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RESEARCH ARTICLE

On blow-up for the supercritical defocusing nonlinear wave equation

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Abstract
In this paper, we consider the defocusing nonlinear wave equation $-\partial_t^2 u + \Delta u = |u|^{p-1}u$ in $\mathbb{R}\times\mathbb{R}^d$. Building on our companion work (*Self-similar imploding solutions of the relativistic Euler equations*, arXiv:2403.11471), we prove that for $d = 4$, $p \geq 29$ and $d \geq 5$, $p \geq 17$, there exists a smooth complex-valued solution that blows up in finite time.

Contents

1. Introduction

In this paper, we consider the defocusing nonlinear wave equation

$$\Box u = |u|^{p-1}u, \tag{1.1}$$

where $u : \mathbb{R}^{1+d} \to \mathbb{C}$ is the unknown field, $\Box = \partial^\alpha\partial_\alpha = -\partial_t^2 + \sum_{i=1}^d \partial_i^2$ is the d'Alembertian operator[1] on Minkowski spacetime $\mathbb{R}^{1+d}$ with the standard Minkowski metric

$$m_{00} = -1, \quad m_{ii} = 1 \text{ for all } i \in \mathbb{Z} \cap [1, d], \quad m_{\mu\nu} = 0 \text{ if } \mu, \nu \in \mathbb{Z} \cap [0, d] \text{ with } \mu \neq \nu,$$

and we assume $p \in 2\mathbb{Z}_+ + 1$ for simplicity.

Given smooth initial data $(u|_{t=0}, \partial_t u|_{t=0})$, there exists a local smooth solution on the maximal existence of interval $[0, T)$; $T < +\infty$ if and only if $\limsup_{t\uparrow T} \|u(t)\|_{L^\infty} = +\infty$, see [68, 45]; moreover, there holds the energy conservation

$$E[u(t)] := \int_{\mathbb{R}^d} \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{p+1}|u|^{p+1}\,\mathrm{d}x. \tag{1.2}$$

The class of solutions to (1.1) is invariant under the scaling

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x), \quad \lambda > 0. \tag{1.3}$$

This scaling symmetry preserves the critical norm invariant, i.e.,

$$\|u_\lambda(t,\cdot)\|_{\dot{H}_x^{s_c}} = \|u(\lambda t,\cdot)\|_{\dot{H}_x^{s_c}} \quad \text{where} \quad s_c := \frac{d}{2} - \frac{2}{p-1}.$$

We can split the range of parameters (d, p) into three cases accordingly:

- *Subcritical* case: $s_c < 1 \iff d \le 2$ or $p < 1 + 4/(d-2)$ for $d \ge 3$.
- *Critical* case: $s_c = 1 \iff p = 1 + 4/(d-2)$ and $d \ge 3$.
- *Supercritical* case: $s_c > 1 \iff p > 1 + 4/(d-2)$ and $d \ge 3$.

For the subcritical case, the global well-posedness and propagation of regularity dated back to Jörgens [31] for $d = 3$; see also [19, 20] for the global well-posedness within the energy class $H^1 \times L^2$ for all dimensions; the propagation of regularity holds at least for $d \le 9$ [3]. The critical case is much more difficult. The global regularity result was obtained firstly in [71] for $d = 3$ and spherically symmetric initial data, and then extended to $d \le 9$ for general smooth data in [21, 22, 66], and all dimensions in [67] (in the energy class $H^1 \times L^2$). For the long-time behavior of these global solutions, we refer to [75] and references therein.

For the supercritical case, it is known that the Cauchy problem is ill-posed in some low regularity spaces [6], or even in the energy class [26], despite the global existence of weak solutions [70], as well as the global well-posedness with scattering for small smooth data [44]. The global well-posedness for

[1] Here we use the Einstein's summation convention.

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general smooth data is a long-standing open problem [2, 72]. In the breakthrough work [51], Merle, Raphaël, Rodnianski and Szeftel construct radial and asymptotically self-similar blow-up solutions for the energy supercritical defocusing nonlinear Schrödinger equations (NLS). The goal of this paper is to extend a similar blow-up result for NLS to the defocusing supercritical wave equation.

Before stating our theorem, we recall Tao's blow-up result [73] for the defocusing nonlinear wave system of the form $\Box u = (\nabla_{\mathbb{R}^m} F)(u)$, where $u : \mathbb{R}^{1+d} \to \mathbb{R}^m$ is vector-valued, and $F : \mathbb{R}^m \to \mathbb{R}$ is a smooth potential which is positive and homogeneous of order $p + 1$ outside of the unit ball for some $p > 1$ (letting $m = 2$ and $F(u) = |u|^{p+1}/(p + 1)$ we recover (1.1)). Tao [73] proved that for any supercritical (d, p), and sufficiently large positive integer m, there exists a defocusing $F : \mathbb{R}^m \to \mathbb{R}$ such that the system $\Box u = (\nabla_{\mathbb{R}^m} F)(u)$ has no global smooth solution for some smooth compactly supported initial data. A similar result for the defocusing Schrödinger system was obtained in [74].

1.1. Main results

Roughly speaking, we prove that the defocusing supercritical nonlinear complex-valued wave equation for $d \geq 4$ admits finite time blow-up solutions arising from smooth initial data. The leading order term of blow-up solution is given by a self-similar blow-up solution of the relativistic compressible Euler equation, which is stated here as Assumption 1 (in Section 2). In our companion paper [65], we have verified Assumption 1 for some (d, p).

Theorem 1.1. *Let $d \in \mathbb{Z} \cap [4, +\infty)$ and $p \in 2\mathbb{Z}_+ + 1$ be such that*[2] *$k > \ell + \sqrt{\ell}$, where $k := d - 1$ and $\ell := 1 + 4/(p - 1)$. Assume that there exists $\beta \in (1, k/(\ell + \sqrt{\ell}))$ such that Assumption 1 holds. Then there exist compactly supported smooth functions $u_0, u_1 : \mathbb{R}^d \to \mathbb{R}^2 (= \mathbb{C})$ such that there is no global smooth solution $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^2 (= \mathbb{C})$ to the defocusing nonlinear wave equation (1.1) with initial data $u(0) = u_0$, $\partial_t u(0) = u_1$.*

Corollary 1.2. *If $d = 4$, $p \in (2\mathbb{Z} + 1) \cap [29, +\infty)$ or $d \geq 5$, $p \in (2\mathbb{Z} + 1) \cap [17, +\infty)$, then there exist compactly supported smooth functions $u_0, u_1 : \mathbb{R}^d \to \mathbb{R}^2 (= \mathbb{C})$ such that there is no global smooth solution $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^2 (= \mathbb{C})$ to the defocusing nonlinear wave equation (1.1) with initial data $u(0) = u_0$, $\partial_t u(0) = u_1$.*

Several remarks are in order.

1. For the blow-up solution u we construct in Theorem 1.1, if u blows up at time $T_* \in (0, +\infty)$, then according to our construction, we have the blow-up speed
$$\|u(t, \cdot)\|_{L^\infty} \gtrsim (T_* - t)^{-\frac{2\beta}{p-1}}, \quad \|(u(t), \partial_t u(t))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1}} \gtrsim (T_* - t)^{(1-\beta)\frac{d}{2}}.$$
As $\beta > 1$, our solution is unbounded in the critical space. This is compatible with the results in the literature, which state that the solutions for the supercritical defocusing wave equation that are bounded in the critical space $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1}$ must be global and scattering (at least for real-valued solutions and some supercritical (d, p), see [4, 5, 18, 35, 36]).
2. As in the recent breakthrough work by Merle-Raphaël-Rodnianski-Szeftel [51, 52, 53], the heart of proof of Theorem 1.1 is to study (1.1) in its hydrodynamical formulation, i.e., with respect to its phase and modulus variables, i.e. (2.1). After introducing a front re-normalization (2.2), (2.1) becomes (2.3). Taking the formal limit $b \to 0$, we reveal the underlying relativistic compressible Euler dynamics (2.6). The relativistic Euler dynamics provides us with a self-similar blow-up solution, which has been constructed in our companion paper [65] and which, in turn, acts as the leading order term of the blow-up solution of the defocusing supercritical wave equation (1.1).

[2] In particular, we have $k > \ell$, which is equivalent to $p > 1 + 4/(d - 2)$. So we are in the *supercritical* case. Nevertheless, we can not cover the whole supercritical range using the method of current paper.

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3. In the proof, we first construct a good approximate solution, then solve the perturbation equation backward in time. This method is quite different from [51], and simplifies our proof greatly. Unlike [51], we do not need to analyze the stability of the linearized operator near the leading order profile constructed in [65], hence we only use less information of the leading order profile (see Assumption 1). On the other hand, in [51], the initial data for blow-up form a finite co-dimensional manifold in the class of radial smooth fast-decay functions, we only construct the blow-up solution for one initial data (u_0, u_1) in Theorem 1.1. We believe that the blow-up should hold for a large class of initial data, just as in [51]. This is left to the future work.
4. To prove Corollary 1.2, we just need to verify Assumption 1, which is related to the existence of a smooth global solution to a specific ODE (2.8). If $d = 4, p \geq 29$ or $d = 5, p \geq 17$, Assumption 1 is verified in our companion paper [65]. As a consequence, if one can find some other methods to verify Assumption 1 for smaller p, then one can also get the blow-up for that smaller p. The case $d > 5$ follows from the result for $d = 5$ and truncation, see Subsection 2.5.
5. We emphasize that if Assumption 1 is valid, then we must have $d > \beta(\ell + \sqrt{\ell}) + 1$, where $\ell := 1 + 4/(p-1) > 1$. Using $\beta > 1$, we get $d > 3$. As a result, the case of $d = 3$ is not amenable to our analysis at present, and the existence of blow-up solutions for $d = 3$ remains open. We point out that similar situation happens in [51], where the construction fails for 3-D and 4-D defocusing supercritical NLS.
6. In this work, we can only construct the blow-up for the complex-valued solution. The blow-up for the scalar defocusing supercritical wave equation remains open at this point. We guess that the same blow-up result should hold for the scalar nonlinear wave equation, at least for (d, p) satisfying the same hypothesis as in Theorem 1.1.
7. In this paper, we initiate our exploration of complex-valued blow-up solutions by employing the modulus-phase decomposition $u = we^{\mathrm{i}\Phi}$, as detailed in Section 2. For the $\mathbb{R}$-valued problem, an analogous approach appears promising. Specifically, we propose a decomposition of the form $u = wf(\Phi)$, where $f : \mathbb{R} \to \mathbb{R}^+$ is an unknown real-valued function. Under this framework, the problem reduces to solving the following system of equations:

$$2\partial^\alpha w \partial_\alpha \Phi + w\Box\Phi = 0, \quad f''(\Phi) = -f(\Phi)^p, \quad \Box w = f(\Phi)^{p-1}\left(w^p + w\partial^\alpha\Phi\partial_\alpha\Phi\right).$$

   When compared with (2.1), the above system exhibits a significantly higher level of complexity. Consequently, its thorough investigation is deferred to future work.

The road map of the proof of Theorem 1.1 and Corollary 1.2 can be found in Section 2. The proof is based on Propositions 2.4, 2.7 and 2.8. Our starting point is to introduce a front re-normalization (2.2), relying on a constant $b > 0$; taking the limit $b \to 0$, the defocusing wave equation becomes the relativistic compressible Euler equations.

We first write the desired solution to (1.1) in the form of a power series (see (2.4)) with respect to the constant $b > 0$. The non-degeneracy of the leading order approximation allows us to solve all high-order approximations (ρ_n, ϕ_n), which is exactly the purpose of Proposition 2.4. The proof of Proposition 2.4 is rather technical and can be found in Section 5. One of the key ingredients used is the existence of smooth solutions to the second order ODEs having singular points with a parameter λ, see Appendix B.

Since we do not have enough information on (ρ_n, ϕ_n), especially the estimate uniform in n, we may not have the convergence of the formal series (2.4). To overcome this drawback, we truncate (ρ_n, ϕ_n) in the form of (2.23), and in Proposition 2.7 we prove that the truncated solution is a good approximate solution to the defocusing wave equation. The proof of Proposition 2.7 can be found in Section 3.

Finally, we construct a solution to (1.1) near the truncated approximation solution. This is exactly what Proposition 2.8 says. The proof of Proposition 2.8 can be found in Section 4, where we use the energy method to solve the wave equation in a time-backward direction, and we need to use a technical truncation to avoid the singularity at blow-up time. Such method of solving backward in time has been used in [41, 42, 60]. Let's emphasize that this part does not depend at all on our method of constructing the approximate solutions, and it includes the case $d = 3$ and does not require Assumption 1 or the spherical symmetry of the approximate solutions either.

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1.2. Blow-up phenomenon for related models

Let's review some important results on the blow-up for other related equations.

It is more common to observe the blow-up phenomenon for the focusing nonlinear wave equation, i.e.,

$$\Box u = -|u|^{p-1}u. \tag{1.4}$$

In fact, the spatial independent function $u(t) = C_p(T-t)^{-2/(p-1)}$, where $C_p^{p-1} = 2(p+1)/(p-1)^2$, gives a blow-up solution to (1.4). This ODE-type solution can be further truncated to a smooth compactly supported blow-up solution to (1.4) by using the finite speed of propagation [1, 30, 43]. We will use similar ideas to prove Corollary 1.2 for the case $d > 5$. See also [12, 15, 16, 18, 27, 32, 34, 40, 42, 46, 58] for the construction and classification of blow-up (or global) solutions as well as recent breakthrough [13, 17, 29] on the soliton resolution conjecture.

Other related models such as the nonlinear Schrödinger equation, see [33, 48, 50, 51, 54, 55, 59, 61]; see [14, 38, 39, 41, 62, 64] for the wave map; see [49, 60] for the Schrödinger maps; see [7, 8, 9, 11, 23, 47, 56, 57] for the semilinear heat equation and [10, 28, 37, 63] for the harmonic heat flow.

1.3. Notations and conventions

Unless stated otherwise, we adopt the following notations, abbreviations, and conventions:

- Constants: $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, e is the base of the natural logarithm.
- For any $a \in \mathbb{R}$, we denote $\mathbb{Z}_{\geq a} := \mathbb{Z} \cap [a, +\infty)$ and $\mathbb{Z}_{>a} := \mathbb{Z} \cap (a, +\infty)$. Moreover, we denote $\mathbb{Z}_+ := \mathbb{Z}_{\geq 1}$. Similarly, $\mathbb{R}_{\geq 0} := \mathbb{R} \cap [0, +\infty)$.
- Greek indices run from 0 to d, where $d \in \mathbb{Z}_{\geq 2}$ is the spatial dimension, Latin indices run from 1 to d, and we use the Einstein's summation convention: repeated indices appearing once upstairs and once downstairs are summed over their range.
- $(t, x) = (t, x_1, \cdots, x_d)$ denotes coordinates in spacetime, $r = |x| = (\sum_{j=1}^d x_j^2)^{1/2}$. We write $\partial_0 = -\partial^0 = \partial_t = \frac{\partial}{\partial t}$, $\partial_j = \partial^j = \partial_{x_j} = \frac{\partial}{\partial x_j}$ for $j \in \mathbb{Z} \cap [1, d]$, $\Box = \partial^\alpha \partial_\alpha = -\partial_t^2 + \sum_{j=1}^d \partial_j^2$ and $\Delta = \sum_{j=1}^d \partial_j^2$, then $\Box = -\partial_t^2 + \Delta$.
- We denote $\ell := 1 + 4/(p-1) > 1$, $k := d - 1 \in \mathbb{Z}_+$ and $\gamma := 4\beta/(p-1) + 2 = \beta(\ell - 1) + 2$.
- For a (vector-valued) differentiable function $f = f(t, x)$, we denote

  $$Df := (\partial_t f, \partial_1 f, \partial_2 f, \cdots, \partial_d f) \quad \text{and} \quad D_x f := (\partial_1 f, \partial_2 f, \cdots, \partial_d f) = \nabla_x f,$$

  and $|Df| := (|\partial_t f|^2 + \sum_{j=1}^d |\partial_j f|^2)^{1/2}$, $|D_x f| := (\sum_{j=1}^d |\partial_j f|^2)^{1/2}$. For all $j \in \mathbb{Z}_+$ we denote $D^j f := DD^{j-1} f$, $D_x^j f := D_x D_x^{j-1} f$, $D^0 f = D_x^0 f = f$, noting that $D^{j-1} f$ and $D_x^{j-1} f$ are again vector-valued functions; moreover, $D^{\leq 1} f := (f, Df)$.
- For $(t, x) \in [0, T) \times \mathbb{R}^d$, we let $\tau := -\ln(T - t)$ and $Z := |x|/(T - t) \in [0, +\infty)$.
- For $N \geq 0$, H_x^N denotes the inhomogeneous Sobolev space with the norm $\|\cdot\|_{H_x^N}$ with respect to the spatial variables and $\dot{H}_x^N$ denotes the homogeneous Sobolev space with the norm $\|\cdot\|_{\dot{H}_x^N}$. Moreover, we denote $L_x^2 := H_x^0$.
- A function space is a linear vector space if it is closed under addition and multiplication by a constant. A function space is a ring (algebra) if it contains all the constant functions and is closed under addition and multiplication. Then a ring is also a linear vector space.

2. A roadmap of the proof

We introduce the modulus-phase decomposition $u = w\mathrm{e}^{\mathrm{i}\Phi}$, with $w : \mathbb{R}^{1+d} \to \mathbb{R}_{>0}$ and $\Phi : \mathbb{R}^{1+d} \to \mathbb{R}$. Then

$$\Box u = (\Box w + 2\mathrm{i}\partial^\alpha w \partial_\alpha \Phi + \mathrm{i}w\Box\Phi - w\partial^\alpha \Phi \partial_\alpha \Phi)\mathrm{e}^{\mathrm{i}\Phi},$$

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and (1.1) becomes

$$\Box w = w^p + w\partial^\alpha\Phi\partial_\alpha\Phi, \qquad 2\partial^\alpha w\partial_\alpha\Phi + w\Box\Phi = 0. \tag{2.1}$$

Let $b > 0$ be a positive constant. We re-normalize according to

$$w(t,x) = b^{-\frac{1}{p-1}}\rho(t,x), \qquad \Phi(t,x) = b^{-\frac{1}{2}}\phi(t,x), \tag{2.2}$$

then (2.1) becomes

$$b\Box\rho = \rho^p + \rho\partial^\alpha\phi\partial_\alpha\phi, \qquad 2\partial^\alpha\rho\partial_\alpha\phi + \rho\Box\phi = 0. \tag{2.3}$$

We seek solutions (ρ,ϕ) to (2.3) in the form of

$$\rho(t,x) = \sum_{n=0}^{\infty}\rho_n(t,x)b^n, \qquad \phi(t,x) = \sum_{n=0}^{\infty}\phi_n(t,x)b^n. \tag{2.4}$$

Plugging (2.4) into (2.3), we obtain the following recurrence relation for $n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} \Box\rho_{n-1} &= \sum_{n_1+n_2+\cdots+n_p=n}\rho_{n_1}\rho_{n_2}\cdots\rho_{n_p} + \sum_{n_1+n_2+n_3=n}\rho_{n_1}\partial^\alpha\phi_{n_2}\partial_\alpha\phi_{n_3}, \\ 0 &= 2\sum_{n_1+n_2=n}\partial^\alpha\rho_{n_1}\partial_\alpha\phi_{n_2} + \sum_{n_1+n_2=n}\rho_{n_1}\Box\phi_{n_2}, \end{aligned} \tag{2.5}$$

where we have used the convention that $\rho_{-n'} = \phi_{-n'} = 0$ for all $n' \in \mathbb{Z}_+$. Here (2.4) is only a formal expansion and we will use cutoff functions to construct approximate solutions. Here b plays the role of deriving recurrence relation (2.5), the smallness lies in the functions ρ_n, ϕ_n as $t \uparrow T$ rather than b^n. We will not let $b \downarrow 0$, in fact, we will fix $b = 1$.

2.1. *The leading order term of the blow-up solution*

Letting $n = 0$ in (2.5), we know that (ρ_0,ϕ_0) satisfies the system[3]

$$\rho_0^p + \rho_0\partial^\alpha\phi_0\partial_\alpha\phi_0 = 0, \qquad 2\partial^\alpha\rho_0\partial_\alpha\phi_0 + \rho_0\Box\phi_0 = 0. \tag{2.6}$$

For any $\beta > 1$, the system (2.6) is invariant under the scaling

$$\phi_{0,\lambda}(t,x) = \lambda^{\beta-1}\phi_0(\lambda t,\lambda x), \qquad \rho_{0,\lambda}(t,x) = \lambda^{\frac{2\beta}{p-1}}\rho_0(\lambda t,\lambda x), \qquad \forall\,\lambda > 0.$$

We seek radially symmetric self-similar blow-up solutions to (2.6) of the form

$$\phi_0(t,r) = (T-t)^{1-\beta}\widehat{\phi}_0(Z), \quad \rho_0(t,x) = (T-t)^{-\frac{2\beta}{p-1}}\widehat{\rho}_0(Z), \quad Z = \frac{r}{T-t}, \quad r = |x|, \tag{2.7}$$

where $T > 0$ is the blow-up time and $\beta > 1$ is a constant.[4] Let $v = \partial_r\phi_0/\partial_t\phi_0$, then $v = v(Z)$ solves the ODE[5]

$$\begin{aligned} &\Delta_Z(Z,v)\mathrm{d}v/\mathrm{d}Z = \Delta_v(Z,v), \quad \Delta_v(Z,v) := (1-v^2)[\beta\ell(1-v^2)Z - kv(1-Zv)], \\ &\Delta_Z(Z,v) := Z\left[(1-Zv)^2 - \ell(v-Z)^2\right], \end{aligned} \tag{2.8}$$

where $\ell := 1 + 4/(p-1) > 1$ and $k := d - 1 \in \mathbb{Z}_{\geq 1}$. See Subsection A.1 for the derivation of (2.8).

Recall the following fact from [65] (recalling footnote 4).

[3] System (2.6) is exactly the same as (2.5) and (2.6) in [65] as long as we let $\ell = 1 + 4/(p-1)$ and $\varrho = \rho^{p+1}$.

[4] Note that β in this paper is not the same as β in [65]. In fact, $\beta_{\text{in this paper}} = \beta_{\text{in [40]}}/(\ell+1)$. Hence, $\beta_{\text{in this paper}} > 1$ is equivalent to $\beta_{\text{in [65]}} > \ell + 1$, see Lemma A.7 in [65].

[5] ODE (2.8) is exactly the same as (2.17) in [65], as long as we let $m = \beta\ell$.

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Lemma 2.1 ([65], Lemma 2.1). *If $v(Z):[0,1]\to(-1,1)$ is a C^1 solution to (2.8) with $v(0)=0$ and $\ell>1,\beta>0,k>0$, then $k>\beta(\ell+\sqrt{\ell})$.*

As a consequence, it is natural to restrict the parameters (k,ℓ,β) in the following range:

$$\beta>1,\qquad \ell>1,\qquad k\in\mathbb{Z}\cap[3,+\infty),\qquad k>\beta(\ell+\sqrt{\ell}). \tag{2.9}$$

Assumption 1. There exists a smooth function $v=v(Z)\in(-1,1)$ defined on $Z\in[0,+\infty)$ solving the ODE (2.8) with $v(0)=0$ and $v\in C_{\mathrm{o}}^\infty([0,+\infty))$.

Here we define (with $\mathbb{R}_{\geq 0}:=[0,+\infty)$)

$$C_{\mathrm{e}}^\infty(\mathbb{R}_{\geq 0}):=\left\{f\in C^\infty(\mathbb{R}_{\geq 0}):\exists\,\widetilde{f}\in C^\infty(\mathbb{R}_{\geq 0})\text{ s.t. }f(Z)=\widetilde{f}(Z^2)\ \forall\,Z\in\mathbb{R}_{\geq 0}\right\}, \tag{2.10}$$

$$C_{\mathrm{o}}^\infty(\mathbb{R}_{\geq 0}):=\left\{f\in C^\infty(\mathbb{R}_{\geq 0}):\exists\,\widetilde{f}\in C^\infty(\mathbb{R}_{\geq 0})\text{ s.t. }f(Z)=Z\widetilde{f}(Z^2)\ \forall\,Z\in\mathbb{R}_{\geq 0}\right\}. \tag{2.11}$$

Then $C_{\mathrm{e}}^\infty(\mathbb{R}_{\geq 0})$ is a ring and $C_{\mathrm{o}}^\infty(\mathbb{R}_{\geq 0})$ is a linear vector space.

Remark 2.2. Under Assumption 1 and (2.9), we can show that the solution $v(Z)$ satisfies

- $v(Z)<Z$ and $Zv(Z)<1$ for all $Z\in(0,+\infty)$.
- $\Delta_Z(Z,v(Z))>0$ for $Z\in(0,Z_1)$ and $\Delta_Z(Z,v(Z))<0$ for $Z\in(Z_1,+\infty)$, where $Z_1=\frac{k}{\sqrt{\ell}(k-\beta(\ell-1))}>0$.
- Let $\Delta_0(Z):=\Delta_Z(Z,v(Z))$ for $Z\in[0,+\infty)$, then $\Delta_0'(Z_1)\neq 0$.

See Subsection A.2 for the proof.

In view of Assumption 1, we can define that for $Z\in[0,+\infty)$

$$\begin{aligned}\widehat{\phi}_0(Z)&:=\frac{1}{\beta-1}\exp\left((\beta-1)\int_0^Z\frac{v(s)}{1-sv(s)}\,\mathrm{d}s\right),\\ \widehat{\rho}_0(Z)&:=\frac{(\beta-1)^{\frac{2}{p-1}}\widehat{\phi}_0(Z)^{\frac{2}{p-1}}(1-v(Z)^2)^{\frac{1}{p-1}}}{(1-Zv(Z))^{\frac{2}{p-1}}}.\end{aligned} \tag{2.12}$$

Then $\widehat{\phi}_0(Z)>0$, $\widehat{\rho}_0(0)=1$ and $\widehat{\rho}_0(Z)>0$ for all $Z\in[0,+\infty)$. As a consequence, (ϕ_0,ρ_0) defined by (2.7) solves (2.6) (see Lemma A.1), and $\widehat{\phi}_0,\ \widehat{\rho}_0\in C_{\mathrm{e}}^\infty([0,+\infty))$ (see Lemma A.6). This is the leading order term of our blow-up solution (ρ,ϕ) to (2.3).

2.2. *Solving (ρ_n,ϕ_n) for $n\in\mathbb{Z}_{\geq 1}$*

In Subsection 2.1, under Assumption 1, we construct the leading order blow-up solution (ρ_0,ϕ_0). In view of the expansion (2.4), we construct (ρ_n,ϕ_n) for $n\in\mathbb{Z}_{\geq 1}$. We rewrite the recurrence relation (2.5) for $n\in\mathbb{Z}_{\geq 1}$ as

$$\begin{aligned}&(p\rho_0^{p-1}+\partial^\alpha\phi_0\partial_\alpha\phi_0)\rho_n+2\rho_0\partial^\alpha\phi_0\partial_\alpha\phi_n\\ &\quad=\Box\rho_{n-1}-\sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\cdots,n_p\leq n-1}}\rho_{n_1}\cdots\rho_{n_p}-\sum_{\substack{n_1+n_2+n_3=n\\ n_1,n_2,n_3\leq n-1}}\rho_{n_1}\partial^\alpha\phi_{n_2}\partial_\alpha\phi_{n_3}=:F_n,\end{aligned} \tag{2.13}$$

$$\begin{aligned}&\rho_0\Box\phi_n+2\partial^\alpha\rho_0\partial_\alpha\phi_n+2\partial^\alpha\phi_0\partial_\alpha\rho_n+\Box\phi_0\rho_n\\ &\quad=-2\sum_{\substack{n_1+n_2=n\\ n_1,n_2\leq n-1}}\partial^\alpha\rho_{n_1}\partial_\alpha\phi_{n_2}-\sum_{\substack{n_1+n_2=n\\ n_1,n_2\leq n-1}}\rho_{n_1}\Box\phi_{n_2}=:G_n.\end{aligned} \tag{2.14}$$

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Using the equations for (ρ_0, ϕ_0) given by (2.6), the above recurrence relation becomes

$$(p-1)\rho_0^{p-1}\rho_n + 2\rho_0\partial^\alpha\phi_0\partial_\alpha\phi_n = F_n, \tag{2.15}$$

$$\partial^\alpha(\rho_0^2\partial_\alpha\phi_n) + 2\partial^\alpha(\rho_0\partial_\alpha\phi_0\rho_n) = \rho_0 G_n. \tag{2.16}$$

By (2.15), we have

$$\rho_n = \frac{\rho_0^{1-p}F_n}{p-1} - \frac{2}{p-1}\rho_0^{2-p}\partial^\alpha\phi_0\partial_\alpha\phi_n. \tag{2.17}$$

Substituting the above identity into (2.16), we obtain the following linear equation for ϕ_n:

$$\partial^\alpha\left(\rho_0^2\partial_\alpha\phi_n - \frac{4}{p-1}\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\phi_n\right) = \rho_0 G_n - \frac{2}{p-1}\partial^\alpha\left(\rho_0^{2-p}\partial_\alpha\phi_0 F_n\right) =: H_n. \tag{2.18}$$

We introduce the linearized operator

$$\mathscr{L}(\phi) := \partial^\alpha\left(\rho_0^2\partial_\alpha\phi - \frac{4}{p-1}\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\phi\right), \qquad \phi = \phi(t,x) = \phi(t,r). \tag{2.19}$$

Then our aim is to solve inductively $\mathscr{L}(\phi_n) = H_n$ for each $n \geq 1$.

Indeed, we can show that $\mathscr{L}$ is surjective in some well-chosen functional spaces and then we solve $\mathscr{L}(\phi_n) = H_n$ in these spaces. Letting $\tau = \ln\frac{1}{T-t}$, we define (here $C_{\mathrm{e}}^\infty([0,+\infty))$ is defined in (2.10))

$$\mathscr{X}_0 := \left\{ f(t,x) = \sum_{j=0}^n f_j(Z)\tau^j : n \in \mathbb{Z}_{\geq 0}, f_j \in C_{\mathrm{e}}^\infty([0,+\infty)) \ \ \forall\, j \in \mathbb{Z}\cap[0,n] \right\}, \tag{2.20}$$

$$\mathscr{X}_\lambda := (T-t)^\lambda\mathscr{X}_0 = \left\{ f(t,x) = (T-t)^\lambda g(t,x) = \mathrm{e}^{-\lambda\tau}g(t,x) : g \in \mathscr{X}_0 \right\}, \quad \forall\, \lambda \in \mathbb{C}. \tag{2.21}$$

Then $\mathscr{X}_0$ is a ring (using that $\{f(t,x) = f(Z)\tau^j : f_j \in C_{\mathrm{e}}^\infty([0,+\infty)),\ j \in \mathbb{Z}_{\geq 0}\}$ is closed under multiplication) and $\mathscr{X}_\lambda$ is a linear vector space.

