 in the edge graph of the permutohedron.

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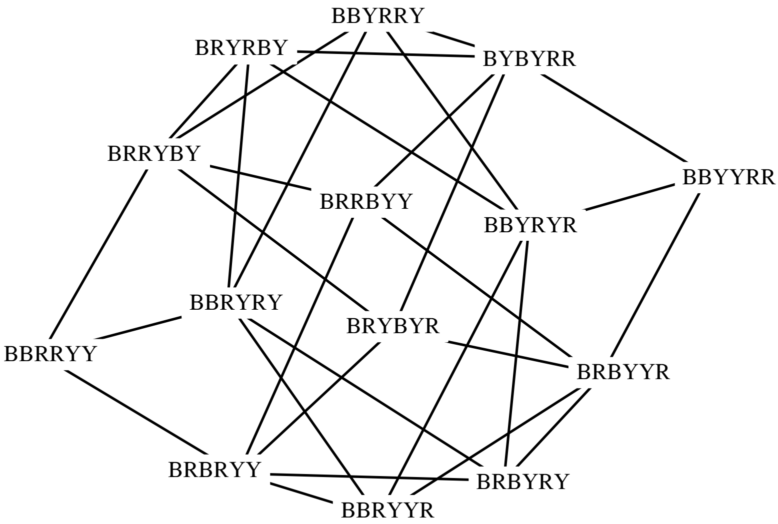

FIGURE 3: Tantrix graph

3. *Properties*

We quickly confirm that our 'Tantrix graph' is bipartite by sorting its vertices into two sets, one of which consists of tiles with only two same-coloured path endings adjacent: {BRBYYR, BRRYBY, BBYRYR, BRBRYY, BYBYRR, BBRYRY}. All edges of the graph go from tiles in this set to tiles not in this set.

As drawn in Figure 3, many of our graph's edges cross. Is there a way to draw the graph so that its vertices are all distinct and its edges never cross? In fact there is not — this graph is non-planar. A computer can quickly confirm this using an algorithm described by Demoucron et al., but you may find it more satisfying to consider different ways of drawing the graph determined by just the tiles BBRRYY, BRBRYY, BRRYBY, BBRYRY, BBRYYR, BRRBYY and BRYBYR (a planar graph with 7 vertices and 9 edges; one drawing of it appears in Figure 4) and convince yourself that there is no way to add the vertex corresponding to BRBYYR without overlapping vertices or crossing edges.* An elementary result from polytope theory [5, p. 134] tells us that the edge graph of any polyhedron can be drawn in the plane without edges crossing. Thus we know to look elsewhere for our Tantrix-tope.

* Students of graph theory may note that the vertices BRBYYR, BRBRYY, BRRYBY, BRRBYY, BRYBYR and BBRYYR support a subgraph homeomorphic to $K_{3,3}$.

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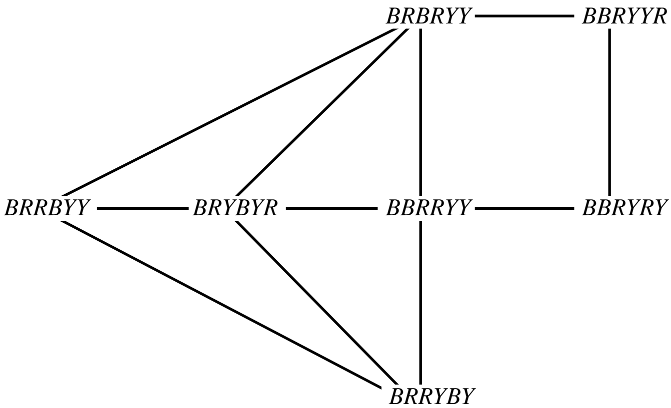

FIGURE 4: There is no way to add vertex BRBYYR without adding edge crossings

The computer program Maple™ can quickly apply the Floyd-Warshall all-pairs shortest path algorithm to determine that the diameter of the graph is 4. In other words, we can get from any vertex in the graph to any other vertex in the graph by following a chain of at most 4 edges. The distance from BBRRYY to BBYYRR is exactly 4.

We also find that the girth of the graph, or the length of its shortest cycle of edges, is 4. We get these cycles by swapping non-overlapping pairs of edges of a tile — for example, BBRRYY - BRBRYY - BRBYRY - BBRYRY - BBRRYY. We know that this must be the shortest cycle because a bipartite graph cannot have a cycle of length 3, and by definition the graph contains no cycles of length 2.

The graph contains no Eulerian cycle visiting every edge exactly once because 8 of its vertices have odd degree. Because it is bipartite with 6 and 8 vertices in each of its parts, it contains no Hamiltonian cycle visiting each vertex exactly once, though its cousin the permutohedron does contain such a cycle [4].

If we remove any two vertices from the graph it will remain connected, but removing vertices BRBRYY, BRRYBY and BBRYRY would leave vertex BBRRYY disconnected from the rest of the graph. Thus, this graph is 3-connected.

For this Tantrix graph we have answered just a few of many possible questions. The remainder of this paper describes a large family of graphs for which these and other questions remain unanswered; we hope that our readers will be inspired to answer some of them.

4. *Generalisations of the Tantrix graph*

As we study the Tantrix graph in Figure 3, we might wonder why its vertices have different degrees. There are 6 pairs of adjacent sides on every hexagonal tile — shouldn't 6 edges meet at each vertex?

