NORMAL CLOSURES OF POWERS OF DEHN TWISTS IN MAPPING CLASS GROUPS

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1. Let $F = F(g, n)$ be an oriented surface of genus $g \geq 1$ with $n < 2$ boundary components and let $M(F)$ be its mapping class group. Then M(F) is generated by Dehn twists about a finite number of non-bounding simple closed curves in F ([**6, 5**]). See [**1**] for the definition of a Dehn twist. Let e be a non-bounding simple closed curve in F and let E denote the isotopy class of the Dehn twist about e. Let N be the normal closure of E^2 in $M(F)$. In this paper we answer a question of Birman [**1**, Qu 28 page 219]:

THEOREM 1. *The subgroup N is of finite index in $M(F)$.*

In fact we prove somewhat more:

THEOREM 2. *If F is closed and has genus two or three, then the normal closure of E^3 is of finite index in $M(F)$.*

THEOREM 3. *If F has genus two and has a single boundary component, then the normal closure of E^2 or E^3 is of finite index in $M(F)$.*

On the other hand we prove:

THEOREM 4. *If F has genus two and has $n \geq 0$ boundary components, then the normal closure of E^k is of infinite idex in $M(F)$ for all $k > 3$.*

The case $g = 1$, $n = 0$ gives the group $M = SL(2, \mathbb{Z})$ [**1**] and a Dehn twist is represented by a matrix conjugate to the parabolic matrix $E = \begin{pmatrix} 1 & 1 \\ O & 1 \end{pmatrix}$. Let N^k be the normal closure of E^k. Then N^k is of index 6, 24, 48, 120 for $k = 2$, 3, 4, 5 (respectively) and is of infinite index if $n > 5$ [**9**]. The case $g = 1$, $n = 1$ gives the group $M = B_3$, the braid group on 3 strings [**1**] and a Dehn twist is represented by one of the standard braid generators σ. Let N^k be the normal closure of σ^k. Then N^k is of index 6, 24, 96, 600 for $k = 2$, 3, 4, 5 (respectively) and is of infinite index if $n > 5$.

2. Proof of Theorem 1. Let $F = F(g, 0)$, $g > 1$. Let $Sp(2g, R)$ be the symplectic group of rank $2g$ matrices with coefficients in the ring $R = \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$. If we think of the underlying symplectic space on which this symplectic group acts as being the homology group $H_1(F, R)$ with its natural symplectic form coming from the algebraic intersection number, then we have a natural map $M(F) \to Sp(2g, R)$ which is actually onto [**7** p. 178]. By a *k-chain* of simple closed curves in F we will mean a sequence $c_0, c_1, c_2, \ldots, c_{k-1}$ of homologically independent simple closed curves in F such that c_i and c_j intersect if and only if $|i - j| = 1$ and then only once geometrically. Theorem 1 will follow from:

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PROPOSITION 2.1. *Let $F = F(g, 0)$ and let N be the normal closure of E^2 in $M(F)$. Then N is the kernel of the natural map*

$$\varphi : M(F) \to Sp(2g, \mathbb{Z}/2\mathbb{Z}).$$

Proof. An easy calculation shows that N is contained in the kernel of the map φ. Let $\varphi' : M(F) \to Sp(2g, \mathbb{Z})$ be the map giving the action of $M(F)$ on $H_1(F, \mathbb{Z})$. Then φ' is surjective ([**7**] p. 178) and φ is the composite of epimorphisms

$$M(F) \to Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z}).$$

We will prove (a) that N contains the kernel of the first map and (b) that the image of N in $Sp(2g, \mathbb{Z})$ is exactly the kernel of the second map. For (a) we note that by [**10**] the kernel I_g of the map φ' is generated by (i) Dehn twists about bounding curves, and (ii) bounding pairs. Here a bounding pair is a product GH^{-1}, where G and H are Dehn twists about disjoint non-bounding simple closed curves in F which together bound in F. This kernel is called the Torelli group. To prove (a) it will suffice to show that N contains all generators of types (i) and (ii). For generators of type (i) we will prove the following more general result:

LEMMA 2.2. *Let F be a surface of genus $g > 0$ with at most one boundary component. Then the normal closure N of E^2 contains the Dehn twists about all bounding curves.*

Proof. First note ([**6**] or [**1**]) that if C is any Dehn twist about a non-bounding curve in F, then there is an element α of $M(F)$ such that $\alpha E \alpha^{-1} = C$. It follows that C^2 belongs to N. Now note that if d is a bounding curve in F with Dehn twist D, then there is another bounding curve d' with Dehn twist D' such that d' bounds a surface of strictly smaller genus (possibly zero) than does d and such that d and d' together bound a genus 1 surface containing a 3-chain of simple closed curves $x, y\ z$ with Dehn twists X, Y, Z. Now x and z are disjoint curves and so X and Z commute. This fact and [**6**, Lemma 3] implies that

$$DD' = XZYXZY^2XZYXZ \qquad (*)$$

lies in N. It easily follows by induction that each such D belongs to N. This proves the Lemma and shows that the subgroup generated by Dehn twists about bounding curves lies in N.