We have the following properties for the functional spaces $\mathscr{X}_\lambda$.

Lemma 2.3.

(i) *Let $\lambda, \mu \in \mathbb{C}$, $f \in \mathscr{X}_\lambda, g \in \mathscr{X}_\mu$. Then $\partial_t f \in \mathscr{X}_{\lambda-1}$, $\Delta f \in \mathscr{X}_{\lambda-2}$, $\Box f \in \mathscr{X}_{\lambda-2}$, $fg \in \mathscr{X}_{\lambda+\mu}$, $\partial^\alpha f\partial_\alpha g \in \mathscr{X}_{\lambda+\mu-2}$, and $\partial^\alpha(f\partial_\alpha g) \in \mathscr{X}_{\lambda+\mu-2}$.*

(ii) *Let $\lambda, \mu \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $\lambda > j + \mu$. If $f \in \mathscr{X}_\lambda$, then $(T-t)^{-\mu}D^j f \in L^\infty(\mathcal{C})$, where $\mathcal{C}$ is the light cone $\mathcal{C} := \left\{(t,x) \in [0,T)\times\mathbb{R}^d : |x| < 2(T-t)\right\}$.*

(iii) *Let $\lambda, \mu \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $\lambda \geq j + \mu$. If $f(t,x) = (T-t)^\lambda\widehat{f}(Z)$ for some $\widehat{f} \in C_{\mathrm{e}}^\infty([0,+\infty))$, then $(T-t)^{-\mu}D^j f \in L^\infty(\mathcal{C})$.*

The proof of Lemma 2.3 can be found in Subsection 5.1.

Proposition 2.4. *The linear operator $\mathscr{L} : \mathscr{X}_\lambda \to \mathscr{X}_{\lambda-\gamma}$ is surjective for all $\lambda \in \mathbb{C}$, where $\gamma := 4\beta/(p-1) + 2 = \beta(\ell-1) + 2$.*

See Section 5 for the proof of Proposition 2.4.

Let

$$\lambda_n := (2n-1)(\beta-1), \qquad \mu_n := 2n(\beta-1) - \frac{2\beta}{p-1}, \qquad \forall\, n \in \mathbb{Z}_{\geq 0}. \tag{2.22}$$

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Recall from (2.7) that

$$\phi_0(t,r)=(T-t)^{\lambda_0}\widehat{\phi}_0(Z),\qquad \rho_0(t,r)=(T-t)^{\mu_0}\widehat{\rho}_0(Z).$$

As $\widehat{\phi}_0(Z),\widehat{\rho}_0(Z)\in C_{\rm e}^\infty([0,+\infty))$, by (2.20), (2.21) we have $\phi_0\in\mathscr{X}_{\lambda_0}$ and $\rho_0\in\mathscr{X}_{\mu_0}$. Similarly, for $a\in\mathbb{R}$ we have $\rho_0(t,r)^a=(T-t)^{a\mu_0}\widehat{\rho}_0(Z)^a$ and $\widehat{\rho}_0(Z)^a\in C_{\rm e}^\infty([0,+\infty))$, then $\rho_0^a\in\mathscr{X}_{a\mu_0}$. Moreover, $\widehat{\phi}_0(Z),\widehat{\rho}_0(Z)$ are real-valued, so are ϕ_0,ρ_0.

Lemma 2.5. *Assume that $n\in\mathbb{Z}_{\geq 1}$, $\phi_j\in\mathscr{X}_{\lambda_j}$, $\rho_j\in\mathscr{X}_{\mu_j}$ are are real-valued for $j\in\mathbb{Z}\cap[0,n-1]$. Let F_n, G_n be defined in (2.13), (2.14). Then there exist real-valued $\phi_n\in\mathscr{X}_{\lambda_n}$ and $\rho_n\in\mathscr{X}_{\mu_n}$ such that (2.15) and (2.16) hold.*

Proof. By Lemma 2.3 (i) and the definition of F_n, we have $F_n\in\mathscr{X}_{\mu_{n-1}-2}$, where we have used the fact that $\mu_{n_1}+\cdots+\mu_{n_p}=\mu_{n-1}-2$ if $n_1+\cdots+n_p=n$ and $\mu_{n_1}+\lambda_{n_2}+\lambda_{n_3}-2=\mu_{n-1}-2$ if $n_1+n_2+n_3=n$. Using Lemma 2.3 (i) and the definition of G_n, we have $G_n\in\mathscr{X}_{(2n-1)(\beta-1)-2\beta/(p-1)-2}$, where we have used the fact that $\mu_{n_1}+\lambda_{n_2}-2=(2n-1)(\beta-1)-2\beta/(p-1)-2$ if $n_1+n_2=n$. It follows from Lemma 2.3 (i) that $\rho_0G_n\in\mathscr{X}_{\mu_0+(2n-1)(\beta-1)-2\beta/(p-1)-2}=\mathscr{X}_{2n(\beta-1)-\beta\ell-1}$ (recall that $\rho_0\in\mathscr{X}_{\mu_0}$, $\ell=1+\frac{4}{p-1}$). Since $\rho_0^{2-p}\in\mathscr{X}_{(2-p)\mu_0}$, $F_n\in\mathscr{X}_{\mu_{n-1}-2}$, by Lemma 2.3 (i) we get $\rho_0^{2-p}F_n\in\mathscr{X}_{\mu_{n-1}-2+(2-p)\mu_0}$, then by $\phi_0\in\mathscr{X}_{\lambda_0}$ we have

$$\partial^\alpha\left(\rho_0^{2-p}\partial_\alpha\phi_0F_n\right)\in\mathscr{X}_{\mu_{n-1}-2+(2-p)\mu_0+\lambda_0-2}=\mathscr{X}_{2n(\beta-1)-\beta\ell-1}.$$

Hence by the definition of H_n in (2.18), we have $H_n\in\mathscr{X}_{2n(\beta-1)-\beta\ell-1}=\mathscr{X}_{\lambda_n-\gamma}$ (recall that $\gamma=\beta(\ell-1)+2$). Moreover, F_n,G_n,H_n are real-valued.

By Proposition 2.4, there exists (real-valued) $\phi_n\in\mathscr{X}_{\lambda_n}$ such that $\mathscr{L}(\phi_n)=H_n$ (otherwise take $\operatorname{Re}\phi_n$), then (2.18) holds. Let ρ_n be defined by (2.17). Then ρ_n is real-valued. Moreover, using (i) of Lemma 2.3, $\rho_0^{1-p}\in\mathscr{X}_{(1-p)\mu_0}$, $\rho_0^{2-p}\in\mathscr{X}_{(2-p)\mu_0}$, $F_n\in\mathscr{X}_{\mu_{n-1}-2}$, $\phi_0\in\mathscr{X}_{\lambda_0}$ and $\phi_n\in\mathscr{X}_{\lambda_n}$, we have

$$\rho_0^{1-p}F_n\in\mathscr{X}_{\mu_0(1-p)+\mu_{n-1}-2}=\mathscr{X}_{\mu_n},\quad \rho_0^{2-p}\partial^\alpha\phi_0\partial_\alpha\phi_n\in\mathscr{X}_{\mu_0(2-p)+\lambda_0+\lambda_n-2}=\mathscr{X}_{\mu_n},$$

hence $\rho_n\in\mathscr{X}_{\mu_n}$. Now (2.15) follows from (2.17), and (2.16) follows from (2.17) and (2.18). □

As $\phi_0\in\mathscr{X}_{\lambda_0}$, $\rho_0\in\mathscr{X}_{\mu_0}$ and ϕ_0,ρ_0 are real-valued, by Lemma 2.5 and the induction, we have the following result.

Proposition 2.6. *Let ϕ_0, ρ_0 be defined in (2.7). For each $n\in\mathbb{Z}_{\geq 1}$, there exist real-valued $\phi_n\in\mathscr{X}_{\lambda_n}$ and $\rho_n\in\mathscr{X}_{\mu_n}$ such that (2.15) and (2.16) hold with F_n, G_n defined in (2.13), (2.14). Hence, (2.5) holds for $n\in\mathbb{Z}_{\geq 0}$.*

Now we briefly explain the ideas in the proof of Proposition 2.4. In the proof of Lemma 2.5, we see that we only need to use the surjectivity of $\mathscr{L}$ from $\mathscr{X}_\lambda$ to $\mathscr{X}_{\lambda-\gamma}$ for $\lambda\in\{\lambda_n:n\in\mathbb{Z}_+\}$. However, this is not easy to solve the equation $\mathscr{L}f=g$ for $f\in\mathscr{X}_\lambda$ even in the simplest case $g=(T-t)^{\lambda-\gamma}\widehat{g}(Z)\in\mathscr{X}_{\lambda-\gamma}$ for some $\widehat{g}\in C_{\rm e}^\infty([0,+\infty))$ (without the logarithm correction τ^j for $j\in\mathbb{Z}_+$), in which process we need to check a non-degenerate property (nonzero of Wronski defined in (5.32)) on the coefficients of $\mathscr{L}_\lambda$ (defined in (5.23)), and it is difficult to check that all λ_n satisfy the non-degenerate property, even for one λ_{n_0}. To overcome this drawback, we solve the equation for all $\lambda\in\mathbb{C}$, not merely for those $\lambda\in\{\lambda_n:n\in\mathbb{Z}_+\}$. It turns out that the non-degenerate property holds for all but countably many $\lambda\in\mathbb{C}$ (these λ are "bad" in some sense) and the solution depends analytically on λ. In this way, we can show that the solution $f=f(\cdot;\lambda)$ is a meromorphic function on λ. For those countably many $\lambda\in\mathbb{C}$ not satisfying the non-degenerate property, the analytic property of f allows us to introduce a logarithm correction to solve the corresponding equation for "bad" λ. See Section 5 and Appendix B for details.

2.3. The approximate solution

Let $(\rho_n,\phi_n)\in\mathscr{X}_{\mu_n}\times\mathscr{X}_{\lambda_n}(n\in\mathbb{Z}_{\geq 0})$ be given by Proposition 2.6. We fix a bump function $\eta\in C_c^\infty(\mathbb{R};[0,1])$ such that $\eta|_{[0,1]}=1$ and $\eta|_{[2,+\infty)}=0$.

Proposition 2.7. *Let $T=b=1$. There exist $N_0\in\mathbb{Z}_+$, $c_0\in(0,T)$ and a sequence $\{T_n\}_{n\geq 0}$ such that $T_n=T$ for $0\leq n<N_0$, $0<T_n\leq T_{n-1}/4$ for all $n\geq N_0$, and for functions*[6]

$$\rho_*(t,x):=\sum_{n=0}^{\infty}\eta\left(\frac{T-t}{T_n}\right)\rho_n(t,x)b^n,\quad \phi_*(t,x):=\sum_{n=0}^{\infty}\eta\left(\frac{T-t}{T_n}\right)\phi_n(t,x)b^n,\tag{2.23}$$

$$E_*:=\rho_*^p+\rho_*\partial^\alpha\phi_*\partial_\alpha\phi_*-b\Box\rho_*,\quad J_*:=2\partial^\alpha\rho_*\partial_\alpha\phi_*+\rho_*\Box\phi_*,\tag{2.24}$$

defined on $(t,x)\in[0,T)\times\mathbb{R}^d$ we have

$$(T-t)^{\beta+j-1}D^j\phi_*\in L^\infty(\mathcal{C}),\quad (T-t)^{\frac{2\beta}{p-1}+j}D^j\rho_*\in L^\infty(\mathcal{C}),\qquad \forall\, j\in\mathbb{Z}_{\geq 0},\tag{2.25}$$

$$(T-t)^\beta(\partial_t\phi_*-|D_x\phi_*|)\geq c_0,\quad (T-t)^{\frac{2\beta}{p-1}}\rho_*\geq c_0,\quad \forall\,(t,x)\in\mathcal{C},\ T-t<c_0,\tag{2.26}$$

$$(T-t)^{-\lambda}(D^jE_*,D^jJ_*)\in L^\infty(\mathcal{C}),\qquad \forall\,\lambda>0,\ \forall\, j\in\mathbb{Z}_{\geq 0}.\tag{2.27}$$

See Section 3 for the proof of Proposition 2.7.

2.4. Solving nonlinear wave equation

Proposition 2.8. *Assume that $T=1$, $w_*\in C^\infty(\mathcal{C}),\Phi_*\in C^\infty(\mathcal{C})$ satisfy*

$$(T-t)^{\beta+j-1}D^j\Phi_*\in L^\infty(\mathcal{C}),\quad (T-t)^{\frac{2\beta}{p-1}+j}D^jw_*\in L^\infty(\mathcal{C}),\qquad \forall\, j\in\mathbb{Z}_{\geq 0},\tag{2.28}$$

$$(T-t)^\beta(\partial_t\Phi_*-|D_x\Phi_*|)\geq c_0,\quad (T-t)^{\frac{2\beta}{p-1}}w_*\geq c_0,\quad \forall\,(t,x)\in\mathcal{C},\ T-t<c_0,\tag{2.29}$$

for some $c_0\in(0,T)=(0,1)$. Suppose that (2.27) holds for E_*, J_* defined as

$$E_*:=w_*^p+w_*\partial^\alpha\Phi_*\partial_\alpha\Phi_*-\Box w_*,\quad J_*:=2\partial^\alpha w_*\partial_\alpha\Phi_*+w_*\Box\Phi_*,\tag{2.30}$$

Then there exist $c_1\in(0,c_0)$ and $u\in C^2((T-c_1,T)\times\mathbb{R}^d;\mathbb{C})$ such that $u(t,\cdot),\partial_tu(t,\cdot)\in C_c^\infty(\mathbb{R}^d;\mathbb{C})$ for $t\in(T-c_1,T)$, $\Box u=|u|^{p-1}u$ for $t\in(T-c_1,T)$, $|x|\leq T-t$, and

$$C^{-1}(T-t)^{-\frac{2\beta}{p-1}}\leq|u(t,x)|\leq C(T-t)^{-\frac{2\beta}{p-1}},\qquad \forall\, t\in(T-c_1,T),\ |x|\leq T-t\tag{2.31}$$

for some constant $C>0$.

In fact, $u=(1+h)w_*\mathrm{e}^{\mathrm{i}\Phi_*}$, $h=O((T-t)^\lambda)$, $\forall\,\lambda>0$. To prove Proposition 2.8, it suffices to solve the equation for h (see (4.1)) and prove that h is small. In view of the singularity of (4.1) at blow-up time T, we take a sequence $\varepsilon_n\downarrow 0$ and then we solve (4.1) (with technical truncation) with zero initial data at $T-\varepsilon_n$ in a backward direction. We denote the solution for each $n\in\mathbb{Z}_+$ by h_n. Using energy estimates and a bootstrap argument, we can show that h_n lives in an interval with a positive lower bound independent of $n\in\mathbb{Z}_+$. Taking the limit $n\to\infty$ we get a desired solution to (4.1) (in the light cone). See Section 4 for details.

[6]For fixed $(t,x)\in[0,T)\times\mathbb{R}^d$, the summations in (2.23) are both finite sums. Indeed, we have $T-t>0$, then $\lim_{n\to\infty}(T-t)/T_n=+\infty$, thus $(T-t)/T_n>2$ for all sufficiently large n and hence $\eta\big((T-t)/T_n\big)=0$ for all sufficiently large n. As a consequence, we have $\rho_*,\phi_*\in C^\infty([0,T)\times\mathbb{R}^d)$.

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2.5. *Proof of main results*

Let's begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $T = b = 1$ and $(\rho_*, \phi_*) \in C^\infty(\mathcal{C})$, $c_0 \in (0, T)$ be given by Proposition 2.7, and $w_* = \rho_*, \Phi_* = \phi_*$. Then $w_*, \Phi_* \in C^\infty(\mathcal{C})$, (2.28) is equivalent to (2.25), (2.29) is equivalent to (2.26). The definitions of E_*, J_* in (2.24) and (2.30) are the same, and (2.27) also follows from Proposition 2.7. It follows from Proposition 2.8 that there exist $c_1 \in (0, c_0) \subset (0, 1)$, $\widetilde{u} \in C^2((T - c_1, T) \times \mathbb{R}^d; \mathbb{C})$ such that $\widetilde{u}(t, \cdot), \partial_t\widetilde{u}(t, \cdot) \in C_c^\infty(\mathbb{R}^d; \mathbb{C})$ for $t \in (T - c_1, T)$, $\Box\widetilde{u} = |\widetilde{u}|^{p-1}\widetilde{u}$ for $t \in (T - c_1, T)$, $|x| \le T - t$, and

$$C_1^{-1}(T-t)^{-\frac{2\beta}{p-1}} \le |\widetilde{u}(t, x)| \le C_1(T-t)^{-\frac{2\beta}{p-1}}, \qquad \forall\, t \in (T - c_1, T),\ |x| \le T - t \tag{2.32}$$

for some constant $C_1 > 0$. Choose initial data $u_0, u_1 : \mathbb{R}^d \to \mathbb{C}$ such that

$$u_0(x) = \widetilde{u}(T - c_1/2, x), \quad u_1(x) = \partial_t\widetilde{u}(T - c_1/2, x), \qquad \forall\, x \in \mathbb{R}^d.$$

Then $u_0, u_1 \in C_c^\infty(\mathbb{R}^d; \mathbb{C})$. Moreover, let $u_*(t, x) = \widetilde{u}(t + T - c_1/2, x)$ for $t \in (-c_1/2, c_1/2)$, $x \in \mathbb{R}^d$ then $u_*(0, x) = u_0(x)$, $\partial_t u_*(0, x) = u_1(x)$ and $\Box u_* = |u_*|^{p-1}u_*$ for $t \in [0, c_1/2)$, $|x| \le c_1/2 - t$. Suppose for contradiction that Theorem 1.1 fails for this initial data u_0, u_1, then there exists a smooth function $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{C}$ such that $\Box u = |u|^{p-1}u$ and $u(0, x) = u_0(x), \partial_t u(0, x) = u_1(x)$ for all $x \in \mathbb{R}^d$. Finite speed of propagation shows that $u = u_*$ in the region $\{(t, x) \in [0, c_1/2) \times \mathbb{R}^d : |x| \le c_1/2 - t\}$. Hence by (2.32) we have

$$|u(t, 0)| = |u_*(t, 0)| = |\widetilde{u}(t + T - c_1/2, 0)| \ge C_1^{-1}(c_1/2 - t)^{-\frac{2\beta}{p-1}}, \qquad \forall\, t \in [0, c_1/2).$$

On the other hand, since u is smooth on $[0, +\infty) \times \mathbb{R}^d$, we have $|u(t, x)| \le C$ for all $|x| \le 2T$ and $t \in [0, c_1/2]$, where $C > 0$ is a constant. This reaches a contradiction. $\square$

The following result was proved in [65] Theorem 2.2 and Lemma A.7 ($\beta > \ell + 1$ in [65] is equivalent to $\beta > 1$ in this paper, recalling footnote 4).

Lemma 2.9. *There exist* $\ell^*(3) = \frac{76 - 4\sqrt{154}}{23} \in (\frac{8}{7}, \frac{7}{6})$ *and* $\ell_1(4) \in (5/4, 4/3)$ *such that if*

$$k = 4,\ 1 < \ell < \ell_1(4) \quad or \quad k = 3,\ 1 < \ell < \ell^*(3), \tag{2.33}$$

Then there exists $\beta \in (1, k/(\ell + \sqrt{\ell}))$ [7]*such that Assumption 1 holds for* $d = k + 1$.

Proof of Corollary 1.2. Let $T = 1$. If $d = 4$, $k = 3$, $p \ge 29$, $\ell = 1 + \frac{4}{p-1}$, then $1 < \ell \le 1 + \frac{4}{29-1} = \frac{8}{7} < \ell^*(3)$. If $d = 5$, $k = 4$, $p \ge 17$, $\ell = 1 + \frac{4}{p-1}$, then $1 < \ell \le 1 + \frac{4}{17-1} = \frac{5}{4} < \ell_1(4)$. Thus, if $d = 4$, $p \ge 29$ or $d = 5$, $p \ge 17$, then (2.33) holds for $k = d - 1$, $\ell = 1 + \frac{4}{p-1}$ and the result follows from Theorem 1.1.

The remaining case is $d > 5$, $p \ge 17$. Then Assumption 1 holds with d replaced by $d' = 5$. By the proof of Theorem 1.1, there exists $c_1 \in (0, 1)$, $\widetilde{u} \in C^2((T - c_1, T) \times \mathbb{R}^5; \mathbb{C})$ such that $\widetilde{u}(t, \cdot), \partial_t\widetilde{u}(t, \cdot) \in C_c^\infty(\mathbb{R}^5; \mathbb{C})$ for $t \in (T - c_1, T)$, $\Box\widetilde{u} = |\widetilde{u}|^{p-1}\widetilde{u}$ for $t \in (T - c_1, T)$, $|x| \le T - t$, and (2.32) holds for some constant $C_1 > 0$. Choose initial data $u_0, u_1 : \mathbb{R}^d \to \mathbb{C}$ such that

$$u_0(x) = \eta(|x|)\widetilde{u}(T - c_1/2, x_1, \cdots, x_5), \quad u_1(x) = \eta(|x|)\partial_t\widetilde{u}(T - c_1/2, x_1, \cdots, x_5),$$

for all $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. Then $u_0, u_1 \in C_c^\infty(\mathbb{R}^d; \mathcal{C})$. Moreover, let $u_*(t, x) = \eta(|x|)\widetilde{u}(t + T - c_1/2, x_1, \cdots, x_5)$ for $t \in (-c_1/2, c_1/2)$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. Then $u_*(0, x) = u_0(x)$, $\partial_t u_*(0, x) = u_1(x)$ and $\Box u_* = |u_*|^{p-1}u_*$ for $t \in [0, c_1/2)$, $|x| \le c_1/2 - t$. Here we used that $\eta(|x|) = 1$ for $|x| \le 1$ and that if $t \in [0, c_1/2)$, $|x| \le c_1/2 - t$ then $|x| \le c_1/2 < 1$. Suppose for contradiction that Corollary 1.2

[7]Note that $1 < \ell^*(3) < \ell_1(4) < 3/2$, thus if (2.33) holds then $\ell + \sqrt{\ell} < 2\ell < 3 \le k$.

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fails for this kind of initial data u_0, u_1, then there exists a smooth function $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{C}$ such that $\Box u = |u|^{p-1}u$ and $u(0, x) = u_0(x), \partial_t u(0, x) = u_1(x)$ for all $x \in \mathbb{R}^d$, and we can get a contradiction as in the proof of Theorem 1.1. □

3. The approximate solution

In this section, we prove Proposition 2.7, i.e., the construction of the approximate solution.

3.1. Construction of the approximate solution

Let $T = b = 1$ and $(\rho_n, \phi_n) \in \mathscr{X}_{\mu_n} \times \mathscr{X}_{\lambda_n} (n \in \mathbb{Z}_{\geq 0})$ be given by Proposition 2.6. For $N \in \mathbb{Z}_+$ and $(t, x) \in [0, T) \times \mathbb{R}^d$, let

$$\begin{aligned}
&\rho_{(N)}(t,x) := \sum_{n=0}^{N} \rho_n(t,x) b^n, \qquad \phi_{(N)}(t,x) := \sum_{n=0}^{N} \phi_n(t,x) b^n,\\
&E_N := \rho_{(N)}^p + \rho_{(N)} \partial^\alpha \phi_{(N)} \partial_\alpha \phi_{(N)} - b\Box\rho_{(N)},\\
&J_N := 2\partial^\alpha \rho_{(N)} \partial_\alpha \phi_{(N)} + \rho_{(N)} \Box \phi_{(N)}.
\end{aligned}$$

It follows from (2.5) that

$$E_N(t,x) = \sum_{n=N+1}^{pN} E_{N,n}(t,x) b^n, \qquad J_N(t,x) = \sum_{n=N+1}^{2N} J_{N,n}(t,x) b^n$$

with

$$\begin{aligned}
E_{N,n} &:= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\cdots,n_p\leq N}} \rho_{n_1}\rho_{n_2}\cdots\rho_{n_p} + \sum_{\substack{n_1+n_2+n_3=n\\ n_1,n_2,n_3\leq N}} \rho_{n_1}\partial^\alpha\phi_{n_2}\partial_\alpha\phi_{n_3} - \Box\rho_N \mathbf{1}_{n=N+1},\\
J_{N,n} &:= 2\sum_{\substack{n_1+n_2=n\\ n_1,n_2\leq N}} \partial^\alpha\rho_{n_1}\partial_\alpha\phi_{n_2} + \sum_{\substack{n_1+n_2=n\\ n_1,n_2\leq N}} \rho_{n_1}\Box\phi_{n_2}.
\end{aligned}$$

Then $E_{N,n} \in \mathscr{X}_{2n(\beta-1)-2p\beta/(p-1)}$, $J_{N,n} \in \mathscr{X}_{(2n-1)(\beta-1)-2\beta/(p-1)-2} = \mathscr{X}_{(2n+1)(\beta-1)-2p\beta/(p-1)}$. Here the proof is similar to Lemma 2.5.

Take $N_0 \in \mathbb{Z}_+$ such that $2N_0(\beta - 1) - 2p\beta/(p - 1) > 3$. We fix such N_0 (which is the same as the one in Proposition 2.7) and a non-decreasing sequence $\{k_N\}_{N\in\mathbb{Z}_{\geq N_0}} \subset \mathbb{Z}_+$ such that

$$2N(\beta-1) - 2p\beta/(p-1) > 3k_N \ \forall\, N \in \mathbb{Z}\cap[N_0,+\infty), \text{ and } \lim_{N\to\infty} k_N = +\infty. \tag{3.1}$$

Then by Lemma 2.3 (ii), we have $(T - t)^{-2k_N}(D^j E_N, D^j J_N) \in L^\infty(\mathcal{C})$ for $0 \leq j \leq k_N$, $N \geq N_0$. Or equivalently, for each $N \in \mathbb{Z} \cap [N_0, +\infty)$ there is a constant $A_N > 0$ satisfying

$$|D^j E_N(t,x)| + |D^j J_N(t,x)| \leq A_N (T-t)^{2k_N}, \quad \forall\, 0 \leq j \leq k_N,\ (t,x) \in \mathcal{C}.$$

In fact for every fixed $n > d/2$, we can use $(\rho_{(N)}, \phi_{(N)})$ as an approximate solution for N large enough (but fixed) to construct blow-up solutions of H^n regularity. But to obtain a blow-up solution of C^∞ initial data, we need to sum all the (ρ_n, ϕ_n) with truncation as in (2.23). Note that for $T - t \in [2T_{N+1}, T_N]$, we have $E_*(t, x) = E_N(t, x)$ and $J_*(t, x) = J_N(t, x)$. The following result extends the above estimate to the case $T - t \in [T_{N+1}, T_N]$ (with a possible different A_N).

Lemma 3.1. *Let $T = b = 1$. Then there exists a sequence $\{A_N\}_{N\in\mathbb{Z}_{\geq N_0-1}}$ such that for all $\{T_n\}_{n\in\mathbb{Z}_{\geq 0}}$ satisfying $T_n = T$ for $0 \leq n < N_0$, $0 < T_n \leq T_{n-1}/4$ for all $n \geq N_0$, if we define ρ_*, ϕ_*, E_*, J_* by (2.23), (2.24), then for $j \in \mathbb{Z} \cap [0, k_N]$ we have*

$$|D^j E_*(t,x)| + |D^j J_*(t,x)| \leq A_N(T-t)^{2k_N}, \quad \forall\, T-t \in [T_{N+1}, T_N],\ (t,x) \in \mathcal{C}. \tag{3.2}$$

Lemma 3.2. *Let $T = b = 1$. There exists a sequence $\{\widetilde{T}_n\}_{n\geq 0}$ satisfying*

$$\widetilde{T}_n = T \text{ for } 0 \leq n < N_0 \quad \text{and} \quad 0 < \widetilde{T}_n \leq \widetilde{T}_{n-1}/4 \text{ for } n \geq N_0,$$

such that for every sequence $\{T_n\}_{n\geq 0}$ with $T_n = T$ for $0 \leq n < N_0$ and $0 < T_n \leq \min(\widetilde{T}_n, T_{n-1}/4)$ for $n \geq N_0$, for ρ_*, ϕ_* defined in (2.23), we have

$$(T-t)^j D^j(\phi_* - \phi_0) \in L^\infty(\mathcal{C}),\ (T-t)^{\frac{2\beta}{p-1}+j-\beta+1} D^j(\rho_* - \rho_0) \in L^\infty(\mathcal{C}),\ \forall\, j \in \mathbb{Z}_{\geq 0}. \tag{3.3}$$

Lemma 3.3. *Let $T = 1$. There exists $\widetilde{c} \in (0,T)$ such that*

$$(T-t)^{\beta-1+j} D^j\phi_0 \in L^\infty(\mathcal{C}), \quad (T-t)^{\frac{2\beta}{p-1}+j} D^j\rho_0 \in L^\infty(\mathcal{C}), \quad \forall\, j \in \mathbb{Z}_{\geq 0}, \tag{3.4}$$

$$(T-t)^\beta(\partial_t\phi_0 - |D_x\phi_0|)(t,x) \geq \widetilde{c}, \quad (T-t)^{\frac{2\beta}{p-1}}\rho_0(t,x) \geq \widetilde{c}, \quad \forall\, (t,x) \in \mathcal{C}. \tag{3.5}$$

Let's first prove Proposition 2.7 by admitting Lemma 3.1~Lemma 3.3 for the moment.

Proof of Proposition 2.7.

Step 1. Construction of the sequence $\{T_n\}_{n\geq 0}$. Let $A_N > 0$ be given by Lemma 3.1 and the sequence $\{\widetilde{T}_n\}_{n\geq 0}$ be given by Lemma 3.2. Let $T_n = T$ for $0 \leq n < N_0$ and $T_n = \min(\widetilde{T}_n, A_n^{-1/k_n}, T_{n-1}/4)$ for $n \geq N_0$. Then $0 < T_n \leq T_{n-1}/4$, $T_n \leq \widetilde{T}_n$, $A_n T_n^{k_n} \leq 1$, for all $n \geq N_0$, and $\lim\limits_{N\to+\infty} T_N = 0$. So, there hold (3.2) for $j \in \mathbb{Z} \cap [0, k_N]$, (3.3), (3.4), and (3.5) with $\widetilde{c} \in (0,T)$ given by Lemma 3.3.