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Recall that we could have added a loop to our graph corresponding to swapping the two ends of the blue path on BBRRYY. If we did, should we include just one loop or also include loops representing the transposition of red and yellow path endings? If we add all three loops, does that bring the degree of the vertex to 9? Or do we consider each edge as directed from one tile to another, so that the out-degree of each vertex is 6 and the in-degree varies between vertices?

We might also include vertices corresponding to the non-existent Tantrix tiles BRYBRY and BYRBYR. If included, these vertices would both be adjacent to each of BRYBYR, BRBYRY and BRYRBY in our graph.

The Tantrix Game Pack contains 56 hexagonal tiles decorated with 4 different colours of path. Swapping path endings will never transform a red path into a green one. To accommodate the fourth colour, should we consider a Tantrix graph with four separate components, or should we introduce a new edge type — for example, one representing a colour swap and joining BBRRYY to BBGGYY?

Tantrix tiles are hexagonal, but we could draw paths joining the edges of any $2n$-gon. We define tantrix tiles (with a lower case t) to be regular $2n$-sided polygons decorated by paths numbered from 1 to n connecting pairs of sides of the polygon — each side associated with exactly one path. Since rotating a Tantrix tile doesn't change the tile, we may consider 112233 to be equivalent to 122331. We do not normally equate 112233 and 113322, but we could decide to treat mirror symmetric tiles as identical.*

As before, edges correspond to swapping adjacent path endpoints. In our notation, this is equivalent to either swapping adjacent digits (112233 $\to$ 121233) or swapping the first and last digit (112233 $\to$ 312231). In order for all possible swaps to correspond to edges we could include asterisk-patterned tiles such as 123123 and 1212.

We may choose to allow loops corresponding to swapping same numbered path ends, and to use directed and/or weighted edges in our graph; Figure 5 shows that there are 4 swaps which take 1212 to 1122 and only 2 swaps which take 1122 to 1212.

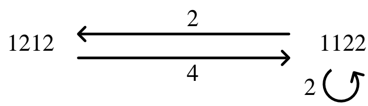

FIGURE 5: The tantrix graph obtained by using square tiles, two ‘colours’, directed edges and allowing loops; edge weights indicate the number of different swaps taking one tile to another

We might label our edges with more or fewer than n ‘colours’, allowing our tantrix graphs to be disconnected and/or defining a colour swapping

* Standard Tantrix tiles are opaque; BBYYRR and BBRRYY are two different tiles.

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edge. Such graphs would have symmetries dependent on permutations of sets of colours and could contain many more or fewer vertices than their n coloured cousins.

We might also choose to differentiate between tiles in which path 1 crosses above path 2 and those on which path 1 crosses below path 2. This would require us to consider what happens in the middle of the tile in addition to path positions at the edges, which might prove quite challenging.

The symmetric group of permutations of $2n$ elements lies at the heart of all of these graphs. Defining a tantrix graph amounts to deciding which permutations will be considered, which are equivalent (a Tantrix tile is still considered the same tile after rotation), and how these permuatations will be associated in a graph. For small tantrix graphs this work can be done by hand during spare moments, but larger graphs or graphs with more complex definitions might best be studied using a computer algegra package. Once the details of a graph — large or small — are encoded, a plethora of algorithms provide nearly instant results to standard queries.

Ideally, a dedicated researcher could formally prove that an entire family of tantrix graphs is bipartite, 3-connected, or has some other property. Alternately, one could generate conjectures by programming a computer to generate the vertices and incidences of graphs in a certain family and applying the algorithms mentioned above.

5. *Research suggestions*

We have described a few tantrix graphs, but there are many more to explore. If you are interested in studying your own tantrix graph, decide which of the options outlined in the previous section you wish to adopt. Then try answering the questions below or exploring questions of your own.

(1) How many vertices does your graph have? (See [6].)

(2) Is your graph connected? How connected is it?

(3) Is your graph Eulerian?

(4) Is your graph Hamiltonian?

(5) Is your graph bipartite?

(6) Is your graph planar? If not, find a maximal planar subgraph.

(7) Can you colour the vertices of your graph so that no pair of adjacent vertices has the same colour? If so, what is the minimum number of colours needed?

(8) Can you colour the edges of your graph so that each vertex has at most one edge of each colour meeting it? If so, what is the minimum number of colours needed?

Question (1) should be fairly simple to answer once one has deciphered the formula published by Perfect in [6]. Questions 2 to 8 can be quickly answered quickly by a computer algebra package (though it is more informative to solve such problems by hand); they are straightforward for small graphs but may be quite challenging for larger graphs.

An undergraduate research project in topology might consider the surfaces formed by identifying the sides of a tile containing the two ends of a path, with or without adding a twist. Associating each surface formed with a vertex of a tantrix graph might — or might not — shed light on relationships between different two-manifolds. One could extend this project by considering the links described by coloured paths under these identifications. What links and knots can be realised on tantrix tiled surfaces? We do not know; indeed, we do not know how challenging this question might be to answer.

A game theory or game design class could play tantrix or work tantrix puzzles with square tiles, on a torus, or on a Klein bottle. How does the number of edges or colours affect the challenge level and enjoyability of the game or puzzle?

A class on algorithms might explore how standard algorithms from graph theory for finding shortest paths, maximum flows, etc. behave when applied to different tantrix graphs. Do these algorithms run quickly (best case), slowly (worst case), or does their speed depend on initial conditions?

Anyone able to read this paper can quickly define and start to study a small tantrix graph. The properties of most such graphs are unknown at present; tantrix graphs provide an extrordinarily accessible and fertile topic for undergraduate research.

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