For generators of type (ii) we first note that for any bounding pair BD^{-1} there is a $(2g+1)$-chain $c_0, c_1, c_2, \ldots, c_{2g}$ such that b and d only intersect some c_k for fixed odd k and then only once. Let

$$w = C_0C_1C_2C_3 \ldots C_{g-2}C_{g-1}C_g^2C_{g-1}C_{g-2} \ldots C_1C_0.$$

Then w is an involution of F [**1**] and satisfies $w(b) = d$. One first notes that w belongs to N and so $BwB^{-1}w^{-1} = BD^{-1}$ also belongs to N. This proves case (ii).

We next show that the image of N in $Sp(2g, \mathbb{Z})$ is equal to the kernel of the natural map $Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z})$. We note that the image of D in $Sp(2g, \mathbb{Z})$ is a primitive symplectic transvection T and that the normal closure of T^2 is a finite index in $Sp(2g, \mathbb{Z})$ since by [**8**] it is equal to the kernel of the map $Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z})$. Theorem 1 now follows.

3. Proof of Theorem 2. Now suppose that $F = F(2, 0)$ and let N be the normal closure in $M(F)$ of E^3. We want to show that N contains I_2. Again [**10**] shows that I_2 is generated by Dehn twists about bounding curves only, since there are no bounding pairs in this case.

LEMMA 3.1. *Let F be a surface of genus g with or without boundary and let N be the normal closure in $M(F)$ of E^3. If D is the Dehn twist about a bounding curve d in F which bounds a genus 1 subsurface, then D lies in N.*

Proof. By the hypothesis we see that there is a 2-chain a, b in F such that d is isotopic to the boundary of a tubular neighbourhood of $a \cup b$. Then one calculates that $D = (ABA)^4$. Now $ABA = BAB$ and so

$$\begin{aligned} D &= ABAABAABAABA = ABAABABABABA \\ &= ABAAABAABABA = (AB)AAA(AB)^{-1}ABBAABABA \\ &= (AB)AAA(AB)^{-1}ABBABABBA = (AB)AAA(AB)^{-1}ABBBABBBA \end{aligned}$$

which clearly belongs to N.

Returning to the case where $g = 2$ and F is closed this lemma shows that N contains all Dehn twists about bounding curves and so contains I_2. Again [**8**] shows that the image of N in $Sp(4, \mathbb{Z}/3\mathbb{Z})$ is equal to the kernel of the natural map $Sp(4, \mathbb{Z}) \to Sp(4, \mathbb{Z}/3\mathbb{Z})$.

Now suppose that $F = F(3, 0)$ and that N is the normal closure of E^3. To show that N has finite index in $M(F)$ it will suffice to show (i) that $I_3 \cap N$ has finite index in I_3 and (ii) that the image of N in $M(F)/I_3 = Sp(6, \mathbb{Z})$ has finite index. In fact this latter fact again follows from [**8**]. For (i) we note that by [**3**] there is a map $\tau: I_3 \to A$ where A is a free abelian group of rank 14 and by [**4**] the kernel K of τ is the subgroup generated by twists on bounding curves. Since F has genus three and is closed we see that any bounding curve bounds a surface of genus 1 and so Lemma 3.1 shows that any Dehn twist about a bounding curve lies in N. Thus N contains K and we now need only show (i)′ $\tau(I_3 \cap N)$ has finite index in A. Since I_3/K is generated by the images of bounding pairs (i)′ will follow from the fact that if BD^{-1} is a bounding pair, then $(BD^{-1})^3 = B^3D^{-3}$ belongs to N. This shows that in fact

$$I_3/(I_3 \cap N) = \tau(I_3)/\tau(I_3 \cap N) = (\mathbb{Z}/3\mathbb{Z})^{14}$$

and so $M(F)/N$ is an extension of $Sp(6, \mathbb{Z}/3\mathbb{Z})$ by $(\mathbb{Z}/3\mathbb{Z})^{14}$. This concludes the proof of Theorem 2.

4. Proof of Theorem 3. Let $F = F(2, 1)$ be a genus 2 surface with a single boundary component and let N be the normal closure of E^2. Let T be the subgroup of the Torelli subgroup $I_{2,1}$ generated by the Dehn twists about bounding curves. Let T_i, $i = 1, 2$, be the subgroup of T generated by the Dehn twists about bounding curves of genus i. Here the genus of a bounding curve is the genus of the surface that it bounds. Clearly T is generated by T_1 and T_2, since F has genus 2. Note that there is only one (isotopy class of) bounding closed curve of genus 2, namely the curve parallel to the boundary component. It follows that T_2 is in the centre of T. Now by Lemma 2.2 we see that N contains all of T. Again [**4**] shows that T is the kernel of the map $\tau: I_{2,1} \to A$ where here A is a free abelian group of rank 4. An argument similar to that in §3 shows that

$$I_{1,2}/(I_{2,1}\cap N)=\tau(I_{2,1})/\tau(I_{2,1}\cap N)=(\mathbb{Z}/2\mathbb{Z})^4.$$

This now shows that $M(F)/N$ is an extension of $Sp(4,\mathbb{Z}/2\mathbb{Z})$ by $(\mathbb{Z}/2\mathbb{Z})^4$ and so is finite.