Step 2. Proof of (2.25). As $\beta > 1$, $(T-t)^{\beta-1} \in L^\infty(\mathcal{C})$, we get by (3.3) that

$$\begin{aligned}
&(T-t)^{\beta-1+j} D^j(\phi_* - \phi_0) = (T-t)^{\beta-1}(T-t)^j D^j(\phi_* - \phi_0) \in L^\infty(\mathcal{C}), \quad \forall\, j \in \mathbb{Z}_{\geq 0},\\
&(T-t)^{\frac{2\beta}{p-1}+j} D^j(\rho_* - \rho_0) = (T-t)^{\beta-1}(T-t)^{\frac{2\beta}{p-1}+j-\beta+1} D^j(\rho_* - \rho_0) \in L^\infty(\mathcal{C}), \quad \forall\, j \in \mathbb{Z}_{\geq 0},
\end{aligned}$$

which, along with with (3.4), implies (2.25).

Step 3. Proof of (2.26). By (3.3), we have (for some $C_1 > 0$)

$$(T-t)(|\partial_t(\phi_* - \phi_0)| + |D_x(\phi_* - \phi_0)|) + (T-t)^{\frac{2\beta}{p-1}-\beta+1}|\rho_* - \rho_0| \leq C_1 \quad \text{in } \mathcal{C}.$$

Now we take $c_0 \in (0,T)$ such that $c_0 + c_0^{\beta-1}C_1 \leq \widetilde{c}$, where the existence of such a c_0 is ensured by $\beta > 1$ and $\widetilde{c} > 0$. Then for $(t,x) \in \mathcal{C}, T-t < c_0$, we get by (3.5) that (as $T-t > 0$)

$$\begin{aligned}
(T-t)^\beta(\partial_t\phi_* - |D_x\phi_*|) \geq& (T-t)^\beta(\partial_t\phi_0 - |D_x\phi_0|) - (T-t)^\beta(|\partial_t(\phi_* - \phi_0)| + |D_x(\phi_* - \phi_0)|)\\
\geq& \widetilde{c} - (T-t)^{\beta-1}C_1 \geq \widetilde{c} - c_0^{\beta-1}C_1 \geq c_0,
\end{aligned}$$

and

$$(T-t)^{\frac{2\beta}{p-1}}\rho_* \geq (T-t)^{\frac{2\beta}{p-1}}\rho_0 - (T-t)^{\frac{2\beta}{p-1}}|\rho_* - \rho_0| \geq \widetilde{c} - (T-t)^{\beta-1}C_1 \geq \widetilde{c} - c_0^{\beta-1}C_1 \geq c_0.$$

Step 4. Proof of (2.27). We fix $\lambda > 0$, $j \in \mathbb{Z}_{\geq 0}$. As $\lim_{N\to+\infty} k_N = +\infty$, there exists $N_1 \in \mathbb{Z}_{\geq N_0}$ such that $k_N > \lambda + j$ for $N \in \mathbb{Z}_{\geq N_1}$. Then by (3.2) and $A_n T_n^{k_n} \leq 1$, we have

$$|D^j E_*(t,x)| + |D^j J_*(t,x)| \leq A_N (T-t)^{2k_N} \leq A_N T_N^{k_N} (T-t)^{k_N} \leq (T-t)^{k_N} \leq (T-t)^{\lambda},$$

for $T - t \in [T_{N+1}, T_N]$, $(t,x) \in \mathcal{C}$, $N \in \mathbb{Z}_{\geq N_1}$. As $\lim_{N\to+\infty} T_N = 0$, we have

$$|D^j E_*(t,x)| + |D^j J_*(t,x)| \leq (T-t)^{\lambda}, \quad \forall\, T-t \in (0, T_{N_1}],\ (t,x) \in \mathcal{C}. \tag{3.6}$$

As $\rho_*, \phi_* \in C^\infty([0,T) \times \mathbb{R}^d)$ (see footnote 6), we have $E_*, J_* \in C^\infty([0,T) \times \mathbb{R}^d)$ by (2.24). Thus, there exists a constant $C(j, T_{N_1}) > 0$ such that

$$|D^j E_*(t,x)| + |D^j J_*(t,x)| \leq C(j, T_{N_1}), \quad \forall\, T-t \in [T_{N_1}, T],\ |x| \leq 2T.$$

Then (recall that $\mathcal{C} = \{(t,x) \in [0,T) \times \mathbb{R}^d : |x| < 2(T-t)\}$)

$$|D^j E_*(t,x)| + |D^j J_*(t,x)| \leq C(j, T_{N_1}) T_{N_1}^{-\lambda} (T-t)^{\lambda}, \quad \forall\, T-t \in [T_{N_1}, T],\ (t,x) \in \mathcal{C},$$

which along with with (3.6) implies (2.27). □

3.2. Proof of main lemmas

We define the following auxiliary spaces

$$\mathscr{Y}_0 := \{f \in C^\infty([0,+\infty)) : f' = 0 \text{ in } [0,1] \cup [2,+\infty)\}, \tag{3.7}$$

$$\mathscr{X}_\lambda^* := \left\{ f(t,x,s) = \sum_{j=0}^{n} f_j(t,x)\eta_j\left(\frac{T-t}{s}\right) : n \in \mathbb{Z}_{\geq 0}, f_j \in \mathscr{X}_\lambda, \eta_j \in \mathscr{Y}_0,\ \forall\, j \right\}. \tag{3.8}$$

Note that $\eta \in \mathscr{Y}_0$, $\mathscr{Y}_0$ is a ring, and $\mathscr{X}_\lambda^*$ is a linear vector space.

Lemma 3.4.

(i) *Let $\lambda, \mu \in \mathbb{C}$, $f \in \mathscr{X}_\lambda^*, g \in \mathscr{X}_\mu^*$. Then $\Box f \in \mathscr{X}_{\lambda-2}^*$, $fg \in \mathscr{X}_{\lambda+\mu}^*$, $\partial^\alpha f \partial_\alpha g \in \mathscr{X}_{\lambda+\mu-2}^*$.*

(ii) *Let $\lambda, \mu \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $\lambda > j + \mu$. If $f \in \mathscr{X}_\lambda^*$, then $(T-t)^{-\mu} D^j f \in L^\infty(\mathcal{C} \times (0,1])$.*

Here the operators $\Box$, ∂_α and D are only acted on (t,x) and not on s.

Lemma 3.5.

(i) *Let $\lambda, \mu \in \mathbb{C}$, $f \in \mathscr{X}_\lambda^*, g \in \mathscr{X}_\mu^*$. Then $\Delta f \in \mathscr{X}_{\lambda-2}^*$, $\partial_t f \in \mathscr{X}_{\lambda-1}^*$, $fg \in \mathscr{X}_{\lambda+\mu}^*$.*

(ii) *Let $\lambda, \mu \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $\lambda > j + \mu$. If $f \in \mathscr{X}_\lambda^*$, then $(T-t)^{-\mu} D_x^j f \in L^\infty(\mathcal{C} \times (0,1])$.*

Proof. By the definition of $\mathscr{X}_\lambda^*$, it suffices to prove the result for $f(t,x,s) = f_1(t,x)\eta_1\left(\frac{T-t}{s}\right)$, $g(t,x,s) = g_1(t,x)\widetilde{\eta}_1\left(\frac{T-t}{s}\right)$ for some $f_1 \in \mathscr{X}_\lambda, g_1 \in \mathscr{X}_\mu, \eta_1, \widetilde{\eta}_1 \in \mathscr{Y}_0$.

In this case, $\Delta f(t,x,s) = \Delta f_1(t,x)\eta_1\left(\frac{T-t}{s}\right)$. By Lemma 2.3 (i), we have $\Delta f_1 \in \mathscr{X}_{\lambda-2}$, thus $\Delta f \in \mathscr{X}_{\lambda-2}^*$. We also have

$$\partial_t f(t,x,s) = \partial_t f_1(t,x)\eta_1\big((T-t)/s\big) + (T-t)^{-1} f_1(t,x)\eta_2\big((T-t)/s\big)$$

with $\eta_2(z) = -z\eta_1'(z) \in \mathscr{Y}_0$ (as $\eta_2 = 0$, $\eta_2' = 0$ in $[0,1] \cup [2,+\infty)$). By Lemma 2.3 (i), we have $\partial_t f_1 \in \mathscr{X}_{\lambda-1}$, $(T-t)^{-1} \in \mathscr{X}_{-1}$, $(T-t)^{-1} f_1 \in \mathscr{X}_{\lambda-1}$, thus $\partial_t f \in \mathscr{X}_{\lambda-1}^*$.

In this case, $(fg)(t,x,s) = (f_1 g_1)(t,x)(\eta_1\widetilde{\eta}_1)\left(\frac{T-t}{s}\right)$. By Lemma 2.3 (i), we have $f_1 g_1 \in \mathscr{X}_{\lambda+\mu}$, as $\mathscr{Y}_0$ is a ring we have $\eta_1\widetilde{\eta}_1 \in \mathscr{Y}_0$, thus $fg \in \mathscr{X}_{\lambda+\mu}^*$. This completes the proof of (i).

Assume that $\alpha_1, \cdots, \alpha_d \in \mathbb{Z}_{\geq 0}$ are such that $\alpha_1 + \cdots + \alpha_d = j$, then we have

$$(T-t)^{-\mu}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d} f(t,x,s) = (T-t)^{-\mu}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d} f_1(t,x)\eta_1((T-t)/s).$$

Then (ii) follows from Lemma 2.3 (ii) and $\eta_1 \in L^\infty([0,+\infty))$. □

Proof of Lemma 3.4. By Lemma 3.5 (i), we have $fg \in \mathscr{X}^*_{\lambda+\mu}$, $\Delta f \in \mathscr{X}^*_{\lambda-2}$, $\partial_t f \in \mathscr{X}^*_{\lambda-1}$, $\partial_t^2 f \in \mathscr{X}^*_{\lambda-2}$, thus $\Box f = -\partial_t^2 f + \Delta f \in \mathscr{X}^*_{\lambda-2}$. As a consequence, we have $\Box(fg) \in \mathscr{X}^*_{(\lambda+\mu)-2}$, $(\Box f)g \in \mathscr{X}^*_{(\lambda-2)+\mu}$, $f\Box g \in \mathscr{X}^*_{\lambda+(\mu-2)}$, hence $\partial^\alpha f \partial_\alpha g = (\Box(fg) - (\Box f)g - f\Box g)/2 \in \mathscr{X}^*_{\lambda+\mu-2}$.

Assume that $\alpha_0, \alpha_1, \cdots, \alpha_d \in \mathbb{Z}_{\geq 0}$ are such that $\alpha_0 + \cdots + \alpha_d = j$. By Lemma 3.5 (i), we have $\partial_t^{\alpha_0} f \in \mathscr{X}^*_{\lambda-\alpha_0}$. Then by Lemma 3.5 (ii) and $\lambda - \alpha_0 > j' + \mu$ (here $j' = \alpha_1 + \cdots + \alpha_d = j - \alpha_0$), we have

$$(T-t)^{-\mu}|\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d} f| \leq (T-t)^{-\mu}|D_x^{j'}\partial_t^{\alpha_0} f| \in L^\infty(\mathcal{C}\times(0,1]).$$

This completes the proof. □

Now we are in a position to prove Lemma 3.1.

Proof of Lemma 3.1. For $t \in [0,T), x \in \mathbb{R}^d, s \in (0,+\infty)$ and $N \in \mathbb{Z}_{\geq 0}$, let

$$\rho_{N*}(t,x,s) := \sum_{n=0}^{N} \rho_n(t,x)b^n + \eta\left(\frac{T-t}{s}\right)\rho_{N+1}(t,x)b^{N+1},$$
$$\phi_{N*}(t,x,s) := \sum_{n=0}^{N} \phi_n(t,x)b^n + \eta\left(\frac{T-t}{s}\right)\phi_{N+1}(t,x)b^{N+1}.$$

Then by (2.23), for all $N \in \mathbb{Z}_{\geq N_0-1}$, we have

$$\rho_*(t,x) = \rho_{N*}(t,x,T_{N+1}), \quad \phi_*(t,x) = \phi_{N*}(t,x,T_{N+1}), \quad \forall\, T-t \in [T_{N+1}, T_N],\ (t,x) \in \mathcal{C}.$$

Let

$$E_{N*} = \rho_{N*}^p + \rho_{N*}\partial^\alpha\phi_{N*}\partial_\alpha\phi_{N*} - b\Box\rho_{N*}, \quad J_{N*} = 2\partial^\alpha\rho_{N*}\partial_\alpha\phi_{N*} + \rho_{N*}\Box\phi_{N*}. \tag{3.9}$$

Then by (2.24), for all $N \in \mathbb{Z}_{\geq N_0-1}$, we have

$$E_*(t,x) = E_{N*}(t,x,T_{N+1}), \quad J_*(t,x) = J_{N*}(t,x,T_{N+1}), \quad \forall\, T-t \in [T_{N+1}, T_N],\ (t,x) \in \mathcal{C}.$$

Now (3.2) is reduced to the proof of

$$(T-t)^{-2k_N}(D^j E_{N*}, D^j J_{N*}) \in L^\infty(\mathcal{C}\times(0,1]), \quad \forall\, j \in \mathbb{Z}\cap[0,k_N],\ N \in \mathbb{Z}_{\geq N_0-1}. \tag{3.10}$$

For $t \in [0,T), x \in \mathbb{R}^d, s \in (0,+\infty)$, let

$$\rho_{N,n}(t,x,s) := \begin{cases} \rho_n(t,x) & n \in \mathbb{Z}\cap[0,N] \\ \rho_n^*(t,x,s) & n = N+1 \\ 0 & n \in \mathbb{Z}_{\geq N+2} \end{cases},$$
$$\phi_{N,n}(t,x,s) := \begin{cases} \phi_n(t,x) & n \in \mathbb{Z}\cap[0,N] \\ \phi_n^*(t,x,s) & n = N+1 \\ 0 & n \in \mathbb{Z}_{\geq N+2} \end{cases}.$$

where

$$\rho_n^*(t,x,s) := \eta((T-t)/s)\,\rho_n(t,x), \ \ \phi_n^*(t,x,s) := \eta((T-t)/s)\,\phi_n(t,x), \quad \forall\, n \in \mathbb{Z}_{\geq 0}. \tag{3.11}$$

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As $\rho_n \in \mathscr{X}_{\mu_n}$, $\phi_n \in \mathscr{X}_{\lambda_n}$ for all $n \in \mathbb{Z}_{\geq 0}$, we have $\rho_n^*, \rho_{N,n} \in \mathscr{X}_{\mu_n}^*$, $\phi_n^*, \phi_{N,n} \in \mathscr{X}_{\lambda_n}^*$ for all $n, N \in \mathbb{Z}_{\geq 0}$. For $t \in [0,T)$, $x \in \mathbb{R}^d$, $s \in (0,+\infty)$ and $N \in \mathbb{Z}_{\geq 0}$, we have

$$\rho_{N*}(t,x,s) := \sum_{n=0}^{N+1} \rho_{N,n}(t,x,s)b^n, \quad \phi_{N*}(t,x,s) := \sum_{n=0}^{N+1} \phi_{N,n}(t,x,s)b^n.$$

Then by (3.9), (2.5) and $\rho_{N,n}(t,x,s) = \rho_n(t,x)$ for $n \in \mathbb{Z} \cap [0,N]$, we have

$$E_{N*}(t,x,s) = \sum_{n=N+1}^{p(N+1)} E_{N,n}^*(t,x,s)b^n, \quad J_{N*}(t,x,s) = \sum_{n=N+1}^{2(N+1)} J_{N,n}^*(t,x,s)b^n, \tag{3.12}$$

with (note that $\rho_{N,n}(t,x,s) = 0$ for $n \in \mathbb{Z}_{\geq N+2}$)

$$\begin{aligned} E_{N,n}^* &= \sum_{n_1+\cdots+n_p=n} \rho_{N,n_1}\rho_{N,n_2}\cdots\rho_{N,n_p} + \sum_{n_1+n_2+n_3=n} \rho_{N,n_1}\partial^\alpha\phi_{N,n_2}\partial_\alpha\phi_{N,n_3} - \Box\rho_{N,n-1}, \\ J_{N,n}^* &= 2\sum_{n_1+n_2=n} \partial^\alpha\rho_{N,n_1}\partial_\alpha\phi_{N,n_2} + \sum_{n_1+n_2=n} \rho_{N,n_1}\Box\phi_{N,n_2}. \end{aligned}$$

By Lemma 3.4 (i), we have $E_{N,n}^* \in \mathscr{X}_{2n(\beta-1)-2p\beta/(p-1)}^*$, $J_{N,n}^* \in \mathscr{X}_{(2n+1)(\beta-1)-2p\beta/(p-1)}^*$, where we have used the facts that $\mu_{n_1} + \cdots + \mu_{n_p} = \mu_{n-1} - 2$ if $n_1 + \cdots + n_p = n$, $\mu_{n_1} + \lambda_{n_2} + \lambda_{n_3} - 2 = \mu_{n-1} - 2 = 2n(\beta-1) - 2p\beta/(p-1)$ if $n_1 + n_2 + n_3 = n$ and $\mu_{n_1} + \lambda_{n_2} - 2 = (2n-1)(\beta-1) - 2\beta/(p-1) - 2 = (2n+1)(\beta-1) - 2p\beta/(p-1)$if $n_1 + n_2 = n$.

If $n \geq N+1$ and $j \in \mathbb{Z} \cap [0, k_N]$, then we get by (3.1) that

$$\begin{aligned} &(2n+1)(\beta-1) - 2p\beta/(p-1) - j > 2n(\beta-1) - 2p\beta/(p-1) - j \\ &> 2N(\beta-1) - 2p\beta/(p-1) - j > 3k_N - j \geq 2k_N. \end{aligned}$$

Thus, by Lemma 3.4 (ii), we have $(T-t)^{-2k_N} D^j E_{N,n}^* \in L^\infty(\mathcal{C} \times (0,1])$ for $n \in \mathbb{Z} \cap [N+1, p(N+1)]$, and $(T-t)^{-2k_N} D^j J_{N,n}^* \in L^\infty(\mathcal{C} \times (0,1])$ for $n \in \mathbb{Z} \cap [N+1, 2(N+1)]$, which along with (3.12) implies (3.10). ◻

Next we prove Lemma 3.2.

Proof of Lemma 3.2. For $(t,x) \in [0,T) \times \mathbb{R}^d$, and any fixed $\{T_n\}_{n\geq 0}$ satisfying $T_n = T$ for $0 \leq n < N_0$ and $0 < T_n \leq T_{n-1}/4$ for $n \geq N_0$, we define ρ_*, ϕ_* by (2.23), then ((ρ_n^*, ϕ_n^*) is defined in (3.11))

$$\begin{aligned} \rho_*(t,x) &= \sum_{n=0}^\infty \rho_n^*(t,x,T_n)b^n, \quad \phi_*(t,x) = \sum_{n=0}^\infty \phi_n^*(t,x,T_n)b^n, \\ (\rho_* - \rho_0)(t,x) &= \sum_{n=1}^\infty \rho_n^*(t,x,T_n)b^n, \quad (\phi_* - \phi_0)(t,x) = \sum_{n=1}^\infty \phi_n^*(t,x,T_n)b^n. \end{aligned}$$

Recall that $\rho_n^* \in \mathscr{X}_{\mu_n}^*$ and $\phi_n^* \in \mathscr{X}_{\lambda_n}^*$ for all $n \in \mathbb{Z}_{\geq 0}$. By (2.22), we have

$$\begin{aligned} \lambda_n - j &= (2n-1)(\beta-1) - j > (n-1)(\beta-1) - j, \quad \forall\, n \in \mathbb{Z}_+, \\ \mu_n - j &= 2n(\beta-1) - \frac{2\beta}{p-1} - j > n(\beta-1) - \frac{2\beta}{p-1} - j, \quad \forall\, n \in \mathbb{Z}_+. \end{aligned}$$

Thus, by Lemma 3.4 (ii), for any $j, n \in \mathbb{Z}_{\geq 0}$, there exists a constant $B_{n,j} > 0$, which is independent of the sequence $\{T_n\}_{n\geq 0}$, such that for all $(t,x) \in \mathcal{C}$, we have

$$(T-t)^{j-(n-1)(\beta-1)}|D^j\phi_n^*(t,x,T_n)| + (T-t)^{\frac{2\beta}{p-1}+j-n(\beta-1)}|D^j\rho_n^*(t,x,T_n)| \leq B_{n,j},$$

which gives (recalling that $\eta((T-t)/T_n)) \neq 0$ implies $T-t \leq 2T_n$)

$$(T-t)^j|D^j\phi_n^*(t,x,T_n)| + (T-t)^{\frac{2\beta}{p-1}+j-\beta+1}|D^j\rho_n^*(t,x,T_n)| \leq B_{n,j}(2T_n)^{(n-1)(\beta-1)}.$$

Let $\widetilde{T}_n := T$ for $0 \leq n < N_0$ and for $n \geq N_0$ we let

$$\widetilde{B}_n := 2^n \max_{0\leq j\leq n-N_0} B_{n,j}, \qquad \widetilde{T}_n := \min\left(\widetilde{B}_n^{-1/[(n-1)(\beta-1)]}/2, \widetilde{T}_{n-1}/4\right).$$

Now we prove that $\{\widetilde{T}_n\}_{n\geq 0}$ is a desired sequence for Lemma 3.2.

Let $\{T_n\}_{n\geq 0}$ be such that $T_n = T$ for $0 \leq n < N_0$ and $0 < T_n \leq \min(\widetilde{T}_n, T_{n-1}/4)$ for $n \geq N_0$. Then $\widetilde{B}_n(2\widetilde{T}_n)^{(n-1)(\beta-1)} \leq 1$ for $n \geq N_0$. Fix $j \in \mathbb{Z}_{\geq 0}$. For any $(t,x) \in \mathcal{C}$, we have

$$\begin{aligned}
&(T-t)^j|D^j(\phi_* - \phi_0)(t,x)| + (T-t)^{\frac{2\beta}{p-1}+j-\beta+1}|D^j(\rho_* - \rho_0)(t,x)| \\
&\leq \sum_{n=1}^{\infty}\left((T-t)^j|D^j\phi_n^*(t,x,T_n)| + (T-t)^{\frac{2\beta}{p-1}+j-\beta+1}|D^j\rho_n^*(t,x,T_n)|\right) \\
&\leq \sum_{n=1}^{\infty} B_{n,j}(2T_n)^{(n-1)(\beta-1)} \leq \sum_{n=1}^{\infty} B_{n,j}(2\widetilde{T}_n)^{(n-1)(\beta-1)} \\
&\leq \sum_{n=1}^{N_0+j-1} B_{n,j}(2\widetilde{T}_n)^{(n-1)(\beta-1)} + \sum_{n=N_0+j}^{\infty} 2^{-n}\widetilde{B}_n(2\widetilde{T}_n)^{(n-1)(\beta-1)} \\
&\leq \sum_{n=1}^{N_0+j-1} B_{n,j}(2\widetilde{T}_n)^{(n-1)(\beta-1)} + \sum_{n=N_0+j}^{\infty} 2^{-n} \leq \sum_{n=1}^{N_0+j-1} B_{n,j}(2\widetilde{T}_n)^{(n-1)(\beta-1)} + 1,
\end{aligned}$$

which implies (3.3), as the right hand side is a finite constant independent of $(t,x) \in \mathcal{C}$. □

Finally, we prove Lemma 3.3.

Proof of Lemma 3.3. By (2.7), Lemma A.6 and Lemma 2.3 (iii), we obtain (3.4). It suffices to prove (3.5). By (2.7), we have

$$\partial_t\phi_0 = (T-t)^{-\beta}\left((\beta-1)\widehat{\phi}_0(Z) + Z\widehat{\phi}_0'(Z)\right), \quad \partial_r\phi_0 = (T-t)^{-\beta}\widehat{\phi}_0'(Z).$$

It follows from (2.12) that

$$\widehat{\phi}_0'(Z) = \frac{(\beta-1)\widehat{\phi}_0(Z)v(Z)}{1-Zv(Z)}, \quad Z\widehat{\phi}_0'(Z) + (\beta-1)\widehat{\phi}_0(Z) = \frac{(\beta-1)\widehat{\phi}_0(Z)}{1-Zv(Z)}. \tag{3.13}$$

Hence,

$$(T-t)^{\beta}\left(\partial_t\phi_0 - |D_x\phi_0|\right) = (\beta-1)\frac{\widehat{\phi}_0(Z)(1-|v(Z)|)}{1-Zv(Z)}. \tag{3.14}$$

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Since $\beta > 1$, $\widehat{\phi}_0(Z) > 0$, $v(Z) \in (-1,1)$, $Zv(Z) < 1$ for all $Z \in [0,+\infty)$ and $\widehat{\phi}_0, v \in C^\infty([0,+\infty))$, we know that the right hand side of (3.14) is strictly positive and continuous. Thus, there exists $\widetilde{c}_1 \in (0,T)$ such that

$$\inf_{Z\in[0,2]} (\beta-1)\frac{\widehat{\phi}_0(Z)(1-|v(Z)|)}{1-Zv(Z)} > \widetilde{c}_1.$$

On the other hand, by (2.7), we have $(T-t)^{2\beta/(p-1)}\rho_0 = \widehat{\rho}_0(Z)$. As $\widehat{\rho}(Z) > 0$ and $\widehat{\rho} \in C([0,+\infty))$, there exists $\widetilde{c}_2 \in (0,T)$ such that $\inf_{Z\in[0,2]} \widehat{\rho}_0(Z) > \widetilde{c}_2$. As a consequence, letting $\widetilde{c} := \min(\widetilde{c}_1,\widetilde{c}_2) \in (0,T)$, we have (3.5). □

4. The blow-up solution of nonlinear wave equation

Fix $T = 1$. Recall that $\mathcal{C} = \{(t,x) \in [0,T)\times\mathbb{R}^d : |x| < 2(T-t)\}$. Let $w_* \in C^\infty(\mathcal{C};\mathbb{R})$, $\Psi_* \in C^\infty(\mathcal{C};\mathbb{R})$ be such that both (2.28) and (2.29) hold; moreover, (2.27) also holds for E_*, J_* defined by (2.30).

4.1. Derivation of the error equation

We construct a blow-up solution u to $\Box u = |u|^{p-1}u$ of the form $u = (1+h)w_*\mathrm{e}^{\mathrm{i}\Phi_*}$, where h is complex-valued. First of all, we deduce the equation for the error h.

Lemma 4.1. *Assume that $u = (1+h)w_*\mathrm{e}^{\mathrm{i}\Phi_*}$ solves $\Box u = |u|^{p-1}u$. Then h satisfies*

$$\Box h + 2\,\mathrm{i}\,\partial^\alpha\Phi_*\partial_\alpha h + 2\frac{\partial^\alpha w_*}{w_*}\partial_\alpha h - (p-1)w_*^{p-1}h_{\mathrm{r}} = w_*^{p-1}\varphi_1(h) + \frac{E_* - \mathrm{i}\,J_*}{w_*}(1+h), \tag{4.1}$$

where $h_{\mathrm{r}} = \operatorname{Re} h = (h+\bar{h})/2$ and

$$\varphi_1(h) = (|1+h|^{p-1} - 1 - (p-1)h_{\mathrm{r}})(1+h) + (p-1)h_{\mathrm{r}}h = O(|h|^2).$$

The converse is also true.

Proof. This is a brute force computation. If $u = (1+h)w_*\mathrm{e}^{\mathrm{i}\Phi_*}$, then for any $\alpha \in \mathbb{Z}\cap[0,d]$, we have

$$\partial_\alpha u = \partial_\alpha h w_*\mathrm{e}^{\mathrm{i}\Phi_*} + (1+h)\partial_\alpha w_*\mathrm{e}^{\mathrm{i}\Phi_*} + \mathrm{i}(1+h)w_*\mathrm{e}^{\mathrm{i}\Phi_*}\partial_\alpha\Phi_*.$$

Hence,

$$\begin{aligned}
(\Box u)\mathrm{e}^{-\mathrm{i}\Phi_*} =& (\Box h + 2\mathrm{i}\partial_\alpha h\partial^\alpha\Phi_*)w_* + 2\partial_\alpha h\partial^\alpha w_*\\
&+ (1+h)(\Box w_* + 2\mathrm{i}\partial_\alpha w_*\partial^\alpha\Phi_* - w_*\partial^\alpha\Phi_*\partial_\alpha\Phi_* + \mathrm{i}\Box\Phi_* w_*)\\
\overset{(2.30)}{=\!=\!=}& (\Box h + 2\mathrm{i}\partial_\alpha h\partial^\alpha\Phi_*)w_* + 2\partial_\alpha h\partial^\alpha w_* + (1+h)(w_*^p + \mathrm{i}J_* - E_*),\\
|u|^{p-1}u =& |1+h|^{p-1}(1+h)w_*^p\mathrm{e}^{\mathrm{i}\Phi_*}.
\end{aligned}$$

By the definition of φ_1, we have $1 + (p-1)h_{\mathrm{r}} + h + \varphi_1(h) = (1+h)|1+h|^{p-1}$. Thus,

$$\begin{aligned}
\Box u = |u|^{p-1}u \iff& \Box h + 2\mathrm{i}\partial^\alpha\Phi_*\partial_\alpha h + 2\frac{\partial^\alpha w_*}{w_*}\partial_\alpha h + (1+h)\left(w_*^{p-1} + \frac{\mathrm{i}J_* - E_*}{w_*}\right)\\
&= \left(1 + (p-1)h_{\mathrm{r}} + h + \varphi_1(h)\right)w_*^{p-1}\\
\iff& \Box h + 2\mathrm{i}\partial^\alpha\Phi_*\partial_\alpha h + 2\frac{\partial^\alpha w_*}{w_*}\partial_\alpha h - (p-1)w_*^{p-1}h_{\mathrm{r}} = w_*^{p-1}\varphi_1(h) + \frac{E_* - \mathrm{i}J_*}{w_*}(1+h).
\end{aligned}$$

This completes the proof of Lemma 4.1. □

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We fix a bump function $\xi \in C_c^\infty(\mathbb{R}; [0,1])$ such that $\operatorname{supp}\xi \subset [-1,1]$ and $\xi_{[0,4/5]} = 1$. We define the vector fields X, Y by[8]

$$X(t,x) := D\Phi_*(t,x)\xi\left(\frac{3|x|}{5(T-t)}\right), \quad Y(t,x) := \frac{Dw_*(t,x)}{w_*(t,x)}\xi\left(\frac{3|x|}{5(T-t)}\right) \tag{4.2}$$

for $(t,x) \in [0,T) \times \mathbb{R}^d$. We also define the functions on $[0,T) \times \mathbb{R}^d$ by

$$V_*(t,x) := (p-1)w_*^{p-1}(t,x)\xi\left(\frac{3|x|}{5(T-t)}\right), \quad N_*(t,x) := \frac{E_* - \mathrm{i}\,J_*}{w_*}(t,x)\xi\left(\frac{3|x|}{4(T-t)}\right). \tag{4.3}$$

Then $X, Y \in C^\infty([0,T)\times\mathbb{R}^d; \mathbb{R}^{d+1})$ and $V_* \in C^\infty([0,T)\times\mathbb{R}^d; \mathbb{R})$, $N_* \in C^\infty([0,T)\times\mathbb{R}^d; \mathbb{C})$. Moreover, we have

$$\operatorname{supp}_x N_*(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \le 4(T-t)/3\}, \quad \forall\, t \in [0,T). \tag{4.4}$$

Let $c_0 \in (0,T)$ satisfy (2.29). Let $\mathcal{C}_1 := \{(t,x) \in [T-c_0,T) \times \mathbb{R}^d : |x| \le 4(T-t)/3\} \subset \mathcal{C}$. Using (2.28), (2.29) and (2.27), we have $X_0(t,x) > 0, V_*(t,x) > 0$ for all $(t,x) \in \mathcal{C}_1$. The following lemma gives more useful properties.

