SUMS OF COMPLEXES IN TORSION FREE ABELIAN GROUPS

H. Heilbronn and P. Scherk

(received February 15, 1969)

Let A, B denote two non-void finite complexes (= subsets) of the torsion free abelian group G,

$$A + B = \{a + b \mid a \in A, \ b \in B\} .$$

Let $d(A), \ldots$ denote the maximum number of linearly independent elements of $A, \ldots$ and let $n = n(A, B)$ denote the number of elements of $A + B$ whose representation in the form $a + b$ is unique. In the preceding paper, Tarwater and Entringer [1] proved that $n \geq d(A)$. We wish to show by an entirely different and perhaps simpler method that $n \geq d = d(A \cup B)$.

Given A and B, we may replace G by the subgroup generated by $A \cup B$. Then G may be interpreted as a d-dimensional vector lattice over the ring of the integers. Imbed G into a d-dimensional real vector space and construct its affine d-space R.

Let $\mathfrak{C}$ denote the set of those points of R whose radius vectors belong to $A + B$. Since $d(A + B) = d(A \cup B) = d$, we have $\dim \mathfrak{C} = d - 1$ or d. The convex closure $\mathfrak{H}(\mathfrak{C})$ of $\mathfrak{C}$ is a convex polytope in R.

If ξ is an extremal point of $\mathfrak{H}(\mathfrak{C})$, ξ is not the barycenter of other points of $\mathfrak{H}(\mathfrak{C})$, in particular of $\mathfrak{C}$. Hence $\xi \in \mathfrak{C}$.

Suppose the radius vector x of the point ξ has two distinct representations $x = a + b = a' + b'$ where $\{a, a'\} \subset A$, $\{b, b'\} \subset B$. The points with the radius vector $a + b'$ and $a' + b$ would lie in $\mathfrak{C}$ and ξ would be the centre of the connecting segment. In particular, ξ could not be an extremal point. Thus every extremal point of $\mathfrak{H}(\mathfrak{C})$ has a radius vector in $A + B$ with a unique representation $a + b$.

Since $\mathfrak{H}(\mathfrak{C})$ is a convex polytope of dimension $\geq d - 1$ and since every convex polytope is the convex closure of the set of its extremal points, $\mathfrak{H}(\mathfrak{C})$ has not less than d extremal points. This proves $n \geq d$.

https://doi.org/10.4153/CMB-1969-061-1 Published online by Cambridge University Press

Remark. If B contains at least two elements, $\mathcal{H}(C)$ has not less than $2(d(A) - 1)$ extremal points. We then have $n \geq 2(d(A) - 1)$.

REFERENCE

1. J.D. Tarwater and R.C. Entringer, Sums of complexes in torsion free abelian groups. Canad. Math. Bull. 12 (1969) 475-478.

University of Toronto

https://doi.org/10.4153/CMB-1969-061-1 Published online by Cambridge University Press