Let F be a genus 2 surface with a single boundary component and let N be the normal closure of E^3. By Lemma 3.1 we see that N contains T_1 and so to conclude the argument we must show that some power of the generator of T_2 belongs to N. If D' is this generator, then by (*) we have

$$D'=D^{-1}XZYXZY^2XZYXZ,$$

where D is the Dehn twist about a genus 1 bounding curve not meeting x, y or z. Thus

$$D'^k=D^{-k}(XZYXZY^2XZYXZ)^k,$$

for all k. Now X, Y and Z satisfy the "braid relations": $XZ=ZX$, $XYX=YXY$, $YZY=ZYZ$ and if we now add in the relations $X^3=Y^3=Z^3=$ identity coming from N, then (by the Todd–Coxeter algorithm) we obtain a group of order $646=2^33^4$ in which the element $ZYXZY^2XZYXZ$ has order 3. By Lemma 3.1 we see that D^3 is in N and so D'^3 belongs to N as required. Thus $T/(T\cap N)$ is finite; in fact it is $\mathbb{Z}/3\mathbb{Z}$. It easily follows that $\tau(I_{2,1})/\tau(I_{2,1}\cap N)$ is a finite abelian 3-group and so $M(F)/N$ is a finite extension of $Sp(4,\mathbb{Z}/3Z)$. This proves Theorem 3.

5. Proof of Theorem 4. The theorem will follow for an arbitrary number n of boundary components if we can prove it for the closed case ($n=0$) since for any g and n there is an epimorphism $M(F(g,n))\to M(F(g,0))$ ([**1,** §4.1)]. The idea for our proof is to use a certain matrix representation of $M=M(F(2,0))$ constructed by Jones [**2**]. By [**1**] M is generated by Dehn Twists $T_1,\ldots,T_5$ and the Jones representation J of M satisfies

$$J(T_1)=\begin{pmatrix}-1&0&0&0&q\\0&-1&1&0&0\\0&0&q&0&0\\0&0&1&-1&0\\0&0&0&0&q\end{pmatrix},$$

$$J(T_2)=\begin{pmatrix}q&0&0&0&0\\0&q&0&0&0\\0&q&-1&0&0\\1&0&0&-1&0\\1&0&0&0&-1\end{pmatrix},$$

$$J(T_3)=\begin{pmatrix}-1&0&0&q&0\\0&-1&1&0&0\\0&0&q&0&0\\0&0&0&q&0\\0&0&1&0&-1\end{pmatrix}$$

where q is an indeterminate. Now note that if $(-1)^kq$ is a kth root of 1, then each of

$(-1)^k J(T_1), \ldots, (-1)^k J(T_5)$ has order k since an induction shows that for each $i \leq 5$, T_i^k has the form

$$(-1)^k Id + (q^{k-1} - q^{k-2} + \ldots + (-1)^{k-2} q + (-1)^{k-1}) W + (q^k - 1) U$$

for some matrices W, U. Thus we obtain a representation J' of M/N, where N is the normal closure of E^k by letting $(-1)^k q$ be a kth root of 1 and putting $J'(T_i) = (-1)^k J(T_i)$. Let b be the bounding curve which is symmetric relative to $T_1, \ldots, T_5$. Then the Dehn twist about b is $B = (T_1 T_2 T_1)^4$. Let $R = J'(BT_3 BT_3^{-1})$. Then interchanging the 2nd and 4th rows and columns of R gives a matrix R' having the form $\begin{pmatrix} X & Y \\ O & tI \end{pmatrix}$ where $X = X(q)$ is a 2×2 matrix, $t = q^{12}$ and I is the 3×3 identity matrix. The characteristic polynomial of X/q^6 is

$$x^2 - x(q^8 - 2q^7 + q^6 + 2q^5 - 2q^4 + 2q^3 + q^2 - 2q + 1)/q^4 + 1.$$

One checks that for $k = 4$, (with $q = i$) X is a non-trivial parabolic; and that if $k = 6$ (with q = primitive cube root of 1) then X has distinct eigenvalues which are not roots of unity (they have absolute values equal to 0·10102... and 1/0·10102...). Thus in both cases we see that R has infinite order. It follows that if k is even and $3 \mid k$, then R has infinite order. If $k = 2(3n \pm 1) > 6$, then letting $q = \exp(4\pi in/k)$, we see that R has infinite order by noticing that the absolute values of the eigenvalues of X rapidly converge to 0·10102... and 1/0·10102... as $n \to \infty$. One similarly deals with the odd cases using $q = \exp(3\pi in/k)$ if $k = 4n \pm 1$ and n is odd and $q = \exp((3n-1)\pi i/k)$ otherwise.

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