Lemma 4.2. *There exists a constant $M > 0$ such that*

$$(T-t)\left(\frac{|DX|}{X_0} + |Y| + \frac{|DV_*|}{V_*}\right) \le M, \tag{4.5}$$

$$\frac{1}{M}(T-t)^{-\beta} \le X_0 \le M(T-t)^{-\beta}, \quad \frac{1}{M}(T-t)^{-2\beta} \le V_* \le M(T-t)^{-2\beta} \tag{4.6}$$

on $\mathcal{C}_1$. Moreover, for any $j \in \mathbb{Z}_{\ge 0}$,

$$(T-t)^{\beta+j}|D^jX| + (T-t)^{1+j}|D^jY| + (T-t)^{2\beta+j}|D^jV_*| + (T-t)^{1+j}|D^jN_*| \in L^\infty(\mathcal{C}_1). \tag{4.7}$$

For any $j \in \mathbb{Z}_{\ge 0}$ and $\lambda > 0$, there exists a constant $M_{j,\lambda} > 0$ such that

$$|D^jN_*| \le M_{j,\lambda}(T-t)^\lambda \quad \text{on} \quad \mathcal{C}_1. \tag{4.8}$$

Proof. On $\mathcal{C}_1$, we have

$$X = D\Phi_*, \quad Y = Dw_*/w_*, \quad V_* = (p-1)w_*^{p-1}. \tag{4.9}$$

By (2.28), we have $(T-t)^\beta X_0 = (T-t)^\beta \partial_t\Phi_* \in L^\infty(\mathcal{C}_1)$ and $(T-t)^{2\beta}V_* \in L^\infty(\mathcal{C}_1)$. By (2.29), we have $(T-t)^\beta X_0 = (T-t)^\beta\partial_t\Phi_* \ge c_0$, $(T-t)^{2\beta}V_* = (p-1)(T-t)^{2\beta}w_*^{p-1} \ge (p-1)c_0^{p-1}$ on $\mathcal{C}_1$. This proves (4.6).

It follows from (2.28) that $(T-t)^{\beta+1}|DX| = (T-t)^{\beta+1}|D^2\Phi_*| \in L^\infty(\mathcal{C}_1)$, hence by $(T-t)^\beta X_0 \ge c_0$ on $\mathcal{C}_1$, we have $(T-t)|DX|/X_0 \in L^\infty(\mathcal{C}_1)$. Similarly, by using (2.28) and (4.6), we get $(T-t)(|Y| + |DV_*|/V_*) \in L^\infty(\mathcal{C}_1)$. This proves (4.5).

Next we prove (4.7) and (4.8). Recall the product rule: for smooth f, g and $(\alpha_0, \alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_{\ge 0}^{d+1}$, we have (see [24])

[8]Here we explain the notations to avoid ambiguities. For a smooth function $f(t,x)$, we denote the action of the vector field X on f by Xf, i.e., $Xf = X_\alpha\partial^\alpha f = X^\alpha\partial_\alpha f$, where $X_0 = \partial_t\Phi_*\xi$, $X^0 = -X_0$ and $X_j = X^j = \partial_j\Phi_*\xi$ for $j \in \mathbb{Z}\cap[1,d]$. The same clarification holds also for Y. Moreover, in (4.2), although $D\Phi_*(t,x)$ is only defined for $(t,x) \in \mathcal{C}$, we just simply let $X(t,x) = 0$ for $(t,x) \in ([0,T)\times\mathbb{R}^d)\setminus\mathcal{C}$, noting that $\xi(3|x|/(5(T-t))) = 0$ near the boundary of $\mathcal{C}$. The same clarification holds also for Y, and V_*, N_* in (4.3).

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$$\frac{\partial^{\alpha_0+\alpha_1+\cdots+\alpha_d}}{\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}}(fg)=\sum_{j=0}^{\alpha_0}\sum_{j_1=0}^{\alpha_1}\cdots\sum_{j_d=0}^{\alpha_d}\binom{\alpha_0}{j_0}\binom{\alpha_1}{j_1}\cdots\binom{\alpha_d}{j_d}$$
$$\times\frac{\partial^{j_0+j_1+\cdots+j_d}f}{\partial_t^{j_0}\partial_{x_1}^{j_1}\cdots\partial_{x_d}^{j_d}}\cdot\frac{\partial^{\alpha_0-j_0+\alpha_1-j_1+\cdots+\alpha_d-j_d}g}{\partial_t^{\alpha_0-j_0}\partial_{x_1}^{\alpha_1-j_1}\cdots\partial_{x_d}^{\alpha_d-j_d}}.$$

Hence,

$$|D^n(fg)|\lesssim_n\sum_{j=0}^{n}|D^jf||D^{n-j}g|,\quad |D_x^n(fg)|\lesssim_n\sum_{j=0}^{n}|D_x^jf||D_x^{n-j}g|,\quad\forall\, n\in\mathbb{Z}_{\geq 0},\tag{4.10}$$

$$|gD^nf|\lesssim_n|D^n(fg)|+\sum_{j=0}^{n-1}|D^jf||D^{n-j}g|,\quad\forall\, n\in\mathbb{Z}_+.\tag{4.11}$$

As $X=D\Phi_*$ on $\mathcal{C}_1$, we get by (2.28) that

$$(T-t)^{\beta+j}|D^jX|\in L^\infty(\mathcal{C}_1),\quad\forall\, j\in\mathbb{Z}_{\geq 0}.\tag{4.12}$$

Now we use the induction argument to prove that

$$(T-t)^{1+j}|D^jY|\in L^\infty(\mathcal{C}_1),\quad\forall\, j\in\mathbb{Z}_{\geq 0}.\tag{4.13}$$

By (4.5), we know that (4.13) holds for $j=0$. Assume that (4.13) holds for all $j\in\mathbb{Z}\cap[0,n-1]$ for some $n\in\mathbb{Z}_+$. Note that $Dw_*=w_*Y$ on $\mathcal{C}_1$, hence by (4.11) we have

$$|w_*D^nY|\lesssim_n|D^n(Dw_*)|+\sum_{j=1}^{n}|D^jw_*||D^{n-j}Y|\quad\text{on}\quad\mathcal{C}_1.$$

Using (2.29), (2.28) and the induction assumption, we obtain

$$\begin{aligned}(T-t)^{1+n}|D^nY|&\leq\frac{1}{c_0}(T-t)^{1+n+\frac{2\beta}{p-1}}|w_*D^nY|\\&\lesssim_n(T-t)^{1+n+\frac{2\beta}{p-1}}|D^{1+n}w_*|+\sum_{j=1}^{n}(T-t)^{\frac{2\beta}{p-1}+j}|D^jw_*|(T-t)^{1+n-j}|D^{n-j}Y|\in L^\infty(\mathcal{C}_1).\end{aligned}$$

This proves (4.13).

Now we prove that

$$(T-t)^{\frac{2\beta}{p-1}m+j}\left|D^j(w_*^m)\right|\in L^\infty(\mathcal{C}_1),\quad\forall\, m\in\mathbb{Z}_+,\ \forall\, j\in\mathbb{Z}_{\geq 0}.\tag{4.14}$$

By (2.28), we know that (4.14) holds for $m=1$. We assume that (4.14) holds for $m-1$, where $m\in\mathbb{Z}\cap[2,+\infty)$. By (4.10), for $j\in\mathbb{Z}_{\geq 0}$ we have

$$\left|D^j(w_*^m)\right|=\left|D^j(w_*^{m-1}w_*)\right|\lesssim_j\sum_{i=0}^{j}\left|D^i(w_*^{m-1})\right|\left|D^{j-i}w_*\right|,$$

which gives

$$\begin{aligned}(T-t)^{\frac{2\beta}{p-1}m+j}\left|D^j(w_*^m)\right|&\lesssim_j\sum_{i=0}^{j}(T-t)^{\frac{2\beta}{p-1}(m-1)+i}\left|D^i(w_*^{m-1})\right|(T-t)^{\frac{2\beta}{p-1}+j-i}\left|D^{j-i}w_*\right|\\&\in L^\infty(\mathcal{C}_1).\end{aligned}$$

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By the induction argument, we have (4.14). Letting $m = p - 1$ in (4.14), we get (using (4.9))

$$(T-t)^{2\beta+j}|D^j V_*| \in L^\infty(\mathcal{C}_1), \quad \forall\, j \in \mathbb{Z}_{\geq 0}. \tag{4.15}$$

Finally, we estimate N_*. Let $\widetilde{\xi}(t,x) := \xi(3|x|/(4(T-t)))$. Then by Lemma 2.3 (iii), we have $(T-t)^j D^j \widetilde{\xi} \in L^\infty(\mathcal{C})$. Let $\widetilde{N}_* := (E_* - \mathrm{i}J_*)\widetilde{\xi}$, then $N_* = \widetilde{N}_*/w_*$. By (4.10), we have

$$\left|D^j \widetilde{N}_*\right| \lesssim_j \sum_{i=0}^{j} \left|D^i(E_* - \mathrm{i}J_*)\right| \left|D^{j-i}\widetilde{\xi}\right|,$$

hence by (2.27), for all $\lambda > 0$ and $j \in \mathbb{Z}_{\geq 0}$ we have

$$(T-t)^{-\lambda}\left|D^j \widetilde{N}_*\right| \lesssim_j \sum_{i=0}^{j} (T-t)^{-(\lambda+j-i)}\left|D^i(E_* - \mathrm{i}J_*)\right| (T-t)^{j-i}\left|D^{j-i}\widetilde{\xi}\right| \in L^\infty(\mathcal{C}). \tag{4.16}$$

Now we use the induction argument to prove that

$$(T-t)^{-\lambda}|D^j N_*| \in L^\infty(\mathcal{C}_1), \quad \forall\, j \in \mathbb{Z}_{\geq 0},\ \forall\, \lambda > 0. \tag{4.17}$$

For $j = 0$, (4.17) follows from (4.16) and (2.29). Assume that (4.17) holds for all $j \in \mathbb{Z} \cap [0, n-1]$ for some $n \in \mathbb{Z}_+$. As $\widetilde{N}_* = w_* N_*$, we get by (4.11) that

$$|w_* D^n N_*| \lesssim_n \left|D^n \widetilde{N}_*\right| + \sum_{j=1}^{n} |D^j w_*||D^{n-j} N_*| \quad \text{on} \quad \mathcal{C}_1.$$

Using (2.27), (2.28), (2.29) and the induction assumption, for any $\lambda > 0$ we obtain

$$\begin{aligned}
(T-t)^{-\lambda}|D^n N_*| &\leq \frac{1}{c_0}(T-t)^{-\lambda+\frac{2\beta}{p-1}}|w_* D^n N_*| \\
\lesssim_n (T-t)^{\frac{2\beta}{p-1}}(T-t)^{-\lambda}\left|D^n \widetilde{N}_*\right| &+ \sum_{j=1}^{n} (T-t)^{\frac{2\beta}{p-1}+j}\left|D^j w_*\right| (T-t)^{-(\lambda+j)}\left|D^{n-j} N_*\right| \in L^\infty(\mathcal{C}_1).
\end{aligned}$$

This proves (4.17) for $j = n$. By induction, we have (4.17), which is equivalent to (4.8).

Taking $\lambda = 1$ in (4.8), we get

$$(T-t)^{1+j}|D^j N_*| \leq T^{j+2}(T-t)^{-1}|D^j N_*| \in L^\infty(\mathcal{C}_1), \quad \forall\, j \in \mathbb{Z}_{\geq 0}. \tag{4.18}$$

Therefore, (4.7) follows from (4.12), (4.13), (4.15) and (4.18). ◻

4.2. *Energy estimates for the linearized wave equation*

Lemma 4.3. *Let* $T_* \in (0, c_0)$ *and* $h \in C_c^\infty([T - T_*, T) \times \mathbb{R}^d; \mathbb{C})$ *be such that* $\operatorname{supp}_x h(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \leq 4(T-t)/3\}$ *for all* $t \in [T - T_*, T)$. *We define the linear operator*

$$\mathcal{L}h := \Box h + 2\,\mathrm{i}\,Xh + 2Yh - V_* h_{\mathrm{r}}, \tag{4.19}$$

where $h_{\mathrm{r}} = (h + \bar{h})/2$ *and energy functionals*

$$E_0[h](t) := \frac{1}{2}\int_{\mathbb{R}^d} \left(|Dh(t,x)|^2 + V_*(t,x)|h_{\mathrm{r}}(t,x)|^2\right) \mathrm{d}x, \quad \forall\, t \in [T - T_*, T), \tag{4.20}$$

$$E_j[h] := E_0[D_x^j h], \quad \forall\, j \in \mathbb{Z}_+. \tag{4.21}$$

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Then there exist positive constants $M_1 > 1$ and $\{C_j\}_{j\in\mathbb{Z}_{\geq 0}}$ such that

$$\sqrt{E_j[h](t)} \leq C_j \int_t^T \left(\frac{T-t}{T-s}\right)^{M_1} \sum_{i=0}^{j} \frac{\left\|D_x^i \mathcal{L}h(s)\right\|_{L^2}}{(T-s)^{(j-i)\beta}} \mathrm{d}s, \quad \forall\, t \in [T-T_*, T),\ \forall\, j \in \mathbb{Z}_{\geq 0}. \tag{4.22}$$

Proof. Let $T_* \in (0, c_0)$ and $h \in C_c^\infty([T-T_*, T) \times \mathbb{R}^d; \mathbb{C})$ be such that

$$\operatorname{supp}_x h(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \leq 4(T-t)/3\}, \quad \forall\, t \in [T-T_*, T).$$

We define the energy momentum tensor $T[h]$ by

$$T[h]_{\mu\nu} := \operatorname{Re}\left(\partial_\mu h \overline{\partial_\nu h}\right) - \frac{1}{2} m_{\mu\nu}\left(\partial^\alpha h \overline{\partial_\alpha h} + V_* h_{\mathrm{r}}^2\right), \quad \forall\, \mu, \nu \in \mathbb{Z} \cap [0,d], \tag{4.23}$$

where we have used the Einstein's convention in $\partial^\alpha h \overline{\partial_\alpha h}$. Then we have

$$E_0[h](t) = \int_{\mathbb{R}^d} T[h]_{00}(t,x)\, \mathrm{d}x, \quad \forall\, t \in [T-T_*, T). \tag{4.24}$$

We define

$$P_\mu^X[h] := T[h]_{\mu\nu} X^\nu, \quad \forall\, \mu \in \mathbb{Z} \cap [0,d]. \tag{4.25}$$

Let's first claim that there exists a constant $\widetilde{c}_0 > 0$ such that

$$P_0^X[h] \leq \widetilde{c}_0 T[h]_{00} X^0 \leq 0 \text{ on } \mathcal{C}_* := \{(t,x) \in [T-T_*, T) \times \mathbb{R}^d : |x| \leq 4(T-t)/3\}; \tag{4.26}$$

and there exists a constant $C_* > 0$ such that

$$\left|\partial^\mu P_\mu^X[h]\right| \leq C_*(T-t)^{-\beta}\left((T-t)^{-1} T[h]_{00} + \sqrt{T[h]_{00}}|\mathcal{L}h|\right) \quad \text{on} \quad \mathcal{C}_*, \tag{4.27}$$

and moreover, for all $j \in \mathbb{Z}_{\geq 0}$,

$$\left\|D_x^j(\mathcal{L}D_x h - D_x \mathcal{L}h)\right\|_{L_x^2} \lesssim_j \sum_{i=0}^{j} (T-t)^{-\beta-1+i-j}\sqrt{E_i[h](t)} \quad \forall\, t \in [T-T_*, T), \tag{4.28}$$

where the implicit constants only depend on X, Y, V_*, N_* (and they are independent of h).

Now we prove (4.22) by the induction argument. We first consider $j = 0$. For all $t \in [T-T_*, T)$, by (4.24), $X^0 = -X_0$, $\operatorname{supp}_x h(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \leq 4(T-t)/3\}$, (4.26) and (4.6), we have

$$E_0[h](t) \leq \int_{\mathbb{R}^d} \frac{-P_0^X[h](t,x)}{\widetilde{c}_0 X_0(t,x)}\, \mathrm{d}x \leq \frac{M}{\widetilde{c}_0}(T-t)^\beta \int_{\mathbb{R}^d} -P_0^X[h](t,x)\, \mathrm{d}x. \tag{4.29}$$

Let

$$\widetilde{E}_0[h](t) = \int_{\mathbb{R}^d} -P_0^X[h](t,x)\, \mathrm{d}x \geq 0, \quad \forall\, t \in [T-T_*, T).$$

By the divergence theorem (recall that $\partial^0 = -\partial_0 = -\partial_t$), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_0[h](t) = \int_{\mathbb{R}^d} \partial^0 P_0^X[h](t,x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \partial^\mu P_\mu^X[h](t,x)\, \mathrm{d}x, \quad \forall\, t \in [T-T_*, T).$$

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Using (4.27), (4.24) and Cauchy's inequality, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_0[h](t)\right| \le C_*(T-t)^{-\beta}\left((T-t)^{-1}E_0[h](t)+\sqrt{E_0[h](t)}\|\mathcal{L}h(t)\|_{L^2_x}\right), \quad \forall\, t\in[T-T_*,T).$$

Hence, by (4.29) and $h\in C_c^\infty([T-T_*,T)\times\mathbb{R}^d;\mathbb{C})$, for all $t\in[T-T_*,T)$ we have

$$\begin{aligned}E_0[h](t) &\le \frac{M}{\widetilde{c}_0}(T-t)^{\beta}\widetilde{E}_0[h](t)\le \frac{M}{\widetilde{c}_0}(T-t)^{\beta}\int_t^T\left|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_0[h](s)\right|\mathrm{d}s\\ &\le \frac{MC_*}{\widetilde{c}_0}(T-t)^{\beta}\int_t^T(T-s)^{-\beta}\left((T-s)^{-1}E_0[h](s)+\sqrt{E_0[h](s)}\|\mathcal{L}h(s)\|_{L^2_x}\right)\mathrm{d}s.\end{aligned}$$

By Grönwall's lemma, we have

$$\sqrt{E_0[h](t)}\le\frac{MC_*}{2\widetilde{c}_0}\int_t^T\left(\frac{T-t}{T-s}\right)^{\frac{MC_0/\widetilde{c}_0+\beta}{2}}\|\mathcal{L}h(s)\|_{L^2_x}\,\mathrm{d}s,\quad\forall\,t\in[T-T_*,T).$$

Letting $M_1:=\frac{MC_*/\widetilde{c}_0+\beta}{2}>0$, we know that (4.22) holds for $j=0$.

Let $n\in\mathbb{Z}_+$. We assume that (4.22) holds for all $j\in\mathbb{Z}\cap[0,n-1]$. Then by (4.22) for $j=n-1$ and (4.28), for $t\in[T-T_*,T)$ we have (also using (4.20) and (4.21))

$$\begin{aligned}\sqrt{E_n[h](t)}&=\sqrt{E_{n-1}[D_xh](t)}\lesssim_n\int_t^T\left(\frac{T-t}{T-s}\right)^{M_1}\sum_{j=0}^{n-1}\frac{\|D_x^j\mathcal{L}D_xh(s)\|_{L^2_x}}{(T-s)^{(n-1-j)\beta}}\,\mathrm{d}s\\ &\lesssim_n\int_t^T\left(\frac{T-t}{T-s}\right)^{M_1}\sum_{j=0}^{n-1}\frac{\|D_x^{j+1}\mathcal{L}h(s)\|_{L^2_x}+\sum_{i=0}^j(T-s)^{-\beta-1+i-j}\sqrt{E_i[h](s)}}{(T-s)^{(n-1-j)\beta}}\,\mathrm{d}s\\ &\lesssim_n\int_t^T\left(\frac{T-t}{T-s}\right)^{M_1}\sum_{j=1}^{n}\frac{\|D_x^j\mathcal{L}h(s)\|_{L^2_x}}{(T-s)^{(n-j)\beta}}\,\mathrm{d}s+I_n(t),\end{aligned}$$

where

$$I_n(t):=\sum_{j=0}^{n-1}\sum_{i=0}^{j}\int_t^T\left(\frac{T-t}{T-s}\right)^{M_1}(T-s)^{-1+i-j-(n-j)\beta}\sqrt{E_i[h](s)}\,\mathrm{d}s.$$

For $T-T_*\le t<s<T$, $j\ge i\ge0$ we have $0<T-s<T_*<c_0<T=1$ and $(T-s)^{-1+i-j-(n-j)\beta}=(T-s)^{-1-(n-i)\beta+(j-i)(\beta-1)}\le(T-s)^{-1-(n-i)\beta}$ (as $\beta>1$). Then

$$I_n(t)\le n\sum_{i=0}^{n-1}\int_t^T\left(\frac{T-t}{T-s}\right)^{M_1}(T-s)^{-1-(n-i)\beta}\sqrt{E_i[h](s)}\,\mathrm{d}s.$$

Using the induction assumption and Fubini's theorem, we have

$$\begin{aligned}I_n(t)&\lesssim_n\sum_{i=0}^{n-1}\sum_{j=0}^{i}\int_t^T\frac{((T-t)/(T-s))^{M_1}}{(T-s)^{1+(n-i)\beta}}\int_s^T\left(\frac{T-s}{T-\tau}\right)^{M_1}\frac{\|D_x^j\mathcal{L}h(\tau)\|_{L^2_x}}{(T-\tau)^{(i-j)\beta}}\,\mathrm{d}\tau\,\mathrm{d}s\\ &=\sum_{i=0}^{n-1}\sum_{j=0}^{i}\int_t^T\left(\frac{T-t}{T-\tau}\right)^{M_1}\frac{\|D_x^j\mathcal{L}h(\tau)\|_{L^2_x}}{(T-\tau)^{(i-j)\beta}}\int_t^\tau\frac{\mathrm{d}s}{(T-s)^{1+(n-i)\beta}}\,\mathrm{d}\tau\end{aligned}$$

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$$\leq \sum_{i=0}^{n-1}\sum_{j=0}^{i}\int_t^T \left(\frac{T-t}{T-\tau}\right)^{M_1} \frac{\|D_x^j \mathcal{L}h(\tau)\|_{L_x^2}}{(T-\tau)^{(i-j)\beta}} \frac{1}{(T-\tau)^{(n-i)\beta}}\,\mathrm{d}\tau$$
$$\leq n\sum_{j=0}^{n-1}\int_t^T \left(\frac{T-t}{T-\tau}\right)^{M_1} \frac{\|D_x^j \mathcal{L}h(\tau)\|_{L_x^2}}{(T-\tau)^{(n-j)\beta}}\,\mathrm{d}\tau.$$

Therefore, we obtain (4.22) for $j=n$. This proves (4.22) for all $j\in\mathbb{Z}_{\geq 0}$.

Thus, it remains to prove (4.26), (4.27) and (4.28). We start with

$$P_0^X[h] = T[h]_{0\nu}X^\nu = T[h]_{00}X^0 + \sum_{i=1}^d T[h]_{0i}X^i.$$

On $\mathcal{C}_* \subset \mathcal{C}_1$, by (4.2), we have $-X^0 = X_0 = \partial_t\Phi_*$ and $X^i = X_i = \partial_i\Phi_*$ for $i\in\mathbb{Z}\cap[1,d]$, hence by (4.23) and Cauchy's inequality,

$$\left|\sum_{i=1}^d T[h]_{0i}X^i\right| \leq \sum_{i=1}^d |\partial_t h||\partial_i h||\partial_i\Phi_*| \leq |\partial_t h||D_x h||D_x\Phi_*| \leq \frac{|\partial_t h|^2+|D_x h|^2}{2}|D_x\Phi_*|$$
$$\leq T[h]_{00}|D_x\Phi_*|.$$

On the other hand, by (2.28) and (2.29), there exists a constant $\widetilde{c_0}\in(0,1)$ such that

$$\partial_t\Phi_* - |D_x\Phi_*| \geq c_0(T-t)^{-\beta} \geq \widetilde{c_0}\partial_t\Phi_* > 0 \quad \text{on} \quad \mathcal{C}_1.$$

Thus, we have $|X| \leq |\partial_t\Phi_*| + |D_x\Phi_*| \leq 2\partial_t\Phi_* = 2X_0$ and

$$\left|\sum_{i=1}^d T[h]_{0i}X^i\right| \leq T[h]_{00}|D_x\Phi_*| \leq T[h]_{00}(1-\widetilde{c_0})\partial_t\Phi_* = T[h]_{00}(1-\widetilde{c_0})X_0,$$

hence

$$P_0^X[h] \leq T[h]_{00}X^0 + T[h]_{00}(1-\widetilde{c_0})X_0 = \widetilde{c_0}T[h]_{00}X^0 \leq 0 \quad \text{on} \quad \mathcal{C}_*.$$

This proves (4.26).

As for (4.27), we compute

$$\begin{aligned}\partial^\mu T[h]_{\mu\nu} &= \mathrm{Re}\left(\Box h\overline{\partial_\nu h}\right) + \mathrm{Re}\left(\partial_\mu h\partial^\mu\overline{\partial_\nu h}\right) - \frac{1}{2}\partial_\nu\left(\partial^\alpha h\overline{\partial_\alpha h} + V_* h_{\mathrm{r}}^2\right)\\ &= \mathrm{Re}\left(\Box h\overline{\partial_\nu h}\right) + \frac{1}{2}\mathrm{Re}\,\partial_\nu\left(\partial_\mu h\overline{\partial^\mu h}\right) - \frac{1}{2}\partial_\nu\left(\partial^\alpha h\overline{\partial_\alpha h}\right) - h_{\mathrm{r}}\partial_\nu h_{\mathrm{r}}V_* - \frac{1}{2}h_{\mathrm{r}}^2\partial_\nu V_*\\ &= \mathrm{Re}\left(\Box h\overline{\partial_\nu h}\right) - h_{\mathrm{r}}\partial_\nu h_{\mathrm{r}}V_* - \frac{1}{2}h_{\mathrm{r}}^2\partial_\nu V_*\end{aligned}$$

for $\nu\in\mathbb{Z}\cap[0,d]$. Hence,

$$\begin{aligned}\partial^\mu P_\mu^X[h] &= T[h]_{\mu\nu}\partial^\mu X^\nu + (\partial^\mu T[h]_{\mu\nu})X^\nu\\ &= T[h]_{\mu\nu}(\pi^X)^{\mu\nu} + \mathrm{Re}\left(\Box h X^\nu\overline{\partial_\nu h}\right) - h_{\mathrm{r}}X^\nu\partial_\nu h_{\mathrm{r}}V_* - \frac{1}{2}h_{\mathrm{r}}^2X^\nu\partial_\nu V_*\\ &= T[h]_{\mu\nu}(\pi^X)^{\mu\nu} + \mathrm{Re}\left(\Box h\overline{Xh}\right) - V_*h_{\mathrm{r}}Xh_{\mathrm{r}} - \frac{1}{2}h_{\mathrm{r}}^2XV_*,\end{aligned}$$

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where we have used the fact that X^ν is real-valued for $\nu \in \mathbb{Z} \cap [0, d]$, $Xh = X^\nu \partial_\nu h$ and we define

$$(\pi^X)^{\mu\nu} := \frac{\partial^\mu X^\nu + \partial^\nu X^\mu}{2}, \quad \forall\, \mu, \nu \in \mathbb{Z} \cap [0, d]. \tag{4.30}$$

Hence, it follows from (4.19) that

$$\begin{aligned}\partial^\mu P^X_\mu[h] &= T[h]_{\mu\nu}(\pi^X)^{\mu\nu} + \mathrm{Re}\left(\Box h\overline{Xh}\right) - V_* h_{\mathrm{r}} X h_{\mathrm{r}} - \frac{1}{2}h_{\mathrm{r}}^2 XV_* \\ &= T[h]_{\mu\nu}(\pi^X)^{\mu\nu} - \frac{1}{2}h_{\mathrm{r}}^2 XV_* + \mathrm{Re}\left(\mathcal{L}h\overline{Xh}\right) - 2\,\mathrm{Re}\left(Yh\overline{Xh}\right).\end{aligned} \tag{4.31}$$

By (4.23), we have $|T[h]_{\mu\nu}| \leq T[h]_{00}$ for all $\mu, \nu \in \mathbb{Z} \cap [0, d]$ and $|Dh|^2 \leq 2T[h]_{00}$. Thus, by (4.31), $|X| \leq 2X_0$, (4.5) and (4.6), on $\mathcal{C}_*$ we have (note that $|XV_*| \leq |X||DV_*|$, $|Xh| \leq |X||Dh|$, $|Yh| \leq |Y||Dh|$, see footnote 8)

$$\begin{aligned}\left|\partial^\mu P^X_\mu[h]\right| &\lesssim T[h]_{00}|DX| + T[h]_{00}X_0\frac{|DV_*|}{V_*} + |\mathcal{L}h||X||Dh| + |Y||Dh||X||Dh| \\ &\lesssim T[h]_{00}(T-t)^{-1}X_0 + |\mathcal{L}h|X_0\sqrt{T[h]_{00}} \\ &\lesssim (T-t)^{-\beta}\left((T-t)^{-1}T[h]_{00} + \sqrt{T[h]_{00}}|\mathcal{L}h|\right),\end{aligned}$$

which gives (4.27).

Finally, we prove (4.28). By (4.19), we have

$$\mathcal{L}D_x h - D_x\mathcal{L}h = -2\mathrm{i}D_x X^\alpha \partial_\alpha h - 2D_x Y^\alpha \partial_\alpha h + D_x V_* \cdot h_{\mathrm{r}}.$$

Let $j \in \mathbb{Z}_{\geq 0}$, by (4.10) and (4.7), for any $t \in [T - T_*, T)$ we have

$$\begin{aligned}\|D_x^j(D_x X^\alpha \partial_\alpha h)(t)\|_{L^2_x} &\lesssim_j \sum_{i=0}^{j}\|D_x^{j-i}D_x X(t)\|_{L^\infty_x}\|DD_x^i h(t)\|_{L^2_x} \\ &\lesssim_j \sum_{i=0}^{j}(T-t)^{-\beta-1+i-j}\sqrt{E_i[h](t)}.\end{aligned}$$

Similarly, we have (recalling $\beta > 1$)

$$\|D_x^j(D_x Y^\alpha \partial_\alpha h)(t)\|_{L^2_x} \lesssim_j \sum_{i=0}^{j}(T-t)^{-2+i-j}\sqrt{E_i[h](t)} \lesssim_j \sum_{i=0}^{j}(T-t)^{-\beta-1+i-j}\sqrt{E_i[h](t)}.$$

By (4.6) and (4.7), we have

$$\begin{aligned}\|D_x^j(D_x V_* \cdot h_r)\|_{L^2_x} &\lesssim_j \sum_{i=0}^{j}\|D_x^{j-i+1}V_*/\sqrt{V_*}\|_{L^\infty_x}\|\sqrt{V_*}D_x^i h_{\mathrm{r}}\|_{L^2_x} \\ &\lesssim_j \sum_{i=0}^{j}(T-t)^{-\beta-1+i-j}\sqrt{E_i[h](t)}.\end{aligned}$$

Hence, we get (4.28). ◻

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4.3. Solving the error equation

Lemma 4.4. *There exists a constant $c_2 \in (0, c_0)$ that depends only on X, Y, V_*, N_* such that for any $f \in C_c^\infty([T - c_0, T) \times \mathbb{R}^d; \mathbb{C})$ satisfying $|D_x^j f| \le |D_x^j N_*|$ for all $j \in \mathbb{Z}_{\ge 0}$, there is a solution $h \in C_c^\infty([T - c_2, T) \times \mathbb{R}^d; \mathbb{C})$ to the error equation*

$$\Box h + 2\,\mathrm{i}\, Xh + 2Yh - V_* h_{\mathrm{r}} - \frac{1}{p-1} V_* \varphi_1(h) - N_* h = f. \tag{4.32}$$

Moreover, $\operatorname{supp}_x h(t, \cdot) \subset \{x \in \mathbb{R}^d : |x| \le 4(T - t)/3\}$ *for all $t \in [T - c_2, T)$, and there exists a constant $C_\Box > 0$ that depends only on X, Y, V_*, N_* ($C_\Box$ does not depend on f) such that*

$$|\Box h(t, x)| \le C_\Box, \quad \forall\, t \in [T - c_2, T),\ \forall\, x \in \mathbb{R}^d, \tag{4.33}$$

and for any $j \in \mathbb{Z}_{\ge 0}$, $\lambda > 0$, there exists a constant $C_{j,\lambda} > 0$ that depends only on X, Y, V_*, N_* ($C_{j,\lambda}$ does not depend on f) such that

$$|D_x^j h(t, x)| + |\partial_t D_x^j h(t, x)| \le C_{j,\lambda}(T - t)^\lambda, \quad \forall\, t \in [T - c_2, T),\ \forall\, x \in \mathbb{R}^d. \tag{4.34}$$

The proof is based on the following lemma.

Lemma 4.5. *Let $f \in C_c^\infty([T - c_0, T) \times \mathbb{R}^d; \mathbb{C})$ be such that $|D_x^j f| \le |D_x^j N_*|$ for all $j \in \mathbb{Z}_{\ge 0}$. Let $T_* \in (0, c_0)$. Assume that $h \in C_c^\infty([T - T_*, T) \times \mathbb{R}^d; \mathbb{C})$ solves (4.32) on $[T - T_*, T) \times \mathbb{R}^d$; moreover,* $\operatorname{supp}_x h(t, \cdot) \subset \{x \in \mathbb{R}^d : |x| \le 4(T - t)/3\}$ *for all $t \in [T - T_*, T)$ and*

$$\|h(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \le (T - t)^{2\beta - 1}, \quad \forall\, t \in [T - T_*, T). \tag{4.35}$$

Then there exists a constant $C_\Box > 0$ that depends only on X, Y, V_*, N_* ($C_\Box$ does not depend on f, T_*) such that

$$|\Box h(t, x)| \le C_\Box, \quad \forall\, t \in [T - T_*, T),\ \forall\, x \in \mathbb{R}^d, \tag{4.36}$$

and for any $j \in \mathbb{Z}_{\ge 0}$, $\lambda > 0$, there exists a constant $C_{j,\lambda} > 0$ that depends only on X, Y, V_*, N_* ($C_{j,\lambda}$ does not depend on f, T_*) such that

$$|D_x^j h(t, x)| + |\partial_t D_x^j h(t, x)| \le C_{j,\lambda}(T - t)^\lambda, \quad \forall\, t \in [T - T_*, T),\ \forall\, x \in \mathbb{R}^d. \tag{4.37}$$

Now we present the proof of Lemma 4.4

Proof of Lemma 4.4. Let $f \in C_c^\infty([T - c_0, T) \times \mathbb{R}^d; \mathbb{C})$ be such that $|D_x^j f| \le |D_x^j N_*|$ for all $j \in \mathbb{Z}_{\ge 0}$. We assume that $\varepsilon \in (0, c_0)$ satisfies $f(t, x) = 0$ for all $(t, x) \in (T - \varepsilon, T) \times \mathbb{R}^d$. By the standard local well-posedness theory (Theorem 6.4.11 in [25]), there is a unique local solution $h \in C^\infty((T - T_+, T) \times \mathbb{R}^d; \mathbb{C})$ to (4.32) with $(h, \partial_t h)|_{t = T - \varepsilon/2} = (0, 0)$, where $\varepsilon < T_+ \le c_0$ corresponds to the left life span of h; moreover, if $T_+ < c_0$, then

$$\limsup_{t \downarrow T - T_+} \|h(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} = +\infty. \tag{4.38}$$

By the uniqueness and $f(t, x) = 0$ for all $(t, x) \in (T - \varepsilon, T) \times \mathbb{R}^d$, we have $h(t, x) = 0$ for all $(t, x) \in (T - \varepsilon, T) \times \mathbb{R}^d$. Moreover, by $|f| \le |N_*|$, (4.4) and the finite speed of propagation, we have

$$\operatorname{supp}_x h(t, \cdot) \subset \{x \in \mathbb{R}^d : |x| \le 4(T - t)/3\}, \quad \forall\, t \in [T - T_+, T).$$

Let $c_2 \in (0, c_0)$ be such that $C_{0,2\beta} \cdot c_2 < 1/2$, where $C_{0,2\beta} > 0$ is given by (4.37). Note that c_2 is independent of f and T_*. We claim that $T_+ \geq c_2$. We assume in contrary that $\varepsilon < T_+ < c_2$. Let

$$\mathscr{E} := \left\{T_0 \in (0, T_+) : \|h(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq (T-t)^{2\beta-1} \text{ for all } t \in [T - T_0, T)\right\}. \tag{4.39}$$

Then $(0, \varepsilon) \subset \mathscr{E}$. Let $T_s := \sup \mathscr{E} \in [\varepsilon, T_+]$. By (4.38), we have $T_s < T_+$, hence $T_s \in \mathscr{E}$ and $T_s < T_+ < c_2$. By (4.37), we have

$$|h(t,x)| \leq C_{0,2\beta}(T-t)^{2\beta} = C_{0,2\beta}(T-t)(T-t)^{2\beta-1} \leq C_{0,2\beta}c_2(T-t)^{2\beta-1} < \frac{1}{2}(T-t)^{2\beta-1}$$

for all $t \in [T - T_s, T) \subset [T - c_2, T)$. Thus, by the continuity we have $T_s + \delta \in \mathscr{E}$ for some $\delta > 0$. This contradicts with $T_s = \sup \mathscr{E}$. Therefore, $T_+ \geq c_2$ and $\|h(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq (T-t)^{2\beta-1}$ for all $t \in [T - c_2, T)$. Now Lemma 4.5 (letting $T_* = c_2$) implies Lemma 4.4. ◻

Let's complete the proof of Lemma 4.5.

Proof of Lemma 4.5. Assume that h solves (4.32). Then ($\mathcal{L}$ is defined in (4.19))

$$\mathcal{L}h = \frac{1}{p-1}V_*\varphi_1(h) + N_*h + f.$$

We claim that for each $j \in \mathbb{Z}_{\geq 0}$, there exists a constant $\widetilde{C}_j > 0$ such that

$$\sqrt{E_j[h](t)} \leq \widetilde{C}_j \int_t^T \left(\frac{T-t}{T-s}\right)^{M_1} \sum_{i=0}^{j} \frac{\|D_x^i f(s)\|_{L_x^2}}{(T-s)^{(j-i)\beta}} \mathrm{d}s, \quad \forall\, t \in [T - T_*, T). \tag{4.40}$$

By the definition of φ_1, we know that φ_1 is a polynomial on $(h, \overline{h})$ of the form $\varphi_1 = \sum_{2 \leq i+j \leq p} c_{i,j} h^i \overline{h}^j$, with $c_{i,j} \in \mathbb{R}$, thus

$$|\varphi_1(h)| \lesssim |h|^2 + |h|^p, \quad \forall\, h \in \mathbb{C}. \tag{4.41}$$

Hence, $\operatorname{supp}_x \varphi_1(h)(t, \cdot) \subset \{x \in \mathbb{R}^d : |x| \leq 4(T-t)/3\}$ for all $t \in [T - T_*, T)$. For $j \in \mathbb{Z}_{\geq 0}$ and $t \in [T - T_*, T)$, by (4.10), (4.7) and Poincaré's inequality, we have

$$\begin{aligned}
\left\|D_x^j(V_*\varphi_1(h))(t)\right\|_{L_x^2} &\lesssim_j \sum_{i=0}^{j} \|D_x^{j-i}V_*(t)\|_{L_x^\infty} \|D_x^i \varphi_1(h)(t)\|_{L_x^2} \\
&\lesssim_j \sum_{i=0}^{j} (T-t)^{-(2\beta+j-i)}(T-t)^{j-i} \|D_x^j \varphi_1(h)(t)\|_{L_x^2} \\
&\lesssim_j (T-t)^{-2\beta} \|D_x^j \varphi_1(h)(t)\|_{L_x^2}
\end{aligned}$$

Using the classical product estimate,

$$\|D_x^n(fg)\|_{L_x^2} \lesssim_n \|f\|_{L_x^\infty} \|D_x^n g\|_{L_x^2} + \|g\|_{L_x^\infty} \|D_x^n f\|_{L_x^2}, \quad \forall\, n \in \mathbb{Z}_{\geq 0}, \tag{4.42}$$

and (4.35), we infer

$$\begin{aligned}
&\|D_x^n(h^i \overline{h}^j)\|_{L_x^2} \lesssim_{n,i,j} \|h\|_{L_x^\infty}^{i+j-1} \|D_x^n h\|_{L_x^2}, \quad \forall\, n, i, j \in \mathbb{Z}_{\geq 0},\ i + j \geq 2, \\
&\|D_x^j \varphi_1(h)(t)\|_{L_x^2} \lesssim_j \left(\|h(t)\|_{L_x^\infty} + \|h(t)\|_{L_x^\infty}^{p-1}\right) \|D_x^j h(t)\|_{L_x^2} \lesssim_j (T-t)^{2\beta-1} \|D_x^j h(t)\|_{L_x^2}, \\
&\left\|D_x^j(V_*\varphi_1(h))(t)\right\|_{L_x^2} \lesssim_j (T-t)^{-2\beta}(T-t)^{2\beta-1} \|D_x^j h(t)\|_{L_x^2} = (T-t)^{-1} \|D_x^j h(t)\|_{L_x^2};
\end{aligned}$$

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Similarly, by (4.7) and Poincaré's inequality, we have

$$\left\|D_x^j(N_*h)(t)\right\|_{L_x^2} \lesssim_j (T-t)^{-1}\|D_x^j h(t)\|_{L_x^2}.$$

Therefore, for each $j \in \mathbb{Z}_{\geq 0}$, there holds

$$\left\|D_x^j \mathcal{L}h(t)\right\|_{L_x^2} \lesssim_j (T-t)^{-1}\|D_x^j h(t)\|_{L_x^2} + \|D_x^j f(t)\|_{L_x^2}. \tag{4.43}$$

By (4.43), (4.22), for any $j \in \mathbb{Z}_{\geq 0}$ and $t \in [T-T_*, T)$ we have

$$\sqrt{E_j[h](t)} \lesssim_j \int_t^T \left(\frac{T-t}{T-s}\right)^{M_1} \sum_{i=0}^{j} \frac{(T-s)^{-1}\|D_x^i h(s)\|_{L_x^2} + \|D_x^i f(s)\|_{L_x^2}}{(T-s)^{(j-i)\beta}} \mathrm{d}s. \tag{4.44}$$

It follows from Poincaré's inequality and $\operatorname{supp}_x h(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \leq 4(T-t)/3\}$ that

$$(T-t)^{-1}\|D_x^j h(t)\|_{L_x^2} \lesssim \|D_x^{j+1} h(t)\|_{L_x^2} \leq \sqrt{2E_j[h](t)}, \quad \forall\, t \in [T-T_*, T),\ j \in \mathbb{Z}_{\geq 0}. \tag{4.45}$$

Here we also used the definitions of E_0 and E_j in (4.20) and (4.21). Next we use the induction argument to prove (4.40).

For $j = 0$, by (4.44) and (4.45), there exists a constant $C_0' > 0$ satisfying

$$\sqrt{E_0[h](t)} \leq C_0' \int_t^T \left(\frac{T-t}{T-s}\right)^{M_1} \left(\sqrt{E_0[h](s)} + \|f(s)\|_{L_x^2}\right) \mathrm{d}s, \quad \forall\, t \in [T-T_*, T).$$

By Grönwall's lemma, we get

$$\begin{aligned}(T-t)^{-M_1}\sqrt{E_0[h](t)} &\leq C_0' \int_t^T (T-s)^{-M_1} \mathrm{e}^{C_0'(s-t)} \|f(s)\|_{L_x^2}\, \mathrm{d}s \\ &\leq C_0' \mathrm{e}^{C_0'T} \int_t^T (T-s)^{-M_1} \|f(s)\|_{L_x^2}\, \mathrm{d}s\end{aligned}$$

for all $t \in [T-T_*, T)$. This proves (4.40) for $j = 0$. Let $n \in \mathbb{Z}_+$, assume that (4.40) holds for $j \in \mathbb{Z} \cap [0, n-1]$. By (4.20), (4.21) and $\beta > 1$, we have

$$(T-s)^{-1}\|D_x^n h(s)\|_{L_x^2} \leq (T-s)^{-1}\sqrt{2E_{n-1}[h](s)} \leq (T-s)^{-\beta}\sqrt{2E_{n-1}[h](s)},$$

for $s \in [T-T_*, T)$. Then by (4.44) for $j = n$, (4.45) for $j = i < n$, and the induction assumption, we have (as $0 < T-t \leq T_* < c_0 < 1$)

$$\begin{aligned}\sqrt{E_n[h](t)} &\lesssim_n \int_t^T \left(\frac{T-t}{T-s}\right)^{M_1} \left(\|D_x^n f(s)\|_{L_x^2} + \sum_{j=0}^{n-1} \frac{\sqrt{E_j[h](s)} + \|D_x^j f(s)\|_{L_x^2}}{(T-s)^{(n-j)\beta}}\right) \mathrm{d}s \\ &\lesssim_n \int_t^T \left(\frac{T-t}{T-s}\right)^{M_1} \sum_{j=0}^{n} \frac{\|D_x^j f(s)\|_{L_x^2}}{(T-s)^{(n-j)\beta}}\, \mathrm{d}s + I_n(t),\end{aligned}$$

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where

$$\begin{aligned}
I_n(t) &:= \sum_{j=0}^{n-1}\int_t^T \frac{\big((T-t)/(T-s)\big)^{M_1}}{(T-s)^{(n-j)\beta}}\int_s^T\left(\frac{T-s}{T-\tau}\right)^{M_1}\sum_{i=0}^{j}\frac{\|D_x^i f(\tau)\|_{L_x^2}}{(T-\tau)^{(j-i)\beta}}\,\mathrm{d}\tau\,\mathrm{d}s\\
&= \sum_{j=0}^{n-1}\sum_{i=0}^{j}\int_t^T\left(\frac{T-t}{T-\tau}\right)^{M_1}\frac{\|D_x^i f(\tau)\|_{L_x^2}}{(T-\tau)^{(j-i)\beta}}\int_t^\tau \frac{\mathrm{d}s}{(T-s)^{(n-j)\beta}}\,\mathrm{d}\tau\\
&\le \sum_{j=0}^{n-1}\sum_{i=0}^{j}\int_t^T\left(\frac{T-t}{T-\tau}\right)^{M_1}\frac{\|D_x^i f(\tau)\|_{L_x^2}}{(T-\tau)^{(j-i)\beta}}\frac{1}{(T-\tau)^{(n-j)\beta}}\,\mathrm{d}\tau\\
&\le n\sum_{i=0}^{n-1}\int_t^T\left(\frac{T-t}{T-\tau}\right)^{M_1}\frac{\|D_x^i f(\tau)\|_{L_x^2}}{(T-\tau)^{(n-i)\beta}}\,\mathrm{d}\tau.
\end{aligned}$$

Thus, (4.40) holds for $j=n$. Therefore, by the the induction, (4.40) holds for all $j\in\mathbb{Z}_{\ge 0}$.

As $|D_x^j f|\le |D_x^j N_*|$ for all $j\in\mathbb{Z}_{\ge0}$, by (4.4) and (4.8), for all $j\in\mathbb{Z}_{\ge0}$ and $\lambda>0$ there exists a constant $\widetilde{M}_{j,\lambda}>0$ which is independent of f and T_* such that $\|D_x^j f(t)\|_{L^2}\le \widetilde{M}_{j,\lambda}(T-t)^\lambda$ for all $t\in[T-T_*,T)$. Using (4.40), (4.21) and (4.45), for all $j\in\mathbb{Z}_{\ge0}$ and $\lambda>0$ there exists a constant $M'_{j,\lambda}>0$ independent of f and T_* (depending on $\widetilde{M}_{j,\lambda'}$ for some $\lambda'>\lambda$) such that

$$\|h(t)\|_{H_x^j}+\|\partial_t h(t)\|_{H_x^j}\le M'_{j,\lambda}(T-t)^\lambda,\quad \forall\, t\in[T-T_*,T).$$

By Sobolev's embedding theorem ($H_x^d(\mathbb{R}^d)\hookrightarrow L_x^\infty(\mathbb{R}^d)$), we have (4.37). It remains to prove (4.36). By (4.41), (4.35), (4.32), (4.37), (4.7) ($j=0$) and (4.8) ($j=0,\lambda=1$), we know that there there exist constants $C_2'>0, C_3'>0$ such that

$$|\Box h|\le C_2'(T-t)^{-2\beta}|D^{\le1}h|+|f|\le C_2'(T-t)^{-2\beta}(C_{0,2\beta}+C_{1,2\beta})(T-t)^{2\beta}+|N_*|\le C_3'$$

on $[T-T_*,T)\times\mathbb{R}^d$, which implies (4.36). □

4.4. Solving nonlinear wave equation

Proof of Proposition 2.8. Let $\xi_1=1-\xi$, then $\xi_1|_{[0,4/5]}=0, \xi_1|_{[1,+\infty)}=1$. Let

$$\varepsilon_n:=c_2/2^n,\quad f_n(t,x):=N_*(t,x)\xi_1((T-t)/\varepsilon_n),\quad \forall\,[T-c_0,T)\times\mathbb{R}^d,\ \forall\, n\in\mathbb{Z}_+.$$

Then for each $n\in\mathbb{Z}_+$, we have $f_n\in C^\infty([T-c_0,T)\times\mathbb{R}^d;\mathbb{C})$ and $\operatorname{supp} f_n\subset\{(t,x)\in[T-c_0, T-4\varepsilon_n/5]\times\mathbb{R}^d : |x|\le 4(T-t)/3\}$, hence $f_n\in C_c^\infty([T-c_0,T)\times\mathbb{R}^d;\mathbb{C})$; as f_n equals to N_* multiplied by a function in t that takes values in $[0,1]$, we have $|D_x^j f_n|\le|D_x^j N_*|$ for all $j\in\mathbb{Z}_+$. By Lemma 4.4, for each $n\in\mathbb{Z}_+$, there exists $h_n\in C_c^\infty([T-c_2,T)\times\mathbb{R}^d;\mathbb{C})$ satisfying

$$\Box h_n+2\mathrm{i}Xh_n+2Yh_n-V_*(h_n)_{\mathrm{r}}-\frac{1}{p-1}V_*\varphi_1(h_n)-N_*h_n=f_n=N_*\xi_1\left(\frac{T-t}{\varepsilon_n}\right)\tag{4.46}$$

on $[T-c_2,T)\times\mathbb{R}^d$. Moreover, for $n\in\mathbb{Z}_+$, we have

$$\operatorname{supp}_x h_n(t,\cdot)\subset\{x\in\mathbb{R}^d : |x|\le 4(T-t)/3\},\quad \forall\, t\in[T-c_2,T),\tag{4.47}$$

and for $j\in\mathbb{Z}_{\ge0}, \lambda>0$ we have

$$|\Box h_n(t,x)|\le C_\Box,\quad \forall\, t\in[T-c_2,T),\ \forall\, x\in\mathbb{R}^d,\tag{4.48}$$

$$|D_x^j h_n(t,x)|+|\partial_t D_x^j h_n(t,x)|\le C_{j,\lambda}(T-t)^\lambda,\quad \forall\, t\in[T-c_2,T),\ \forall\, x\in\mathbb{R}^d,\tag{4.49}$$

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where $C_\Box$ and $C_{j,\lambda}$ are given by Lemma 4.4. By $h_n \in C_c^\infty([T-c_2,T)\times\mathbb{R}^d;\mathbb{C})$, (4.47), (4.48), (4.49) and the Arzelà–Ascoli theorem, there exists a subsequence of $\{h_n\}_{n=1}^\infty$, which is still denoted by $\{h_n\}_{n=1}^\infty$[9], such that $h_n \to h$ in $C^1([T-c_0,T)\times\mathbb{R}^d)$ for some $h \in C^1([T-c_0,T)\times\mathbb{R}^d;\mathbb{C})$ and (here $D^{\le 1}f := (f, Df)$)

$$\left\|D^{\le 1}(h_n - h_{n+1})\right\|_{L^\infty([T-c_0,T)\times\mathbb{R}^d)} \le 2^{-n}, \quad \forall\, n \in \mathbb{Z}_+. \tag{4.50}$$

Letting $j=2, \lambda=1$ in (4.49), by (4.50) and the Gagliardo–Nirenberg inequality, we have

$$\|D_x D(h_n - h_{n+1})\|_{L^\infty([T-c_0,T)\times\mathbb{R}^d)} \le \widetilde{C}_0 2^{-n/2}, \quad \forall\, n \in \mathbb{Z}_+ \tag{4.51}$$

for some constant $\widetilde{C}_0 > 0$ which is independent of $n \in \mathbb{Z}_+$. Letting $j=0$ and $\lambda=1$ in (4.49), by the definition of φ_1, there exists a constant $\widetilde{C}_1 > 0$ such that for all $n \in \mathbb{Z}_+$ and $(t,x) \in [T-c_2,T)\times\mathbb{R}^d$, we have

$$|\varphi_1(h_n) - \varphi_1(h_{n+1})| \le \widetilde{C}_1 |h_n - h_{n+1}|.$$

Combining this with (4.46), (4.49), (4.7) ($j=0$), and (4.8) ($j=0, \lambda=1$), we know that there exist constants $\widetilde{C}_2 > 0, \widetilde{C}_3 > 0$ such that for all $n \in \mathbb{Z}_+$, we have[10]

$$\begin{aligned} |\Box h_n - \Box h_{n+1}| &\le \widetilde{C}_2 (T-t)^{-2\beta} \left|D^{\le 1}(h_n - h_{n+1})\right| + \widetilde{C}_2 \varepsilon_n \\ &\underset{(4.50)}{\overset{(4.49)}{\le}} \widetilde{C}_2 (T-t)^{-2\beta} \min\left(2^{-n}, 2(C_{0,4\beta} + C_{1,4\beta})(T-t)^{4\beta}\right) + \widetilde{C}_2 \varepsilon_n \\ &\le \widetilde{C}_3 (2^{-n/2} + \varepsilon_n) \end{aligned}$$

on $[T-c_2,T)\times\mathbb{R}^d$. Hence, by (4.50) and (4.51) we know that $\{h_n\}_{n=1}^\infty$ is Cauchy in $C^2([T-c_2,T)\times\mathbb{R}^d;\mathbb{C})$, hence $h \in C^2([T-c_2,T)\times\mathbb{R}^d;\mathbb{C})$ and $h_n \to h$ in C^2. Moreover, by (4.49), (4.50) and the Gagliardo–Nirenberg inequality we know that $\{D_x^j D^{\le 1} h_n\}$ is Cauchy in $L^\infty([T-c_2,T)\times\mathbb{R}^d)$ for all $j \in \mathbb{Z}_{\ge 0}$. Hence (also using (4.47)) $h(t,\cdot), \partial_t h(t,\cdot) \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}_x h(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \le 4(T-t)/3\}$. Moreover, h solves the equation (as $f_n \to N_*$)

$$\Box h + 2\mathrm{i}Xh + 2Yh - V_* h_{\mathrm{r}} - \frac{1}{p-1} V_* \varphi_1(h) - N_* h = N_* \quad \text{on} \quad [T-c_2,T)\times\mathbb{R}^d. \tag{4.52}$$

By (4.49), we have

$$|h(t,x)| \le C_{0,1}(T-t), \quad \forall\, (t,x) \in [T-c_2,T)\times\mathbb{R}^d.$$

Let $c_1 \in (0,c_2) \subset (0,T)$ be such that $C_{0,1}c_1 < 1/2$, hence $|h(t,x)| < 1/2$ for all $(t,x) \in [T-c_1,T)\times\mathbb{R}^d$. Let

$$u(t,x) := (1 + h(t,x)) w_*(t,x) \xi\left(\frac{3|x|}{5(T-t)}\right) \exp\left(\mathrm{i}\Phi_*(t,x)\xi\left(\frac{3|x|}{5(T-t)}\right)\right)$$

for $(t,x) \in [T-c_1,T)\times\mathbb{R}^d$. Then $u \in C^2([T-c_1,T)\times\mathbb{R}^d;\mathbb{C})$ with $u(t,\cdot), \partial_t u(t,\cdot) \in C_c^\infty(\mathbb{R}^d)$ and $\operatorname{supp}_x u(t,\cdot), \operatorname{supp}_x \partial_t u(t,\cdot) \subset \{x \in \mathbb{R}^d : |x| \le 5(T-t)/3\}$ for $t \in [T-c_1,T)$. Moreover, on

[9]Then $\{\varepsilon_n\}$ becomes its subsequence satisfying $\varepsilon_n \le c_2/2^n$.

[10]Here, we need to estimate $|f_n - f_{n+1}|$, which is achieved by combining (4.8) ($j=0, \lambda=1$) and

$$\left|\xi_1\left(\frac{T-t}{\varepsilon_n}\right) - \xi_1\left(\frac{T-t}{\varepsilon_m}\right)\right| \le \int_{\varepsilon_m}^{\varepsilon_n} \frac{T-t}{\varepsilon^2} \left|\xi_1'\left(\frac{T-t}{\varepsilon}\right)\right| \mathrm{d}\varepsilon \le \|z^2 \xi_1'(z)\|_{L^\infty} (T-t)^{-1} \varepsilon_n$$

for all positive integers $m > n$, where we have used the fact that $\operatorname{supp} \xi_1' \subset [-1,1]$.

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$\mathcal{C}_0 := \{(t,x) \in (T-c_1,T)\times\mathbb{R}^d : |x| \leq T-t\}$ we have $u = (1+h)w_* e^{\mathrm{i}\Phi_*}$, and by (4.2), (4.3) and (4.52), we know that h satisfies (4.1) on $\mathcal{C}_0$, hence by Lemma 4.1 we know that $\Box u = |u|^{p-1}u$ on $\mathcal{C}_0$. Finally, by $|h| < 1/2$, (2.28) (for $j=0$) and (2.29) we have (2.31) on $\mathcal{C}_0$.

This completes the proof of Proposition 2.8. □

5. The linearized operator $\mathscr{L}$

5.1. *Functional spaces*

In this subsection, we define some functional spaces consisting of smooth functions. Let $I \subset [0,+\infty)$ be an interval. We denote $I^2 := \{x^2 : x \in I\}$ and[11]

$$C_{\mathrm{e}}^\infty(I) := \left\{ f \in C^\infty(I;\mathbb{C}) : \exists \widetilde{f} \in C^\infty(I^2) \text{ s.t. } f(Z) = \widetilde{f}(Z^2), \quad \forall\, Z \in I \right\}, \tag{5.1}$$

$$C_{\mathrm{o}}^\infty(I) := \left\{ f \in C^\infty(I;\mathbb{C}) : \exists \widetilde{f} \in C^\infty(I^2) \text{ s.t. } f(Z) = Z\widetilde{f}(Z^2), \quad \forall\, Z \in I \right\}. \tag{5.2}$$

Then $C_{\mathrm{e}}^\infty(I)$ is a ring, and $C_{\mathrm{o}}^\infty(I)$ is a linear vector space. Note that when $I = [0,+\infty)$, the definitions in (5.1), (5.2) are the same as in (2.10), (2.11). For example, we have $f(Z) = Z \in C_{\mathrm{o}}^\infty([0,+\infty)) \setminus C_{\mathrm{e}}^\infty([0,+\infty))$ and $f(Z) = Z^2 \in C_{\mathrm{e}}^\infty([0,+\infty)) \setminus C_{\mathrm{o}}^\infty([0,+\infty))$.

Lemma 5.1. *Let $f \in C_{\mathrm{e}}^\infty([0,+\infty))$. Define $F(x) = f(|x|)$ for $x \in \mathbb{R}^d$, then $F \in C^\infty(\mathbb{R}^d)$.*

Proof. As $f \in C_{\mathrm{e}}^\infty([0,+\infty))$, there exists a function $\widetilde{f} \in C^\infty([0,+\infty))$ such that $f(Z) = \widetilde{f}(Z^2)$ for all $Z \in [0,+\infty)$, hence $F(x) = \widetilde{f}(|x|^2)$ for all $x \in \mathbb{R}^d$. The smoothness of F follows from the smoothness of $\widetilde{f}$ and $x \mapsto |x|^2$. □

We also have the following fundamental properties. Let $I \subset [0,+\infty)$ be an interval, then

$$f \in C_{\mathrm{e}}^\infty(I) \Longrightarrow f' \in C_{\mathrm{o}}^\infty(I); \tag{5.3}$$

$$f \in C_{\mathrm{o}}^\infty(I) \Longrightarrow f' \in C_{\mathrm{e}}^\infty(I); \tag{5.4}$$

$$f_1 \in C_{\mathrm{e}}^\infty(I), f_2 \in C_{\mathrm{e}}^\infty(I) \Longrightarrow f_1 f_2 \in C_{\mathrm{e}}^\infty(I); \tag{5.5}$$

$$f_1 \in C_{\mathrm{o}}^\infty(I), f_2 \in C_{\mathrm{o}}^\infty(I) \Longrightarrow f_1 f_2 \in C_{\mathrm{e}}^\infty(I); \tag{5.6}$$

$$f_1 \in C_{\mathrm{e}}^\infty(I), f_2 \in C_{\mathrm{o}}^\infty(I) \Longrightarrow f_1 f_2 \in C_{\mathrm{o}}^\infty(I). \tag{5.7}$$

Moreover, if $\Omega \subset \mathbb{C}$ is open, $\varphi \in C^\infty(\Omega;\mathbb{C})$ (not necessary to be holomorphic), and $f \in C_{\mathrm{e}}^\infty(I)$ with $f(Z) \in \Omega$ for all $Z \in I$, then the composition $\varphi \circ f \in C_{\mathrm{e}}^\infty(I)$. In particular,

$$f \in C_{\mathrm{e}}^\infty(I) \text{ with } f(Z) \neq 0\ \forall\, Z \in I \Longrightarrow 1/f \in C_{\mathrm{e}}^\infty(I), \tag{5.8}$$

$$f \in C_{\mathrm{e}}^\infty(I) \Longrightarrow \exp f \in C_{\mathrm{e}}^\infty(I), \tag{5.9}$$

$$a \in \mathbb{R}, f \in C_{\mathrm{e}}^\infty(I) \text{ with } f(Z) > 0 \text{ for all } Z \in I \Longrightarrow f^a \in C_{\mathrm{e}}^\infty(I). \tag{5.10}$$

Lemma 5.2. *If $f \in \mathscr{X}_0$, then $\partial_\tau f, Z\partial_Z f, \partial_Z^2 f, \partial_Z f/Z \in \mathscr{X}_0$.*

Proof. By the definition of $\mathscr{X}_0$ in (2.20), it suffices to prove the result for $f = \widehat{f}(Z)\tau^n$ for some $\widehat{f} \in C_{\mathrm{e}}^\infty([0,+\infty))$ and some $n \in \mathbb{Z}_{\geq 0}$.

As $\partial_\tau f = n\widehat{f}(Z)\tau^{n-1}$, we have $\partial_\tau f \in \mathscr{X}_0$ ($\partial_\tau f = 0$ for $n = 0$).

Note that $Z\partial_Z f = Z\widehat{f}'(Z)\tau^n, \partial_Z^2 f = \widehat{f}''(Z)\tau^n, \partial_Z f/Z = \frac{1}{Z}\widehat{f}'(Z)\tau^n$. As $\widehat{f} \in C_{\mathrm{e}}^\infty([0,+\infty))$, by (5.3) we have $\widehat{f}' \in C_{\mathrm{o}}^\infty([0,+\infty))$, then by the definitions of $C_{\mathrm{e}}^\infty([0,+\infty))$ and $C_{\mathrm{o}}^\infty([0,+\infty))$

[11] In (5.1), "e" stands for "even"; In (5.2), "o" stands for "odd". Please don't confuse $C_{\mathrm{o}}^\infty(I)$ with "$C_0^\infty(I)$".

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we have $Z\widehat{f}'(Z), \frac{1}{Z}\widehat{f}'(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$, by (5.4) we have $\widehat{f}'' \in C_{\mathrm{e}}^\infty([0,+\infty))$. Thus, $Z\partial_Z f$, $\partial_Z^2 f$, $\partial_Z f/Z \in \mathscr{X}_0$. □

Now we prove Lemma 2.3.

Proof.

(i) Let $f(t,x) = (T-t)^\lambda \widehat{f}(\tau, Z)$ for some $\widehat{f} \in \mathscr{X}_0$ and $g(t,x) = (T-t)^\mu \widehat{g}(\tau, Z)$ for some $\widehat{g} \in \mathscr{X}_0$. Then $(fg)(t,x) = (T-t)^{\lambda+\mu}(\widehat{f}\widehat{g})(\tau, Z)$. As $\mathscr{X}_0$ is a ring, we have $\widehat{f}\widehat{g} \in \mathscr{X}_0$ and then $fg \in \mathscr{X}_{\lambda+\mu}$. It is direct to check that

$$\partial_t f = (T-t)^{\lambda-1}\left(\partial_\tau + Z\partial_Z - \lambda\right)\widehat{f}, \quad \Delta f = (T-t)^{\lambda-2}\left(\partial_Z^2 + (k/Z)\partial_Z\right)\widehat{f}.$$

Since $\widehat{f} \in \mathscr{X}_0$, by Lemma 5.2 we have $\partial_\tau \widehat{f}, Z\partial_Z \widehat{f}, \partial_Z^2 \widehat{f}, \partial_Z \widehat{f}/Z \in \mathscr{X}_0$, which gives

$$\left(\partial_\tau + Z\partial_Z - \lambda\right)\widehat{f} \in \mathscr{X}_0, \quad \left(\partial_Z^2 + (k/Z)\partial_Z\right)\widehat{f} \in \mathscr{X}_0.$$

Thus, $\partial_t f \in \mathscr{X}_{\lambda-1}$, $\Delta f \in \mathscr{X}_{\lambda-2}$. As a consequence, we have $\partial_t^2 f \in \mathscr{X}_{\lambda-2}$, $\Box f = -\partial_t^2 f + \Delta f \in \mathscr{X}_{\lambda-2}$ and (also using $fg \in \mathscr{X}_{\lambda+\mu}$ for $f \in \mathscr{X}_\lambda$, $g \in \mathscr{X}_\mu$)

$$\Box(fg) \in \mathscr{X}_{(\lambda+\mu)-2}, \quad (\Box f)g \in \mathscr{X}_{(\lambda-2)+\mu}, \quad f\Box g \in \mathscr{X}_{\lambda+(\mu-2)},$$

hence,

$$\begin{aligned} \partial^\alpha f \partial_\alpha g &= [\Box(fg) - (\Box f)g - f\Box g]/2 \in \mathscr{X}_{\lambda+\mu-2}, \\ \partial^\alpha(f\partial_\alpha g) &= \partial^\alpha f\partial_\alpha g + f\Box g \in \mathscr{X}_{\lambda+\mu-2}. \end{aligned}$$

(iii) Let $\lambda, \mu \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $\lambda \geq j + \mu$. Let $f(t,x) = (T-t)^\lambda \widehat{f}(Z)$ for some $\widehat{f} \in C_{\mathrm{e}}^\infty([0,+\infty))$. Assume that $\alpha_0, \alpha_1, \cdots, \alpha_d \in \mathbb{Z}_{\geq 0}$ are such that $\alpha_0 + \cdots + \alpha_d = j$. We only need to prove that

$$(T-t)^{-\mu}\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d} f \in L^\infty(\mathcal{C}). \tag{5.11}$$

Let $\widetilde{f}(x) := \widehat{f}(|x|)$ for $x \in \mathbb{R}^d$, then by Lemma 5.1 we have $\widetilde{f} \in C^\infty(\mathbb{R}^d)$. Let $j' = \alpha_1 + \cdots + \alpha_d \in \mathbb{Z} \cap [0, j]$, and we let

$$f_\alpha := (-(\lambda - j') + x\cdot\nabla_x)(-(\lambda - j' - 1) + x\cdot\nabla_x)\cdots(-(\lambda - j + 1) + x\cdot\nabla_x)\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}\widetilde{f}.$$

Then $f_\alpha \in C^\infty(\mathbb{R}^d)$ and one can check by direct computation that

$$\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d} f(t,x) = (T-t)^{\lambda-j} f_\alpha(x/(T-t)), \quad \forall\,(t,x) \in [0,T)\times\mathbb{R}^d.$$

As $|x/(T-t)| < 2$ for $(t,x) \in \mathcal{C}$ and $\lambda - j - \mu \geq 0$, we have (5.11).

(ii) Let $\lambda, \mu \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $\lambda > j + \mu$. By the definitions of $\mathscr{X}_0$ and $\mathscr{X}_\lambda$, it suffices to prove $(T-t)^{-\mu}D^j f \in L^\infty(\mathcal{C})$ for $f(t,x) = (T-t)^\lambda \widehat{f}(Z)\tau^n$ for some $\widehat{f} \in C_{\mathrm{e}}^\infty([0,+\infty))$ and some $n \in \mathbb{Z}_{\geq 0}$. Let $P(\tau) := \tau^n$ and $\widetilde{P}(t) := P(\tau) = P(-\ln(T-t))$. Then by the induction, for any $i \in \mathbb{Z}_{\geq 0}$, there is a polynomial $P_i(\tau)$ such that $\widetilde{P}^{(i)}(t) = (T-t)^{-i}P_i(\tau)$. Hence,

$$(T-t)^{i+\varepsilon}\widetilde{P}^{(i)}(t) \in L^\infty([0,T)), \quad \forall\, i \in \mathbb{Z}_{\geq 0},\ \forall\,\varepsilon > 0. \tag{5.12}$$

Let $\widetilde{f}(t,x) := (T-t)^\lambda \widehat{f}(Z)$ for $(t,x) \in [0,T)\times\mathbb{R}^d$, then $f(t,x) = \widetilde{f}(t,x)\widetilde{P}(t)$ for $(t,x) \in [0,T)\times\mathbb{R}^d$, and by (iii) we have

$$(T-t)^{i-\lambda}D^i\widetilde{f} \in L^\infty(\mathcal{C}), \quad \forall\, i \in \mathbb{Z}_{\geq 0}. \tag{5.13}$$

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Assume that $\alpha_0, \alpha_1, \cdots, \alpha_d \in \mathbb{Z}_{\geq 0}$ are such that $\alpha_0 + \cdots + \alpha_d = j$. It suffices to prove

$$(T-t)^{-\mu}\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widetilde{f}(t,x)\widetilde{P}(t)) \in L^\infty(\mathcal{C}). \tag{5.14}$$

By Leibnitz's product rule, we have

$$\begin{aligned}
&(T-t)^{-\mu}\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widetilde{f}(t,x)\widetilde{P}(t))\\
&= (T-t)^{-\mu}\sum_{i=0}^{\alpha_0}\binom{\alpha_0}{i}\widetilde{P}^{(i)}(t)\partial_t^{\alpha_0-i}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}\widetilde{f}(t,x)\\
&= \sum_{i=0}^{\alpha_0}\binom{\alpha_0}{i}(T-t)^{\lambda-j-\mu+i}\widetilde{P}^{(i)}(t)\cdot(T-t)^{j-i-\lambda}\partial_t^{\alpha_0-i}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}\widetilde{f}(t,x).
\end{aligned}$$

Then (5.14) follows from $\lambda - j - \mu > 0$, (5.12) and (5.13).

This completes the proof of Lemma 2.3. □

5.2. $\mathscr{L}$ acting on $\mathscr{X}_\lambda$

Let's first compute the linear operator $\mathscr{L}_\lambda$ induced by $\mathscr{L}$ acting on $\mathscr{X}_\lambda$. The following lemma relies highly on the properties of the leading order profile (ρ_0, ϕ_0). For readers' convenience, we recall some notations. The linear operator $\mathscr{L}$ is defined in (2.19):

$$\mathscr{L}(\phi) := \partial^\alpha\left(\rho_0^2\partial_\alpha\phi - \frac{4}{p-1}\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\phi\right), \quad \phi = \phi(t,x) = \phi(t,r),$$

where according to (2.7),

$$\phi_0(t,r) = (T-t)^{1-\beta}\widehat{\phi}_0(Z), \quad \rho_0(t,x) = (T-t)^{-\frac{2\beta}{p-1}}\widehat{\rho}_0(Z), \quad Z = \frac{r}{T-t}, \quad r = |x|.$$

By (3.13), $\widehat{\phi}_0$ satisfies

$$\widehat{\phi}_0'(Z) = \frac{(\beta-1)\widehat{\phi}_0(Z)v(Z)}{1-Zv(Z)}, \quad Z\widehat{\phi}_0'(Z) + (\beta-1)\widehat{\phi}_0(Z) = \frac{(\beta-1)\widehat{\phi}_0(Z)}{1-Zv(Z)},$$

and $\widehat{\rho}_0$ is defined by (2.12):

$$\widehat{\rho}_0(Z) := \frac{(\beta-1)^{\frac{2}{p-1}}\widehat{\phi}_0(Z)^{\frac{2}{p-1}}(1-v(Z)^2)^{\frac{1}{p-1}}}{(1-Zv(Z))^{\frac{2}{p-1}}}.$$

Lemma 5.3. *There exist real-valued $A_0, \widetilde{B}_0, D_1, D_2 \in C_{\mathrm{e}}^\infty([0,+\infty))$ and $\widehat{B}_0 \in C_{\mathrm{o}}^\infty([0,+\infty))$ with*

$$A_0(Z) = \widehat{\rho}_0(Z)^2\frac{(1-Zv(Z))^2 - \ell(v(Z)-Z)^2}{1-v(Z)^2} = \widehat{\rho}_0(Z)^2\frac{\Delta_Z(Z,v(Z))}{Z(1-v(Z)^2)}, \tag{5.15}$$

such that if we define

$$B_0(Z;\lambda) := Z^{-1}\widetilde{B}_0(Z) + \lambda\widehat{B}_0(Z), \quad D_0(Z;\lambda) := \lambda D_1(Z) + \lambda^2 D_2(Z)$$

and

$$(\mathscr{L}_\lambda f)(Z) := A_0(Z)f''(Z) + B_0(Z;\lambda)f'(Z) + D_0(Z;\lambda)f(Z), \tag{5.16}$$

then there hold (here $\mathscr{L}$ is defined in (2.19) and $\gamma := 4\beta/(p-1)+2 = \beta(\ell-1)+2$)

$$\mathscr{L}((T-t)^\lambda f(Z)) = (T-t)^{\lambda-\gamma}(\mathscr{L}_\lambda f)(Z) \quad \textit{for } f \in C_{\mathrm{e}}^\infty([0,+\infty)), \quad \lambda \in \mathbb{C},$$

and

$$A_0(0) = 1, \quad A_0(Z_1) = 0, \quad A_0'(Z_1) < 0, \tag{5.17}$$

$$A_0(Z) > 0 \;\forall\, Z \in [0, Z_1), \quad A_0(Z) < 0 \;\forall\, Z \in (Z_1, +\infty), \tag{5.18}$$

$$\widehat{B}_0(Z) > 0 \textit{ for all } Z > 0, \quad \widetilde{B}_0(0) = k \in \mathbb{Z}_+. \tag{5.19}$$

Proof. We first consider the functions in the form of $(T-t)^\lambda f(\tau, Z)$, where f is a smooth function and

$$\tau = \ln\frac{1}{T-t}, \qquad Z = \frac{r}{T-t}, \qquad r = |x|. \tag{5.20}$$

Let $\gamma := 4\beta/(p-1)+2 = \beta(\ell-1)+2$, and let $f = f(\tau, Z)$ and $\lambda \in \mathbb{C}$. Now we compute $\mathscr{L}\big((T-t)^\lambda f\big)$.

We will use the following identities: for $\lambda, \mu \in \mathbb{C}$, $f = f(\tau, Z)$, $g = g(\tau, Z)$,

$$\begin{aligned}
&\partial^\alpha \left((T-t)^\lambda f\right)\partial_\alpha\left((T-t)^\mu g\right)\\
&\quad = (T-t)^{\lambda+\mu-2}\Big[-(\partial_\tau f + Z\partial_Z f - \lambda f)(\partial_\tau g + Z\partial_Z g - \mu g) + \partial_Z f\partial_Z g\Big],
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
\partial^\alpha \left((T-t)^\lambda f\partial_\alpha\left((T-t)^\mu g\right)\right) = (T-t)^{\lambda+\mu-2}\Big[&-\big(\partial_\tau + Z\partial_Z - (\lambda+\mu-1)\big)\big(f(\partial_\tau g\\
&+ Z\partial_Z g - \mu g)\big) + (\partial_Z + k/Z)(f\partial_Z g)\Big].
\end{aligned} \tag{5.22}$$

Readers can check (5.21) and (5.22) by using direct computation.

By (2.7), (5.22) and $\gamma = 4\beta/(p-1)+2$, we have

$$\begin{aligned}
\partial^\alpha\left(\rho_0^2\partial_\alpha\left((T-t)^\lambda f\right)\right) = (T-t)^{\lambda-\gamma}\big\{&-\big(\partial_\tau + Z\partial_Z - (\lambda-\gamma+1)\big)\left[\widehat{\rho}_0(Z)^2(\partial_\tau + Z\partial_Z - \lambda)f\right]\\
&+ (\partial_Z + k/Z)\left(\widehat{\rho}_0(Z)^2\partial_Z f\right)\big\}.
\end{aligned}$$

By (2.7) and (5.21), we have

$$\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\left((T-t)^\lambda f\right) = (T-t)^{\lambda-\beta-1}\left[-(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0)(\partial_\tau + Z\partial_Z - \lambda)f + \partial_Z\widehat{\phi}_0\partial_Z f\right].$$

It follows from (3.13) that

$$\begin{aligned}
-(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0)(\partial_\tau + Z\partial_Z - \lambda)f + \partial_Z\widehat{\phi}_0\partial_Z f &= \frac{(\beta-1)\widehat{\phi}_0}{1-Zv(Z)}\big(-\partial_\tau + (v-Z)\partial_Z + \lambda\big)f\\
&=: g(\tau, Z),
\end{aligned}$$

and then $\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\left((T-t)^\lambda f\right) = (T-t)^{\lambda-\beta-1}g(\tau, Z)$. By (2.7) and (5.22), we have

$$\begin{aligned}
\partial^\alpha\left(\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\left((T-t)^\lambda f\right)\right) &= \partial^\alpha\left((T-t)^\mu\widehat{\rho}_0^{3-p}g\partial_\alpha\left((T-t)^{1-\beta}\widehat{\phi}_0\right)\right)\\
&= (T-t)^{\lambda-\gamma}\Big\{-\big(\partial_\tau + Z\partial_Z - (\lambda-\gamma+1)\big)\left[\widehat{\rho}_0^{3-p}g\big(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0\big)\right]\\
&\qquad + (\partial_Z + k/Z)\left(\widehat{\rho}_0^{3-p}g\partial_Z\widehat{\phi}_0\right)\Big\},
\end{aligned}$$

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where $\mu = -\frac{2\beta}{p-1}(3-p) + \lambda - \beta - 1$ satisfies $\mu + (1-\beta) - 2 = \lambda - \gamma$. By (2.12) and (3.13),

$$\widehat{\rho}_0(Z)^{3-p}\frac{(\beta-1)\widehat{\phi}_0(Z)}{1-Zv(Z)}\big(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0\big) = \widehat{\rho}_0(Z)^2\widehat{\rho}_0(Z)^{1-p}\frac{(\beta-1)^2\widehat{\phi}_0(Z)^2}{(1-Zv(Z))^2} = \frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2},$$
$$\widehat{\rho}_0(Z)^{3-p}\frac{(\beta-1)\widehat{\phi}_0(Z)}{1-Zv(Z)}\partial_Z\widehat{\phi}_0 = \frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2},$$

thus

$$\widehat{\rho}_0^{3-p}g\big(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0\big) = \frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}\big(-\partial_\tau + (v(Z)-Z)\partial_Z + \lambda\big)f,$$
$$\widehat{\rho}_0^{3-p}g\partial_Z\widehat{\phi}_0 = \frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}\big(-\partial_\tau + (v(Z)-Z)\partial_Z + \lambda\big)f.$$

Therefore,

$$\begin{aligned}
&\partial^\alpha\left(\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\left((T-t)^\lambda f\right)\right)\\
&= (T-t)^{\lambda-\gamma}\Bigg\{-\big(\partial_\tau + Z\partial_Z - (\lambda-\gamma+1)\big)\left[\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}\big(-\partial_\tau + (v(Z)-Z)\partial_Z + \lambda\big)f\right]\\
&\qquad + \left(\partial_Z + \frac{k}{Z}\right)\left[\frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}\big(-\partial_\tau + (v(Z)-Z)\partial_Z + \lambda\big)f\right]\Bigg\}.
\end{aligned}$$

Finally, recall that $\mathscr{L}(\phi) = \partial^\alpha\left(\rho_0^2\partial_\alpha\phi - \frac{4}{p-1}\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\phi\right)$, $\frac{4}{p-1} = \ell - 1$, we obtain

$$\begin{aligned}
\mathscr{L}\big((T-t)^\lambda f\big) = (T-t)^{\lambda-\gamma}\Bigg\{&-\big(\partial_\tau + Z\partial_Z - (\lambda-\gamma+1)\big)\left[\widehat{\rho}_0(Z)^2(\partial_\tau + Z\partial_Z - \lambda)f\right]\\
&+ (\partial_Z + k/Z)\left(\widehat{\rho}_0(Z)^2\partial_Z f\right)\\
&+ (\ell-1)\big(\partial_\tau + Z\partial_Z - (\lambda-\gamma+1)\big)\left[\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}\big(-\partial_\tau + (v(Z)-Z)\partial_Z + \lambda\big)f\right]\\
&- (\ell-1)\left(\partial_Z + \frac{k}{Z}\right)\left[\frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}\big(-\partial_\tau + (v(Z)-Z)\partial_Z + \lambda\big)f\right]\Bigg\}.
\end{aligned}$$

For any $\lambda \in \mathbb{C}$, we define a linear operator $\mathscr{L}_\lambda$ by

$$\begin{aligned}
(\mathscr{L}_\lambda f)(Z) :=& -\big(Z\partial_Z - (\lambda-\gamma+1)\big)\left[\widehat{\rho}_0(Z)^2(Z\partial_Z - \lambda)f\right] + (\partial_Z + k/Z)\left(\widehat{\rho}_0(Z)^2\partial_Z f\right)\\
&+ (\ell-1)\big(Z\partial_Z - (\lambda-\gamma+1)\big)\left[\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}\big((v(Z)-Z)\partial_Z + \lambda\big)f\right]\\
&- (\ell-1)\left(\partial_Z + \frac{k}{Z}\right)\left[\frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}\big((v(Z)-Z)\partial_Z + \lambda\big)f\right],
\end{aligned}\tag{5.23}$$

where $f = f(Z)$ depends only on $Z \in [0, +\infty)$ (not on τ). Assume that $f = f(Z) = f(Z;\lambda)$ satisfies $(\mathscr{L}_\lambda f)(Z) = g(Z) = g(Z;\lambda)$, then (here $\mathscr{L}$ and $\mathscr{L}_\lambda$ do not act on λ)

$$\mathscr{L}\big((T-t)^\lambda f\big) = (T-t)^{\lambda-\gamma}g(Z). \tag{5.24}$$

Now it is enough to prove that $\mathscr{L}_\lambda$ defined in (5.23) can be written in the form of (5.16) with $B_0(Z;\lambda) = Z^{-1}\widetilde{B}_0(Z) + \lambda\widehat{B}_0(Z)$, $D_0(Z;\lambda) = \lambda D_1(Z) + \lambda^2 D_2(Z)$ and $A_0(Z), \widetilde{B}_0(Z), D_1(Z), D_2(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$, $\widehat{B}_0 \in C_{\mathrm{o}}^\infty([0,+\infty))$ satisfying (5.15), (5.17), (5.18), (5.19).

Comparing the coefficients of ∂_Z^j $(j = 0, 1, 2)$ in (5.23) and (5.16), we find

$$\begin{aligned}
A_0(Z) &= \widehat{\rho}_0(Z)^2(1-Z^2) + (\ell-1)\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}Z(v(Z)-Z) - (\ell-1)\frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}(v(Z)-Z)\\
&= \widehat{\rho}_0(Z)^2\left((1-Z^2) - (\ell-1)\frac{(v(Z)-Z)^2}{1-v(Z)^2}\right) = \widehat{\rho}_0(Z)^2\frac{(1-Zv(Z))^2 - \ell(v(Z)-Z)^2}{1-v(Z)^2},\\
B_0(Z;\lambda) &= -Z^2\partial_Z(\widehat{\rho}_0^2) - Z\widehat{\rho}_0^2 + \lambda Z\widehat{\rho}_0^2 + (\lambda-\gamma+1)Z\widehat{\rho}_0^2 + \partial_Z(\widehat{\rho}_0^2) + (k/Z)\widehat{\rho}_0^2\\
&\quad + (\ell-1)Z\partial_Z\left(\frac{\widehat{\rho}_0^2}{1-v^2}\right)(v-Z) + \frac{(\ell-1)\widehat{\rho}_0^2}{1-v^2}(Zv' - Z - (\lambda-\gamma+1)(v-Z) + \lambda Z)\\
&\quad - (\ell-1)\partial_Z\left(\frac{\widehat{\rho}_0^2 v}{1-v^2}\right)(v-Z) - (\ell-1)\frac{\widehat{\rho}_0^2 v}{1-v^2}(v' - 1 + (k/Z)(v-Z) + \lambda),\\
D_0(Z;\lambda) &= -\big(Z\partial_Z - (\lambda-\gamma+1)\big)\left(-\lambda\widehat{\rho}_0(Z)^2\right) + (\ell-1)\big(Z\partial_Z - (\lambda-\gamma+1)\big)\left(\lambda\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}\right)\\
&\quad - (\ell-1)\left(\partial_Z + \frac{k}{Z}\right)\left(\lambda\frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}\right).
\end{aligned}$$

Then (5.23) and (5.16) are equivalent and A_0 satisfies (5.15).

By the expression of $B_0(Z;\lambda)$, we have $B_0(Z;\lambda) = Z^{-1}\widetilde{B}_0(Z) + \lambda\widehat{B}_0(Z)$ with

$$\begin{aligned}
\widehat{B}_0(Z) &= 2Z\widehat{\rho}_0^2 + \frac{(\ell-1)\widehat{\rho}_0^2}{1-v^2}(2Z-v) - \frac{(\ell-1)\widehat{\rho}_0^2 v}{1-v^2} = \frac{2\widehat{\rho}_0^2}{1-v^2}\big(Z(1-v^2) + (\ell-1)(Z-v)\big).\\
\widetilde{B}_0(Z) &= k\widehat{\rho}_0^2 + (1-Z^2)Z\partial_Z(\widehat{\rho}_0^2) - \gamma Z^2\widehat{\rho}_0^2 + (\ell-1)(v-Z)\left[Z^2\partial_Z\left(\frac{\widehat{\rho}_0^2}{1-v^2}\right) - Z\partial_Z\left(\frac{\widehat{\rho}_0^2 v}{1-v^2}\right)\right]\\
&\quad + (\ell-1)\frac{\widehat{\rho}_0^2}{1-v^2}\left(Z(Z-v)v' + (k+\gamma)Zv - \gamma Z^2 - kv^2\right).
\end{aligned}$$

By the expression of $D_0(Z;\lambda)$, we have $D_0(Z;\lambda) = \lambda D_1(Z) + \lambda^2 D_2(Z)$ with

$$\begin{aligned}
D_1(Z) =& \big(Z\partial_Z + \gamma - 1\big)\left(\widehat{\rho}_0(Z)^2\right) + (\ell-1)\big(Z\partial_Z + \gamma - 1\big)\left(\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2}\right)\\
&- (\ell-1)\left(\partial_Z + \frac{k}{Z}\right)\left(\frac{\widehat{\rho}_0(Z)^2 v(Z)}{1-v(Z)^2}\right),\\
D_2(Z) =& -\widehat{\rho}_0(Z)^2 - (\ell-1)\frac{\widehat{\rho}_0(Z)^2}{1-v(Z)^2} = -\widehat{\rho}_0(Z)^2\frac{\ell - v(Z)^2}{1-v(Z)^2}.
\end{aligned}$$

It remains to prove that $A_0(Z), \widetilde{B}_0(Z), D_1(Z), D_2(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$, $\widehat{B}_0 \in C_{\mathrm{o}}^\infty([0,+\infty))$ and (5.17), (5.18), (5.19).

By Lemma A.6 we have $\widehat{\rho}_0 \in C_{\mathrm{e}}^\infty([0,+\infty)), v \in C_{\mathrm{o}}^\infty([0,+\infty))$; by (5.5) and (5.6) we have $\widehat{\rho}_0^2 \in C_{\mathrm{e}}^\infty([0,+\infty))$, $(v(Z)-Z)^2 \in C_{\mathrm{e}}^\infty([0,+\infty))$ and $1-v(Z)^2 \in C_{\mathrm{e}}^\infty([0,+\infty))$; by (5.6) we have $Zv(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$, hence $1 - Zv(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$, then using (5.5) we get $(1-Zv(Z))^2 \in C_{\mathrm{e}}^\infty([0,+\infty))$. Therefore, by $v \in (-1,1)$ (see Assumption 1), (5.8) and (5.15) we have $A_0(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$.

Similarly, by Lemma A.6 and (5.3)–(5.8) we have $\widehat{B}_0(Z) \in C_{\mathrm{o}}^\infty([0,+\infty))$ and $\widetilde{B}_0(Z)$, $D_1(Z)$, $D_2(Z) \in C_{\mathrm{e}}^\infty([0,+\infty))$.

It follows from $\widehat{\rho}_0(0) = 1$ and $v(0) = 0$ that $A_0(0) = 1$. By Remark 2.2 and $\widehat{\rho}_0(Z) > 0$ for all $Z \in [0, +\infty)$, we have $A_0(Z_1) = 0$ and (5.18). Let $\Delta_0(Z) = \Delta_Z(Z, v(Z))$, then by $\Delta_0(Z_1) = 0$ and Remark 2.2, $A_0'(Z_1) = \widehat{\rho}_0(Z_1)^2\Delta_0'(Z_1)/(Z_1(1 - v(Z_1)^2)) \neq 0$. This along with $A_0(Z_1) < A_0(Z)$ for all $Z \in [0, Z_1)$ implies $A_0'(Z_1) < 0$. So we have (5.17).

As $v(Z) \in (0, 1)$, $v(Z) < Z$ for all $Z > 0$ (see Remark 2.2) and $\widehat{\phi}_0(Z) > 0$ for all $Z \in [0, +\infty)$, we have $\widehat{B}_0(Z) > 0$ for all $Z > 0$; as $\widehat{\rho}_0(0) = 1$ and $v(0) = 0$ we have $\widehat{B}_0(0) = k \in \mathbb{Z}_+$. This proves (5.19). ◻

Next we compute the dual operator of $\mathscr{L}_\lambda$. For any $\lambda \in \mathbb{C}$, we define an operator $\mathscr{L}_\lambda^*$, called the *dual operator* of $\mathscr{L}_\lambda$, by

$$\int_0^\infty (\mathscr{L}_\lambda f)(Z)g(Z)Z^k\,\mathrm{d}Z = \int_0^\infty f(Z)(\mathscr{L}_\lambda^* g)(Z)Z^k\,\mathrm{d}Z, \quad \forall\, f, g \in C_c^\infty((0, +\infty)). \tag{5.25}$$

Lemma 5.4. *For any $\lambda \in \mathbb{C}$, we have $\mathscr{L}_\lambda^* = \mathscr{L}_{-\lambda+\gamma-k-2}$.*

Proof. By the definition (5.25), it is enough to prove that

$$\int_0^\infty (\mathscr{L}_\lambda f)(Z)g(Z)Z^k\,\mathrm{d}Z = \int_0^\infty f(Z)(\mathscr{L}_{-\lambda+\gamma-k-2} g)(Z)Z^k\,\mathrm{d}Z \tag{5.26}$$

for all $\lambda \in \mathbb{C}$ and $f, g \in C_c^\infty((0, +\infty))$. We fix $\lambda \in \mathbb{C}$ and $f, g \in C_c^\infty((0, +\infty))$. Let

$$\widetilde{f}(t, x) := (T - t)^\lambda f(Z), \quad \widetilde{g}(t, x) := (T - t)^{-\lambda+\gamma-k-2} g(Z), \quad \forall\, (t, x) \in [0, T) \times \mathbb{R}^d,$$

recalling $Z = |x|/(T - t)$. Then by Lemma 5.3, we have $\mathscr{L}\widetilde{f}(t, x) = (T - t)^{\lambda-\gamma}(\mathscr{L}_\lambda f)(Z)$ and $\mathscr{L}\widetilde{g}(t, x) = (T - t)^{-\lambda-k-2}(\mathscr{L}_{-\lambda+\gamma-k-2} g)(Z)$, thus $(\mathscr{L}\widetilde{f} \cdot \widetilde{g})(t, x) = (T - t)^{-k-2}(\mathscr{L}_\lambda f)(Z)g(Z)$, and $(\widetilde{f} \cdot \mathscr{L}\widetilde{g})(t, x) = (T - t)^{-k-2} f(Z)(\mathscr{L}_{-\lambda+\gamma-k-2} g)(Z)$. Recall that $d = k + 1$, $Z = |x|/(T - t)$, then we have (here $|S^k|$ is the area of the unit sphere S^k in $\mathbb{R}^d = \mathbb{R}^{k+1}$)

$$\begin{aligned}
\int_{\mathbb{R}^d} (\mathscr{L}\widetilde{f} \cdot \widetilde{g})(t, x)dx &= (T - t)^{-1}|S^k| \int_0^\infty (\mathscr{L}_\lambda f)(Z)g(Z)Z^k\,\mathrm{d}Z,\\
\int_{\mathbb{R}^d} (\widetilde{f} \cdot \mathscr{L}\widetilde{g})(t, x)dx &= (T - t)^{-1}|S^k| \int_0^\infty f(Z)(\mathscr{L}_{-\lambda+\gamma-k-2} g)(Z)Z^k\,\mathrm{d}Z,
\end{aligned}$$

for all $t \in [0, T)$. Thus, it is enough to prove that

$$\int_{\mathbb{R}^d} (\mathscr{L}\widetilde{f} \cdot \widetilde{g})(t, x)\,\mathrm{d}x = \int_{\mathbb{R}^d} (\widetilde{f} \cdot \mathscr{L}\widetilde{g})(t, x)\,\mathrm{d}x, \quad \forall\, t \in [0, T). \tag{5.27}$$

Let $\mathcal{J} := \mathscr{L}\widetilde{f} \cdot \widetilde{g} - \widetilde{f} \cdot \mathscr{L}\widetilde{g}$. Then (5.27) is further reduced to

$$\int_{\mathbb{R}^d} \mathcal{J}(t, x)\,\mathrm{d}x = 0, \quad \forall\, t \in [0, T). \tag{5.28}$$

By the definition of $\mathscr{L}$ in (2.19), we can write $\mathcal{J}$ in the divergence form $\mathcal{J} = \partial^\alpha P_\alpha$ with

$$P_\alpha := \rho_0^2(\partial_\alpha \widetilde{f}\widetilde{g} - \widetilde{f}\partial_\alpha \widetilde{g}) - \frac{4}{p-1}\rho_0^{3-p}\partial_\alpha\phi_0(\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\widetilde{f}\widetilde{g} - \widetilde{f}\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\widetilde{g}), \quad \forall\, \alpha \in \mathbb{Z} \cap [0, d].$$

Let $E(t) := \int_{\mathbb{R}^d} P_0(t, x)\,\mathrm{d}x$ for $t \in [0, T)$. By the divergence theorem (recalling $\partial^0 = -\partial_0 = -\partial_t$ and the fact that $\mathrm{supp}_x\, P_\alpha(t, \cdot)$ is compact for each t and α), we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_{\mathbb{R}^d} \partial^0 P_0(t, x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \partial^\alpha P_\alpha(t, x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{J}(t, x)\,\mathrm{d}x, \quad \forall\, t \in [0, T). \tag{5.29}$$

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Thus, it is enough to prove that $E(t)$ is constant in t. We can write $P_0 = P_{0,1} - \frac{4}{p-1}P_{0,2}$ with

$$P_{0,1} := \rho_0^2(\partial_t\widetilde{f}\widetilde{g} - \widetilde{f}\partial_t\widetilde{g}), \quad P_{0,2} := \rho_0^{3-p}\partial_t\phi_0(\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\widetilde{f}\widetilde{g} - \widetilde{f}\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\widetilde{g}).$$

As $\widetilde{f}(t,x) = (T-t)^{\lambda}f(Z)$, $\widetilde{g}(t,x) = (T-t)^{-\lambda+\gamma-k-2}g(Z)$, we have

$$\begin{aligned}
&\partial_t\widetilde{f}(t,x) = (T-t)^{\lambda-1}f_1(Z) \text{ with } f_1(Z) := -\lambda f(Z) + Zf'(Z),\\
&\partial_t\widetilde{g}(t,x) = (T-t)^{-\lambda+\gamma-k-3}g_1(Z) \text{ with } g_1(Z) := -(-\lambda+\gamma-k-2)g(Z) + Zg'(Z).
\end{aligned}$$

Then by (2.7), $\gamma = \frac{4\beta}{p-1} + 2$ and $d = k+1$, we get

$$\begin{aligned}
P_{0,1}(t,x) &= (T-t)^{-\frac{4\beta}{p-1}+\gamma-k-3}\widehat{\rho}_0(Z)^2[f_1(Z)g(Z) - f(Z)g_1(Z)]\\
&= (T-t)^{-d}\widehat{\rho}_0(Z)^2[f_1(Z)g(Z) - f(Z)g_1(Z)].
\end{aligned}$$

As $\widetilde{f}(t,x) = (T-t)^{\lambda}f(Z)$, $\widetilde{g}(t,x) = (T-t)^{-\lambda+\gamma-k-2}g(Z)$, we get by (2.7) and (5.21) that

$$\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\widetilde{f}(t,x) = (T-t)^{\lambda-\beta-1}f_2(Z), \quad \partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\widetilde{g}(t,x) = (T-t)^{-\lambda+\gamma-k-3-\beta}g_2(Z),$$

where

$$\begin{aligned}
f_2 &:= -(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0)(Z\partial_Z - \lambda)f + \partial_Z\widehat{\phi}_0\partial_Z f,\\
g_2 &:= -(Z\partial_Z\widehat{\phi}_0 - (1-\beta)\widehat{\phi}_0)(Z\partial_Z + \lambda - \gamma + k + 2)g + \partial_Z\widehat{\phi}_0\partial_Z g.
\end{aligned}$$

Then by (2.7), (A.2) and $\gamma = \frac{4\beta}{p-1} + 2 = \frac{2(3-p)\beta}{p-1} + 2\beta + 2$, $d = k+1$, we have

$$\begin{aligned}
&P_{0,2}(t,x)\\
&= (T-t)^{-\frac{2(3-p)\beta}{p-1}-2\beta+\gamma-k-3}\widehat{\rho}_0(Z)^{3-p}[(\beta-1)\widehat{\phi}_0(Z) + Z\widehat{\phi}_0'(Z)][f_2(Z)g(Z) - f(Z)g_2(Z)]\\
&= (T-t)^{-d}\widehat{\rho}_0(Z)^{3-p}[(\beta-1)\widehat{\phi}_0(Z) + Z\widehat{\phi}_0'(Z)][f_2(Z)g(Z) - f(Z)g_2(Z)].
\end{aligned}$$

As $P_0 = P_{0,1} - \frac{4}{p-1}P_{0,2}$, we have $P_0(t,x) = (T-t)^{-d}H(Z)$ with

$$\begin{aligned}
H(Z) := {}&\widehat{\rho}_0(Z)^2[f_1(Z)g(Z) - f(Z)g_1(Z)]\\
&- \frac{4}{p-1}\widehat{\rho}_0(Z)^{3-p}[(\beta-1)\widehat{\phi}_0(Z) + Z\widehat{\phi}_0'(Z)][f_2(Z)g(Z) - f(Z)g_2(Z)].
\end{aligned}$$

Then by $d = k+1$, $Z = |x|/(T-t)$, we have $E(t) = \int_{\mathbb{R}^d} P_0(t,x)\,\mathrm{d}x = |S^k|\int_0^{\infty} H(Z)Z^k\,\mathrm{d}Z$, which is constant in t. By (5.29), we have (5.28), thus (5.27) and (5.26). $\square$

5.3. *Surjection of $\mathscr{L}$*

This subsection is devoted to the proof of Proposition 2.4, i.e., $\mathscr{L} : \mathscr{X}_\lambda \to \mathscr{X}_{\lambda-\gamma}$ is surjective for all $\lambda \in \mathbb{C}$. For this, it suffices to show that

Lemma 5.5. *If $R \in (k,+\infty)$, then the linear operator $\mathscr{L} : \mathscr{X}_\lambda \to \mathscr{X}_{\lambda-\gamma}$ is surjective for all $\lambda \in B_R := \{\lambda \in \mathbb{C} : |\lambda| < R\}$, where $\gamma := 4\beta/(p-1) + 2 = \beta(\ell-1) + 2$.*

From here until the end of this section, we fix an $R \in (k,+\infty)$.

We consider functions depending analytically on a complex number λ. Let $I \subset [0,+\infty)$ be an interval and let $\Omega \subset \mathbb{C}$ be an open set. We define

$$\begin{aligned}
&\mathrm{Hol}(\Omega) := \{\text{all holomorphic function on } \Omega\},\\
&\mathcal{H}_I(\Omega) := \{f = f(Z;\lambda) \in C^\infty(I\times\Omega;\mathbb{C}) : f(Z;\cdot) \in \mathrm{Hol}(\Omega) \text{ for all } Z \in I\},\\
&\mathcal{H}_I^{\mathrm{e}}(\Omega) := \{f \in C^\infty(I\times\Omega) : \exists\, \widetilde{f} \in \mathcal{H}_{I^2}(\Omega), \text{ s.t. } f(Z;\lambda) = \widetilde{f}(Z^2;\lambda) \quad \forall\, Z \in I,\ \lambda \in \Omega\}.
\end{aligned}$$

Then $\mathrm{Hol}(\Omega)$, $\mathcal{H}_I(\Omega)$, $\mathcal{H}_I^{\mathrm{e}}(\Omega)$ are rings. Moreover, we have

$$\mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(\Omega) = \mathcal{H}^{\mathrm{e}}_{[0,a_2)}(\Omega) \cap \mathcal{H}_{(a_1,+\infty)}(\Omega), \quad \forall\, 0 < a_1 < a_2 < +\infty.$$

The proof of Lemma 5.5 is based on the following result, which will be proved in next subsection.

Lemma 5.6. *There exists $\varphi \in \mathrm{Hol}(B_R) \setminus \{0\}$ such that if $g \in C^\infty_{\mathrm{e}}([0,+\infty))$, then there exists $f = f(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ such that $\mathscr{L}_\lambda f(\cdot;\lambda) = \varphi(\lambda)\cdot g$ on $(0,+\infty)$ (for all $\lambda \in B_R$).*

Proof of Lemma 5.5. We first prove that $\mathscr{L}$ maps $\mathscr{X}_\lambda$ to $\mathscr{X}_{\lambda-\gamma}$.

Recall that $\mathscr{L}(\phi) = \partial^\alpha\left(\rho_0^2\partial_\alpha\phi - \frac{4}{p-1}\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\phi\right)$, $\rho_0^2 \in \mathscr{X}_{2\mu_0}$, $\rho_0^{3-p} \in \mathscr{X}_{(3-p)\mu_0}$, $\phi_0 \in \mathscr{X}_{\lambda_0}$, $\lambda_0 = 1-\beta$, $\mu_0 = -\frac{2\beta}{p-1}$ (see (2.22)) and $\gamma = 4\beta/(p-1)+2$. If $\phi \in \mathscr{X}_\lambda$, by Lemma 2.3 (i), we have $\partial^\alpha(\rho_0^2\partial_\alpha\phi) \in \mathscr{X}_{\lambda-\gamma}$, $\partial^\alpha(\rho_0^{3-p}\partial_\alpha\phi_0\partial^{\widetilde{\alpha}}\phi_0\partial_{\widetilde{\alpha}}\phi) \in \mathscr{X}_{\lambda-\gamma}$, where we have used that $\lambda+2\mu_0-2 = \lambda-\gamma = \lambda+\lambda_0-2+(3-p)\mu_0+\lambda_0-2$, thus $\mathscr{L}(\phi) \in \mathscr{X}_{\lambda-\gamma}$.

Now we prove that $\mathscr{L}$ is surjective. By the definitions of $\mathscr{X}_0$ and $\mathscr{X}_\lambda$, it suffices to prove that for every $g \in C^\infty_{\mathrm{e}}([0,+\infty))$, $n \in \mathbb{Z}_{\geq 0}$ and $\lambda_* \in B_R$, there exists $F_n \in \mathscr{X}_{\lambda_*}$ such that $\mathscr{L}F_n(t,x) = (T-t)^{\lambda_*-\gamma}g(Z)\tau^n/n!$. Now we fix $g \in C^\infty_{\mathrm{e}}([0,+\infty))$ and $\lambda_* \in B_R$.

By Lemma 5.6, there exist $\varphi \in \mathrm{Hol}(B_R)\setminus\{0\}$ and a function $f = f(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ such that $\mathscr{L}_\lambda f(Z;\lambda) = \varphi(\lambda)g(Z)$ for $Z \in (0,+\infty)$, $\lambda \in B_R$. As $f \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ there exists $\widetilde{f} \in \mathcal{H}_{[0,+\infty)}(B_R)$ such that $f(Z;\lambda) = \widetilde{f}(Z^2;\lambda)$ for $Z \in [0,+\infty)$, $\lambda \in B_R$. As $\lambda_* \in B_R$, there exist $\delta_* > 0$ and $m_* \in \mathbb{Z}_{\geq 0}$ such that $B_{2\delta_*}(\lambda_*) \subset B_R$ and

$$\varphi(\lambda) = (\lambda-\lambda_*)^{m_*}\widetilde{\varphi}(\lambda) \text{ with } \widetilde{\varphi}(\lambda) \neq 0, \quad \forall\, \lambda \in \Omega_* := B_{2\delta_*}(\lambda_*), \text{ where } \widetilde{\varphi} \in \mathrm{Hol}(\Omega_*).$$

Here $B_{2\delta_*}(\lambda_*) := \{\lambda \in \mathbb{C} : |\lambda-\lambda_*| < 2\delta_*\}$ and we have used the fact that if $\varphi \in \mathrm{Hol}(\Omega)\setminus\{0\}$, then the zero set $\mathcal{Z}(\varphi) := \{\lambda \in \Omega : \varphi(\lambda) = 0\}$ is discrete.

Let $\widetilde{F}(Z;\lambda) := \widetilde{f}(Z;\lambda)/\widetilde{\varphi}(\lambda)$, $F(Z;\lambda) := \widetilde{F}(Z^2;\lambda)$ for $Z \in [0,+\infty)$, $\lambda \in B_R$. Then $\widetilde{F} \in \mathcal{H}_{[0,+\infty)}(\Omega_*)$, $F \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(\Omega_*)$, $F(Z;\lambda) = f(Z;\lambda)/\widetilde{\varphi}(\lambda)$, and

$$\mathscr{L}_\lambda F(Z;\lambda) = \varphi(\lambda)g(Z)/\widetilde{\varphi}(\lambda) = (\lambda-\lambda_*)^{m_*}g(Z), \quad \forall\, Z \in (0,+\infty), \lambda \in \Omega_*.$$

By Lemma 5.3, we have

$$\mathscr{L}((T-t)^\lambda F(Z;\lambda)) = (T-t)^{\lambda-\gamma}\mathscr{L}_\lambda F(Z;\lambda) = (T-t)^{\lambda-\gamma}(\lambda-\lambda_*)^{m_*}g(Z)$$

for all $\lambda \in \Omega_*$ and $Z \in (0,+\infty)$. Let

$$F_*(t,x;\lambda) := (T-t)^\lambda F(Z;\lambda) = (T-t)^\lambda\widetilde{F}(Z^2;\lambda), \quad G(t,x;\lambda) := (T-t)^{\lambda-\gamma}(\lambda-\lambda_*)^{m_*}g(Z).$$

Then $F_*, G \in C^\infty([0,T)\times\mathbb{R}^d\times\Omega_*)$ (as $Z^2 = |x|^2/(T-t)^2$ is smooth on $[0,T)\times\mathbb{R}^d$) and $\mathscr{L}F_*(t,x;\lambda) = G(t,x;\lambda)$ on $[0,T)\times\mathbb{R}^d\times\Omega_*$ (the case $Z=0$ follows by continuity).

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Recall that $\tau = \ln\frac{1}{T-t}$ and then

$$G(t,x;\lambda) = (T-t)^{\lambda_*-\gamma}\mathrm{e}^{-(\lambda-\lambda_*)\tau}(\lambda-\lambda_*)^{m_*}g(Z) = \sum_{n=0}^{\infty}(T-t)^{\lambda_*-\gamma}\frac{(-\tau)^n}{n!}(\lambda-\lambda_*)^{m_*+n}g(Z)$$

locally uniformly on $[0,T)\times\mathbb{R}^d\times B_{\delta_*}(\lambda_*)$. By Cauchy's integration formula (Theorem 4.4 in Chapter 2 of [69]), we have (for $n\in\mathbb{Z}_{\geq 0}$)

$$\begin{aligned}(T-t)^{\lambda_*-\gamma}g(Z)\frac{\tau^n}{n!} &= \frac{(-1)^n}{2\pi\mathrm{i}}\oint_{|\lambda-\lambda_*|=\delta_*}\frac{G(t,x;\lambda)}{(\lambda-\lambda_*)^{m_*+n+1}}\,\mathrm{d}\lambda\\ &= \frac{\delta_*^{-m_*-n}}{2\pi(-1)^n}\int_0^{2\pi}G(t,x;\lambda_*+\delta_*\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}(m_*+n)\theta}\,\mathrm{d}\theta.\end{aligned}$$

Now let (for $n\in\mathbb{Z}_{\geq 0}$)

$$\begin{aligned}F_n(t,x) &:= \frac{(-1)^n}{2\pi\mathrm{i}}\oint_{|\lambda-\lambda_*|=\delta_*}\frac{F_*(t,x;\lambda)}{(\lambda-\lambda_*)^{m_*+n+1}}\,\mathrm{d}\lambda\\ &= \frac{\delta_*^{-m_*-n}}{2\pi(-1)^n}\int_0^{2\pi}F_*(t,x;\lambda_*+\delta_*\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}(m_*+n)\theta}\,\mathrm{d}\theta.\end{aligned}$$

Then $F_n\in C^\infty([0,T)\times\mathbb{R}^d)$ and $\mathscr{L}F_n(t,x) = (T-t)^{\lambda_*-\gamma}g(Z)\tau^n/n!$. It remains to prove that $F_n\in\mathscr{X}_{\lambda_*}$.

As $\tau = \ln\frac{1}{T-t}$, $F_*(t,x;\lambda) = (T-t)^{\lambda}\widetilde{F}(Z^2;\lambda)$ then

$$F_*(t,x;\lambda) = (T-t)^{\lambda_*}\mathrm{e}^{-(\lambda-\lambda_*)\tau}\widetilde{F}(Z^2;\lambda) = \sum_{j=0}^{\infty}(T-t)^{\lambda_*}\frac{(-\tau)^j}{j!}(\lambda-\lambda_*)^j\widetilde{F}(Z^2;\lambda),$$

locally uniformly on $[0,T)\times\mathbb{R}^d\times B_{\delta_*}(\lambda_*)$, so we have

$$\begin{aligned}F_n(t,x) &= \sum_{j=0}^{\infty}(T-t)^{\lambda_*}\frac{(-\tau)^j}{j!}F_{n,j}(Z^2), \qquad \text{where}\\ F_{n,j}(\widetilde{Z}) &:= \frac{(-1)^n}{2\pi\mathrm{i}}\oint_{|\lambda-\lambda_*|=\delta_*}\frac{(\lambda-\lambda_*)^j\widetilde{F}(\widetilde{Z};\lambda)}{(\lambda-\lambda_*)^{m_*+n+1}}\,\mathrm{d}\lambda\\ &= \frac{\delta_*^{j-m_*-n}}{2\pi(-1)^n}\int_0^{2\pi}\widetilde{F}(\widetilde{Z};\lambda_*+\delta_*\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}(j-m_*-n)\theta}\,\mathrm{d}\theta.\end{aligned}$$

As $\widetilde{F}\in\mathcal{H}_{[0,+\infty)}(\Omega_*)\subset C^\infty([0,+\infty)\times\Omega_*)$, we have $F_{n,j}\in C^\infty([0,+\infty))$, $Z\mapsto F_{n,j}(Z^2)\in C_{\mathrm{e}}^\infty([0,+\infty))$ for every $n,j\in\mathbb{Z}_{\geq 0}$; moreover by Cauchy's theorem (Corollary 2.3 in Chapter 2 of [69]), we have $F_{n,j}=0$ for $j>m_*+n$, $n,j\in\mathbb{Z}_{\geq 0}$. Thus,

$$F_n(t,x) = \sum_{j=0}^{m_*+n}(T-t)^{\lambda_*}\frac{(-\tau)^j}{j!}F_{n,j}(Z^2)\in\mathscr{X}_{\lambda_*}.$$

This completes the proof of Lemma 5.5. □

5.4. Solvability of $\mathscr{L}_\lambda$

In this subsection, we prove Lemma 5.6.

Lemma 5.7. *For $g\in C_{\mathrm{e}}^\infty([0,Z_1))$, there exists $F=F(Z;\lambda)\in\mathcal{H}^{\mathrm{e}}_{[0,Z_1)}(B_R)$ satisfying $\mathscr{L}_\lambda F = g$ on $(0,Z_1)$ and $F(0;\lambda)=1$ for all $\lambda\in B_R$.*

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Proof. By Lemma 5.3, we have $\widehat{B}_0 \in C_{\mathrm{o}}^\infty([0,+\infty))$ and A_0, $\widetilde{B}_0$, D_1, $D_2 \in C_{\mathrm{e}}^\infty([0,+\infty))$. Thus, there exist $\widetilde{A}_0, B_1, B_2, \widetilde{D}_1, \widetilde{D}_2 \in C^\infty([0,+\infty))$ such that $A_0(Z) = \widetilde{A}_0(Z^2), \widetilde{B}_0(Z) = B_1(Z^2), \widehat{B}_0(Z) = ZB_2(Z^2), D_1(Z) = \widetilde{D}_1(Z^2), D_2(Z) = \widetilde{D}_2(Z^2)$. Then $B_0(Z;\lambda) = Z^{-1}\widetilde{B}_0(Z) + \lambda\widehat{B}_0(Z) = Z^{-1}B_1(Z^2) + \lambda ZB_2(Z^2)$, $D_0(Z;\lambda) = \lambda D_1(Z) + \lambda^2 D_2(Z) = \lambda\widetilde{D}_1(Z^2) + \lambda^2\widetilde{D}_2(Z^2)$. Let $\widetilde{Z} := Z^2$. Then for $f(Z) = \widetilde{f}(Z^2) = \widetilde{f}(\widetilde{Z})$, we have $f'(Z) = 2Z\widetilde{f}'(\widetilde{Z})$, $f''(Z) = 2\widetilde{f}'(\widetilde{Z}) + 4Z^2\widetilde{f}''(\widetilde{Z})$, and by (5.16),

$$\begin{aligned}(\mathscr{L}_\lambda f)(Z) &= A_0(Z)f''(Z) + B_0(Z;\lambda)f'(Z) + D_0(Z;\lambda)f(Z)\\ &= \widetilde{A}_0(Z^2)[2\widetilde{f}'(\widetilde{Z}) + 4Z^2\widetilde{f}''(\widetilde{Z})] + [Z^{-1}B_1(Z^2) + \lambda ZB_2(Z^2)]\cdot 2Z\widetilde{f}'(\widetilde{Z})\\ &\quad + [\lambda\widetilde{D}_1(Z^2) + \lambda^2\widetilde{D}_2(Z^2)]\widetilde{f}(\widetilde{Z})\\ &= 4\widetilde{Z}\widetilde{A}_0(\widetilde{Z})\widetilde{f}''(\widetilde{Z}) + 2[\widetilde{A}_0(\widetilde{Z}) + B_1(\widetilde{Z}) + \lambda\widetilde{Z}B_2(\widetilde{Z})]\widetilde{f}'(\widetilde{Z}) + [\lambda\widetilde{D}_1(\widetilde{Z}) + \lambda^2\widetilde{D}_2(\widetilde{Z})]\widetilde{f}(\widetilde{Z}).\end{aligned}$$

Let

$$\widetilde{A}(\widetilde{Z}) = 4\widetilde{Z}\widetilde{A}_0(\widetilde{Z}),\ \widetilde{B}(\widetilde{Z};\lambda) = 2[\widetilde{A}_0(\widetilde{Z}) + B_1(\widetilde{Z}) + \lambda\widetilde{Z}B_2(\widetilde{Z})],\ \widetilde{D}(\widetilde{Z};\lambda) = \lambda\widetilde{D}_1(\widetilde{Z}) + \lambda^2\widetilde{D}_2(\widetilde{Z}).$$

Then we get

$$(\mathscr{L}_\lambda f)(Z) = \widetilde{A}(\widetilde{Z})\widetilde{f}''(\widetilde{Z}) + \widetilde{B}(\widetilde{Z};\lambda)\widetilde{f}'(\widetilde{Z}) + \widetilde{D}(\widetilde{Z};\lambda)\widetilde{f}(\widetilde{Z}),\ \text{for } f(Z) = \widetilde{f}(\widetilde{Z}),\ \widetilde{Z} = Z^2. \tag{5.30}$$

Let $I_1 = [0, Z_1^2)$. As $\widetilde{A}_0, B_1, B_2, \widetilde{D}_1, \widetilde{D}_2 \in C^\infty([0,+\infty))$, we have $\widetilde{A} \in C^\infty(I_1)$, $\widetilde{B}, \widetilde{D} \in \mathcal{H}_{I_1}(\mathbb{C})$, and $\widetilde{A}'(0) = 4\widetilde{A}_0(0) = 4A_0(0) = 4 \neq 0$ (using (5.17)). By (5.18), we have $\widetilde{A}_0(Z^2) = A_0(Z) > 0$ for $Z \in [0, Z_1)$. Thus, $\widetilde{A}(\widetilde{Z}) = 4\widetilde{Z}\widetilde{A}_0(\widetilde{Z}) = 0$ has a unique solution $\widetilde{Z} = 0$ in $I_1 = [0, Z_1^2)$.

Moreover, we have $\widetilde{B}(\widetilde{Z};\lambda) = \widetilde{B}_1(\widetilde{Z}) + \lambda\widetilde{B}_2(\widetilde{Z})$, where $\widetilde{B}_1(\widetilde{Z}) := 2[\widetilde{A}_0(\widetilde{Z}) + B_1(\widetilde{Z})], \widetilde{B}_2(\widetilde{Z}) := 2\widetilde{Z}B_2(\widetilde{Z})$, then $\widetilde{B}_1(0) = 2[\widetilde{A}_0(0) + B_1(0)] = 2[A_0(0) + \widetilde{B}_0(0)] = 2(1+k) > 0$ (using (5.17) and (5.19)) and $\widetilde{B}_2(0) = 0$. As a consequence, for any $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$ we have $n\widetilde{A}'(0) + \widetilde{B}(0;\lambda) = 4n + 2(1+k) \neq 0$. As $g \in C_{\mathrm{e}}^\infty([0, Z_1))$, there exists $\widetilde{g} \in C^\infty([0, Z_1^2))$ such that $g(Z) = \widetilde{g}(Z^2)$. By Proposition B.4, there exists $\widetilde{F} = \widetilde{F}(\widetilde{Z};\lambda) \in \mathcal{H}_{I_1}(B_R)$ satisfying

$$\widetilde{A}(\widetilde{Z})\widetilde{F}''(\widetilde{Z};\lambda) + \widetilde{B}(\widetilde{Z};\lambda)\widetilde{F}'(\widetilde{Z};\lambda) + \widetilde{D}(\widetilde{Z};\lambda)\widetilde{F}(\widetilde{Z};\lambda) = \widetilde{g}(\widetilde{Z}), \quad \widetilde{F}(0;\lambda) = 1,$$

where the prime $'$ denotes the derivative with respect to $\widetilde{Z}$. Now we define

$$F(Z;\lambda) := \widetilde{F}(Z^2;\lambda), \qquad \forall\, Z \in [0, Z_1), \forall\, \lambda \in B_R,$$

then $F \in \mathcal{H}^{\mathrm{e}}_{[0,Z_1)}(B_R)$, $F(0;\lambda) = \widetilde{F}(0;\lambda) = 1$ and $\mathscr{L}_\lambda F = g$ in $(0, Z_1)$ by recalling (5.30). □

In view of Lemma 5.3 and Proposition B.4, we let

$$\Lambda_* := \{\lambda \in \mathbb{C} : nA_0'(Z_1) + B_0(Z_1;\lambda) = 0 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}. \tag{5.31}$$

By $B_0(Z_1;\lambda) = Z_1^{-1}\widetilde{B}_0(Z_1) + \lambda\widehat{B}_0(Z_1)$ and $\widehat{B}_0(Z_1) > 0$, we know that $\Lambda_* \subset \mathbb{C}$ is a non-empty (countable) discrete set.

Lemma 5.8. *There exists a nonzero polynomial $\psi_1(\lambda)$ satisfying $\{\lambda \in B_R : \psi_1(\lambda) = 0\} = \Lambda_* \cap B_R$ such that for $g \in C^\infty((0,+\infty))$, there exists a function $F = F(Z;\lambda) \in \mathcal{H}_{(0,+\infty)}(B_R)$ satisfying $\mathscr{L}_\lambda F = \psi_1(\lambda)\cdot g$ on $(0,+\infty)$ and $F(Z_1;\lambda) = \psi_1(\lambda)$ for all $\lambda \in B_R$.*

Proof. By Lemma 5.3, we have $A_0(Z_1) = 0$, $A_0'(Z_1) \neq 0$, $\widehat{B}_0(Z_1) > 0$ and Z_1 is the unique solution of $A_0(Z) = 0$ in $(0,+\infty)$. Hence Lemma 5.8 follows from Proposition B.4. □

Taking $g = 0$ in Lemma 5.7 we know that there exists $\Psi_1 = \Psi_1(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,Z_1)}(B_R)$ satisfying $\mathscr{L}_\lambda\Psi_1 = 0$ on $(0, Z_1)$ and $\Psi_1(0;\lambda) = 1$ for all $\lambda \in B_R$. Taking $g = 0$ in Lemma 5.8 we know that there

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exists $\Psi_2 = \Psi_2(Z;\lambda) \in \mathcal{H}_{(0,+\infty)}(B_R)$ satisfying $\mathscr{L}_\lambda \Psi_2 = 0$ on $(0,+\infty)$ and $\Psi_2(Z_1;\lambda) = \psi_1(\lambda)$ for all $\lambda \in B_R$. We define the Wronski

$$W(Z;\lambda) := \Psi_1(Z;\lambda)\Psi_2'(Z;\lambda) - \Psi_1'(Z;\lambda)\Psi_2(Z;\lambda), \quad \forall\, Z \in (0,Z_1),\ \forall\, \lambda \in B_R, \tag{5.32}$$

where the prime $'$ denotes the derivative with respect to Z. Then we have

$$A_0(Z)W'(Z;\lambda) + B_0(Z;\lambda)W(Z;\lambda) = 0, \qquad \forall\, Z \in (0,Z_1), \quad \lambda \in B_R. \tag{5.33}$$

Lemma 5.9. *Fix $Z_0 \in (0,Z_1)$. Let $\psi_2(\lambda) := W(Z_0;\lambda)$ for all $\lambda \in B_R$ and $\lambda_0^* := \gamma - k - 2$. Then $\psi_2 \in \mathrm{Hol}(B_R)$, $0 < -\lambda_0^* < k < R$ and $\psi_2(\lambda_0^*) \neq 0$.*

Proof. As $\Psi_1(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,Z_1)}(B_R)$, $\Psi_2(Z;\lambda) \in \mathcal{H}_{(0,+\infty)}(B_R)$, by (5.32) we have $W(Z;\lambda) \in \mathcal{H}_{(0,Z_1)}(B_R)$, then by $Z_0 \in (0,Z_1)$ we have $\psi_2(\lambda) = W(Z_0;\lambda) \in \mathrm{Hol}(B_R)$. As $\beta > 0$, $\ell > 1$, $\gamma = \beta(\ell-1)+2$, $\lambda_0^* = \gamma - k - 2$, we get by (2.9) that

$$\lambda_0^* = \gamma - k - 2 = \beta(\ell-1) + 2 - k - 2 = \beta(\ell-1) - k < \beta(\ell-1) - \beta(\ell+\sqrt{\ell}) < 0, \tag{5.34}$$

and $R > k > k - \beta(\ell-1) = -\lambda_0^* > 0$, then $\lambda_0^* \in B_R$. It remains to prove that $\psi_2(\lambda_0^*) \neq 0$.

We consider the dual $\mathscr{L}_0^*$ of $\mathscr{L}_0$, defined by (5.25). On one hand, we get by Lemma 5.4 that

$$\mathscr{L}_0^* = \mathscr{L}_{\lambda_0^*} = A_0\partial_Z^2 + B_0(\cdot;\lambda_0^*)\partial_Z + D_0(\cdot;\lambda_0^*). \tag{5.35}$$

On the other hand, by (recalling that $D_0(\cdot;0) = 0$)

$$\mathscr{L}_0 f = A_0\partial_Z^2 + B_0(\cdot;0)\partial_Z = A_0\partial_Z^2 + Z^{-1}\widetilde{B}_0\partial_Z$$

and (5.25), we compute that

$$(\mathscr{L}_0^* f)(Z) = \frac{1}{Z^k}\left(\partial_Z^2(Z^k A_0 f)(Z) - \partial_Z(Z^{k-1}\widetilde{B}_0 f)(Z)\right), \quad \forall\, Z \in (0,+\infty). \tag{5.36}$$

Comparing the coefficients of ∂_Z in (5.35) and (5.36), we obtain

$$\begin{aligned} Z^{-1}\widetilde{B}_0(Z) + \lambda_0^*\widehat{B}_0(Z) = B_0(Z;\lambda_0^*) &= [2\partial_Z(Z^k A_0)(Z) - Z^{k-1}\widetilde{B}_0(Z)]/Z^k \\ &= 2kZ^{-1}A_0(Z) + 2A_0'(Z) - Z^{-1}\widetilde{B}_0(Z) \end{aligned}$$

for all $Z \in (0,+\infty)$. Letting $Z = Z_1$, we get(as $A_0(Z_1) = 0$, see (5.17))

$$Z_1^{-1}\widetilde{B}_0(Z_1) + \lambda_0^*\widehat{B}_0(Z_1)/2 = A_0'(Z_1). \tag{5.37}$$

For any $n \in \mathbb{Z}_{\geq 0}$, by (5.37), (5.17), (5.19) and $\lambda_0^* < 0$ (i.e. (5.34)), we have

$$\begin{aligned} nA_0'(Z_1) + B_0(Z_1;\lambda_0^*) &= nA_0'(Z_1) + Z_1^{-1}\widetilde{B}_0(Z_1) + \lambda_0^*\widehat{B}_0(Z_1) \\ &= (n+1)A_0'(Z_1) + \lambda_0^*\widehat{B}_0(Z_1)/2 < 0, \end{aligned} \tag{5.38}$$

It follows from (5.31) and (5.38) that $\lambda_0^* \notin \Lambda_*$. Then by $\lambda_0^* \in B_R$ and Lemma 5.8, we have $\psi_1(\lambda_0^*) \neq 0$. Let $f_1 = \Psi_1(\cdot;\lambda_0^*)$ and $f_2 = \Psi_2(\cdot;\lambda_0^*)$, then $f_1 \in C_{\mathrm{e}}^\infty([0,Z_1))$, $f_2 \in C^\infty((0,+\infty))$ and $(\mathscr{L}_{\lambda_0^*} f_j)(Z) = 0$ for $Z \in (0,Z_1)$, $j \in \{1,2\}$. By (5.35) and (5.36), we get

$$\partial_Z^2(Z^k A_0 f_j)(Z) - \partial_Z(Z^{k-1}\widetilde{B}_0 f_j)(Z) = 0, \quad \forall\, Z \in (0,Z_1),\ j \in \{1,2\}.$$

By $f_1, A_0, \widetilde{B}_0 \in C_{\mathrm{e}}^\infty([0, Z_1))$, $k \geq 3$ we have $[\partial_Z(Z^k A_0 f_1)(Z) - Z^{k-1}\widetilde{B}_0(Z) f_1(Z)]|_{Z=0} = 0$, so

$$\partial_Z(Z^k A_0 f_1)(Z) - Z^{k-1}\widetilde{B}_0(Z) f_1(Z) = 0, \quad \forall\, Z \in (0, Z_1). \tag{5.39}$$

For f_2, since $A_0(Z_1) = 0$ (see (5.17)), we have

$$\begin{aligned}
&\partial_Z(Z^k A_0 f_2)(Z) - Z^{k-1}\widetilde{B}_0(Z) f_2(Z) = [\partial_Z(Z^k A_0 f_2)(Z) - Z^{k-1}\widetilde{B}_0(Z) f_2(Z)]|_{Z=Z_1}\\
&\quad = Z_1^k A_0'(Z_1) f_2(Z_1) - Z_1^{k-1}\widetilde{B}_0(Z_1) f_2(Z_1) = Z_1^k (A_0'(Z_1) - Z_1^{-1}\widetilde{B}_0(Z_1))\psi_1(\lambda_0^*) =: C'
\end{aligned} \tag{5.40}$$

for all $Z \in (0, Z_1)$, where we have used $f_2(Z_1) = \Psi_2(Z_1; \lambda_0^*) = \psi_1(\lambda_0^*)$ (recalling Lemma 5.8). Moreover, by (5.37), $\lambda_0^* < 0$ (in (5.34)) and $\widehat{B}_0(Z_1) > 0$ (in (5.19)), we have $A_0'(Z_1) - Z_1^{-1}\widetilde{B}_0(Z_1) = \lambda_0^* \widehat{B}_0(Z_1)/2 < 0$, then by $\psi_1(\lambda_0^*) \neq 0$ we have $C' \neq 0$. We claim that

$$f_1(Z) \neq 0, \quad \forall\, Z \in (0, Z_1). \tag{5.41}$$

Indeed, if $f_1(Z^*) = 0$ for some $Z^* \in (0, Z_1)$, by the uniqueness of solutions to (5.39) in $(0, Z_1)$ with $f_1(Z^*) = 0$, we have $f_1(Z) = 0$ for all $Z \in (0, Z_1)$, which contradicts with $1 = f_1(0) = \lim_{Z\to 0+} f_1(Z)$. This proves (5.41). As $f_1 = \Psi_1(\cdot; \lambda_0^*)$, $f_2 = \Psi_2(\cdot; \lambda_0^*)$, by (5.32), (5.39), (5.40), (5.41) and $C' \neq 0$, we have

$$\begin{aligned}
Z^k A_0(Z) W(Z; \lambda_0^*) &= Z^k A_0(Z)[f_1(Z) f_2'(Z) - f_1'(Z) f_2(Z)]\\
&= f_1(Z)\partial_Z(Z^k A_0 f_2)(Z) - \partial_Z(Z^k A_0 f_1)(Z) f_2(Z)\\
&= f_1(Z)[Z^{k-1}\widetilde{B}_0(Z) f_2(Z) + C'] - Z^{k-1}\widetilde{B}_0(Z) f_1(Z) f_2(Z) = C' f_1(Z) \neq 0,
\end{aligned}$$

for all $Z \in (0, Z_1)$. Thus, $W(Z; \lambda_0^*) \neq 0$ for all $Z \in (0, Z_1)$, and $\psi_2(\lambda_0^*) = W(Z_0; \lambda_0^*) \neq 0$. □

Now we fix $Z_0 \in (0, Z_1)$, $\psi_2(\lambda) = W(Z_0; \lambda)$, $\lambda_0^* := \gamma - k - 2$. Let $\psi_1(\lambda)$ be given by Lemma 5.8 and $\varphi(\lambda) := \psi_1(\lambda)\psi_2(\lambda)$ for all $\lambda \in B_R$. Let $g \in C_{\mathrm{e}}^\infty([0, +\infty))$, we need to prove that there exists a function $f = f(Z; \lambda) \in \mathcal{H}_{[0,+\infty)}^{\mathrm{e}}(B_R)$ such that $\mathscr{L}_\lambda f(\cdot; \lambda) = \varphi(\lambda) \cdot g$ on $(0, +\infty)$.

We first consider the case when g is supported near $Z = 0$.

Lemma 5.10. *Assume that* $g \in C_{\mathrm{e}}^\infty([0, +\infty))$ *satisfies* $\operatorname{supp} g \subset [0, Z_1)$*, then there exists a function* $f = f(Z; \lambda) \in \mathcal{H}_{[0,+\infty)}^{\mathrm{e}}(B_R)$ *such that* $\mathscr{L}_\lambda f(\cdot; \lambda) = \varphi(\lambda) \cdot g$ *on* $(0, +\infty)$.

Proof. By Lemma 5.7, there exists $f_0 \in \mathcal{H}_{[0,Z_1)}^{\mathrm{e}}(B_R)$ such that $\mathscr{L}_\lambda f_0 = g$ on $(0, Z_1)$ with $f_0(0; \lambda) = 1$ for all $\lambda \in B_R$. We assume that $\operatorname{supp} g \subset [0, \delta)$ for some $\delta \in (0, Z_1)$, then $(\mathscr{L}_\lambda f_0)(Z) = 0$ for $Z \in [\delta, Z_1)$. For $\lambda \in B_R$, let

$$C_1(\lambda) := f_0(\delta; \lambda)\Psi_2'(\delta; \lambda) - f_0'(\delta; \lambda)\Psi_2(\delta; \lambda) \in \mathbb{C}, \tag{5.42}$$

$$C_2(\lambda) := f_0'(\delta; \lambda)\Psi_1(\delta; \lambda) - f_0(\delta; \lambda)\Psi_1'(\delta; \lambda) \in \mathbb{C}. \tag{5.43}$$

Then C_1, C_2 are holomorphic functions on B_R and for all $\lambda \in B_R$ there holds

$$\begin{aligned}
W(\delta; \lambda) f_0(\delta; \lambda) &= C_1(\lambda)\Psi_1(\delta; \lambda) + C_2(\lambda)\Psi_2(\delta; \lambda),\\
W(\delta; \lambda) f_0'(\delta; \lambda) &= C_1(\lambda)\Psi_1'(\delta; \lambda) + C_2(\lambda)\Psi_2'(\delta; \lambda).
\end{aligned}$$

By the uniqueness of the solution on $[\delta, Z_1)$, we have

$$W(\delta; \lambda) f_0(Z; \lambda) = C_1(\lambda)\Psi_1(Z; \lambda) + C_2(\lambda)\Psi_2(Z; \lambda), \qquad \forall\, Z \in [\delta, Z_1),\ \forall\, \lambda \in B_R.$$

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For $\lambda \in B_R$, let

$$f_*(Z;\lambda) := \begin{cases} W(\delta;\lambda) f_0(Z;\lambda) - C_1(\lambda)\Psi_1(Z;\lambda) & \text{if } Z \in [0, Z_1), \\ C_2(\lambda)\Psi_2(Z;\lambda) & \text{if } Z \in [\delta, +\infty). \end{cases}$$

Then $f_* \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ and $\mathscr{L}_\lambda f_* = W(\delta;\lambda) \cdot g$ on $(0,+\infty)$. By (5.33), we have $W(Z;\lambda) = W(Z_0;\lambda)\mathrm{e}^{-A_*(Z;\lambda)} = \psi_2(\lambda)\mathrm{e}^{-A_*(Z;\lambda)}$ with $A_*(Z;\lambda) := \int_{Z_0}^{Z} \frac{B_0(Z;\lambda)}{A_0(Z)}\mathrm{d}Z \in \mathcal{H}_{(0,Z_1)}(B_R)$ (using Lemma 5.3). Recall that $\varphi = \psi_1\psi_2$, ψ_1 is a polynomial, then $\varphi(\lambda) = \psi_1(\lambda)W(\delta;\lambda)\mathrm{e}^{A_*(\delta;\lambda)}$, and the result follows by taking $f(Z;\lambda) := \psi_1(\lambda)\mathrm{e}^{A_*(\delta;\lambda)} f_*(Z;\lambda)$ for $Z \in [0,+\infty), \lambda \in B_R$. □

Now we consider the case when g is supported away from $Z = 0$.

Lemma 5.11. *Assume that $g \in C^\infty_{\mathrm{e}}([0,+\infty))$ satisfies* $\operatorname{supp} g \subset (0,+\infty)$, *then there exists a function $f = f(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ such that $\mathscr{L}_\lambda f(\cdot;\lambda) = \varphi(\lambda) \cdot g$ on* $(0,+\infty)$.

Proof. By Lemma 5.8, there exists $f_0 = f_0(Z;\lambda) \in \mathcal{H}_{(0,+\infty)}(B_R)$ such that $\mathscr{L}_\lambda f_0 = \psi_1(\lambda)\cdot g$ on $(0,+\infty)$. We assume that $\operatorname{supp} g \subset (\delta,+\infty)$ for some $\delta \in (0, Z_1)$, then $(\mathscr{L}_\lambda f_0)(Z) = 0$ for $Z \in (0,\delta]$. For $\lambda \in B_R$, let $C_1(\lambda), C_2(\lambda)$ be defined by (5.42) and (5.43) respectively. For the same reason as in the proof of Lemma 5.10, we have

$$W(\delta;\lambda) f_0(Z;\lambda) = C_1(\lambda)\Psi_1(Z;\lambda) + C_2(\lambda)\Psi_2(Z;\lambda), \qquad \forall\, Z \in (0,\delta],\ \forall\, \lambda \in B_R.$$

For $\lambda \in B_R$, let

$$f_*(Z;\lambda) := \begin{cases} W(\delta;\lambda) f_0(Z;\lambda) - C_2(\lambda)\Psi_2(Z;\lambda) & \text{if } Z \in (0, +\infty), \\ C_1(\lambda)\Psi_1(Z;\lambda) & \text{if } Z \in [0, \delta]. \end{cases}$$

Then $f_* \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ and $\mathscr{L}_\lambda f_* = W(\delta;\lambda)\psi_1(\lambda) \cdot g$ on $(0,+\infty)$. As in the proof of Lemma 5.10, we have $\varphi(\lambda) = \psi_1(\lambda)W(\delta;\lambda)\mathrm{e}^{A_*(\delta;\lambda)}$ and $A_*(Z;\lambda) \in \mathcal{H}_{(0,Z_1)}(B_R)$, then the result follows by taking $f(Z;\lambda) := \mathrm{e}^{A_*(\delta;\lambda)} f_*(Z;\lambda)$ for $Z \in [0,+\infty), \lambda \in B_R$. □

Now we are in a position to prove Lemma 5.6.

Proof of Lemma 5.6. We fix $Z_0 \in (0, Z_1)$, $\psi_2(\lambda) = W(Z_0;\lambda)$, $\lambda_0^* := \gamma - k - 2$. Let $\psi_1(\lambda)$ be given by Lemma 5.8 and $\varphi(\lambda) := \psi_1(\lambda)\psi_2(\lambda)$ for all $\lambda \in B_R$. By Lemma 5.9, we have $\lambda_0^* \in B_R$ and $\psi_2 \in \mathrm{Hol}(B_R) \setminus \{0\}$. By Lemma 5.8 we have $\psi_1 \in \mathrm{Hol}(B_R) \setminus \{0\}$. Thus, $\varphi = \psi_1\psi_2 \in \mathrm{Hol}(B_R) \setminus \{0\}$.

Let $\zeta \in C^\infty(\mathbb{R}; [0,1])$ satisfy $\operatorname{supp}\zeta \subset (Z_1/2, +\infty)$ and $\zeta(Z) = 1$ for $Z \in [3Z_1/4, +\infty)$. Let $g_1(Z) = g(Z)(1 - \zeta(Z)), g_2(Z) = g(Z)\zeta(Z)$ for all $Z \in [0,+\infty)$. Then

$$\operatorname{supp} g_1 \subset [0, 3Z_1/4], \quad \operatorname{supp} g_2 \subset [Z_1/2, +\infty), \quad g_1, g_2 \in C^\infty_{\mathrm{e}}([0,+\infty)), \quad g = g_1 + g_2.$$

By Lemma 5.10, there exists $f_1 = f_1(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ such that $\mathscr{L}_\lambda f_1 = \varphi(\lambda)g_1$ on $(0,+\infty)$. By Lemma 5.11, there exists $f_2 = f_2(Z;\lambda) \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ such that $\mathscr{L}_\lambda f_2 = \varphi(\lambda)g_2$ on $(0,+\infty)$. Let $f = f_1 + f_2$, then $f \in \mathcal{H}^{\mathrm{e}}_{[0,+\infty)}(B_R)$ satisfies $\mathscr{L}_\lambda f = \varphi(\lambda)g$ on $(0,+\infty)$. □

A. The derivation and properties of ODE (2.8)

A.1. The derivation of ODE (2.8)

Lemma A.1. *Let $\beta > 1$ and $v = v(Z) \in C^\infty([0,+\infty); (-1,1))$ be given by Assumption 1. We define $\widehat{\phi}_0(Z), \widehat{\rho}_0(Z)$ according to (2.12) and we define $\phi_0(t,x), \rho_0(t,x)$ by (2.7). Then (ϕ_0, ρ_0) solves the leading order equation (2.6).*

Proof. Recall that $Z = r/(T-t)$ with $r = |x|$, we know that $\phi_0 = \phi_0(t,r)$ and $\rho_0 = \rho_0(t,r)$ are radially symmetric. Hence, (2.6) is equivalent to

$$\rho_0^{p-1} - |\partial_t\phi_0|^2 + |\partial_r\phi_0|^2 = 0, \qquad -\partial_t(\rho_0^2\partial_t\phi_0) + \partial_r(\rho_0^2\partial_r\phi_0) + \frac{k}{r}\rho_0^2\partial_r\phi_0 = 0, \tag{A.1}$$

where $k = d-1 \in \mathbb{Z}_{\geq 1}$. It follows from (2.7) that

$$\partial_t\phi_0(t,x) = (T-t)^{-\beta}\big((\beta-1)\widehat{\phi}_0(Z) + Z\widehat{\phi}_0'(Z)\big), \quad \partial_r\phi_0(t,x) = (T-t)^{-\beta}\widehat{\phi}_0'(Z), \tag{A.2}$$

where the prime $'$ stands for the derivative with respect to Z. By (2.12), we have

$$\widehat{\phi}_0'(Z) = \frac{(\beta-1)\widehat{\phi}_0(Z)v(Z)}{1-Zv(Z)} \Longleftrightarrow \big((\beta-1)\widehat{\phi}_0(Z) + Z\widehat{\phi}_0'(Z)\big)v(Z) = \widehat{\phi}_0'(Z), \tag{A.3}$$

thus

$$\partial_t\phi_0(t,x) = \frac{(T-t)^{-\beta}(\beta-1)\widehat{\phi}_0(Z)}{1-Zv(Z)}, \qquad \partial_r\phi_0(t,x) = \frac{(T-t)^{-\beta}(\beta-1)\widehat{\phi}_0(Z)v(Z)}{1-Zv(Z)},$$

and then we have

$$|\partial_t\phi_0|^2 - |\partial_r\phi_0|^2 = \frac{(T-t)^{-2\beta}(\beta-1)^2\widehat{\phi}_0(Z)^2(1-v(Z)^2)}{(1-Zv(Z))^2}.$$

Using (2.7) and (2.12) for ρ_0 and $\widehat{\rho}_0$, we obtain the first equation in (A.1).

Now we define

$$\ell = \frac{4}{p-1} + 1 > 1, \quad \widetilde{\phi}_0(Z) := \frac{\widehat{\phi}_0(Z)^{\ell}(1-v(Z)^2)^{\frac{2}{p-1}}}{(1-Zv(Z))^{\ell}} = \frac{\widehat{\phi}_0(Z)^{\ell}(1-v(Z)^2)^{\frac{\ell-1}{2}}}{(1-Zv(Z))^{\ell}} > 0. \tag{A.4}$$

Then we compute that

$$\begin{aligned}
\rho_0^2\partial_t\phi_0(t,x) &= (T-t)^{-\beta\ell}(\beta-1)^{\ell}\widetilde{\phi}_0(Z),\\
\rho_0^2\partial_r\phi_0(t,x) &= (T-t)^{-\beta\ell}(\beta-1)^{\ell}\widetilde{\phi}_0(Z)v(Z),\\
\partial_t(\rho_0^2\partial_t\phi_0)(t,x) &= (T-t)^{-\beta\ell-1}(\beta-1)^{\ell}\big(\beta\ell\widetilde{\phi}_0(Z) + Z\widetilde{\phi}_0'(Z)\big),\\
\partial_r(\rho_0^2\partial_r\phi_0)(t,x) &= (T-t)^{-\beta\ell-1}(\beta-1)^{\ell}(\widetilde{\phi}_0 v)'(Z),\\
\frac{k}{r}\rho_0^2\partial_r\phi_0(t,x) &= (T-t)^{-\beta\ell-1}(\beta-1)^{\ell}\frac{k}{Z}(\widetilde{\phi}_0 v)(Z).
\end{aligned}$$

Therefore, the second equation in (A.1) is equivalent to

$$\beta\ell\widetilde{\phi}_0 + Z\widetilde{\phi}_0' = (\widetilde{\phi}_0 v)' + \frac{k}{Z}(\widetilde{\phi}_0 v) \Longleftrightarrow (\beta\ell - v' - kv/Z)\widetilde{\phi}_0 = (v-Z)\widetilde{\phi}_0'. \tag{A.5}$$

Recall from (A.3) and (A.4) that

$$\begin{aligned}
\frac{\widetilde{\phi}_0'}{\widetilde{\phi}_0} &= \ell\frac{\widehat{\phi}_0'}{\widehat{\phi}_0} - (\ell-1)\frac{vv'}{1-v^2} + \ell\frac{v+Zv'}{1-Zv}\\
&= \ell\frac{(\beta-1)v}{1-Zv} + \ell\frac{v}{1-Zv} - (\ell-1)\frac{vv'}{1-v^2} + \ell\frac{Zv'}{1-Zv}\\
&= \frac{\beta\ell v}{1-Zv} + \frac{vv'}{1-v^2} + \frac{\ell(Z-v)v'}{(1-v^2)(1-Zv)},
\end{aligned}$$

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hence (A.5) is equivalent to

$$\beta\ell - v' - \frac{kv}{Z} = (v-Z)\frac{\widetilde{\phi}_0'}{\widetilde{\phi}_0} = \frac{\beta\ell v(v-Z)}{1-Zv} + \frac{v(v-Z)v'}{1-v^2} - \frac{\ell(v-Z)^2v'}{(1-v^2)(1-Zv)},$$

or equivalently,

$$\begin{aligned}\beta\ell - \frac{\beta\ell v(v-Z)}{1-Zv} - \frac{kv}{Z} &= v' + \frac{v(v-Z)v'}{1-v^2} - \frac{\ell(v-Z)^2v'}{(1-v^2)(1-Zv)},\\ \frac{\beta\ell(1-v^2)}{1-Zv} - \frac{kv}{Z} &= \frac{(1-Zv)v'}{1-v^2} - \frac{\ell(v-Z)^2v'}{(1-v^2)(1-Zv)},\end{aligned}$$

which is a direct consequence of (2.8). □

A.2. Properties of solutions to ODE (2.8)

In this part, we prove Remark 2.2.

Lemma A.2. *Under Assumption 1, we have $v(Z_1) = v_1$, where*

$$Z_1 = \frac{k}{\sqrt{\ell}(k-\beta(\ell-1))}, \qquad v_1 = \frac{\beta\sqrt{\ell}}{k-\beta\ell}. \tag{A.6}$$

Proof. We define a function $F_0 \in C^\infty([0,+\infty))$ by

$$F_0(Z) := 1 - Zv(Z) + \sqrt{\ell}(v(Z)-Z), \quad \forall\, Z \in [0,+\infty). \tag{A.7}$$

Then $F_0(0) = 1$ and $F_0(1) = (\sqrt{\ell}-1)(v(1)-1) < 0$, where we have used that $v(1) \in (-1,1)$, recalling Assumption 1. By the intermediate value theorem, there exists $Z_0 \in (0,1)$ such that $F_0(Z_0) = 0$. Thus, $\Delta_Z(Z_0, v(Z_0)) = Z_0F_0(Z_0)\big(1 - Z_0v(Z_0) - \sqrt{\ell}(v(Z_0)-Z_0)\big) = 0$. Then we have $\Delta_v(Z_0, v(Z_0)) = \Delta_Z(Z_0, v(Z_0))v'(Z_0) = 0$, i.e., $\Delta_v(Z_0, v(Z_0)) = \Delta_Z(Z_0, v(Z_0)) = 0$. On the other hand, it is direct to check that

$$\{(Z,v) \in (0,+\infty)\times(-1,1) : \Delta_v(Z,v) = \Delta_Z(Z,v) = 0\} = \{(Z_1,v_1)\}, \tag{A.8}$$

where Z_1, v_1 are given by (A.6). Moreover, it follows from (2.9) that $0 < v_1 < Z_1 < 1$. Hence, we must have $(Z_0, v(Z_0)) = (Z_1, v_1)$, which implies that $v(Z_1) = v_1$. □

Lemma A.3. *Under Assumption 1, let $\Delta_0(Z) := \Delta_Z(Z, v(Z))$ for $Z \in [0,+\infty)$, then we have $\Delta_0(Z) > 0$ for $Z \in (0, Z_1)$, $\Delta_0(Z) < 0$ for $Z \in (Z_1,+\infty)$ and $\Delta_0'(Z_1) \neq 0$.*

Proof. By the definition of $\Delta_0(Z)$, we have $\Delta_0 \in C^\infty([0,+\infty))$ and (see (2.8))

$$\Delta_0(Z) = ZF_0(Z)\widetilde{F}_0(Z), \quad \forall\, Z \in [0,+\infty), \tag{A.9}$$

where $F_0 \in C^\infty([0,+\infty))$ is defined by (A.7) and $\widetilde{F}_0 \in C^\infty([0,+\infty))$ is defined by

$$\widetilde{F}_0(Z) := 1 - Zv(Z) - \sqrt{\ell}(v(Z)-Z), \quad \forall\, Z \in [0,+\infty). \tag{A.10}$$

If $\Delta_0(Z_*) = 0$ for some $Z_* \in (0,+\infty)$, then $\Delta_v(Z_*, v(Z_*)) = \Delta_0(Z_*)v'(Z_*) = 0$, and by (A.8) we obtain $Z_* = Z_1$, hence (using Lemma A.2, (A.8) and (A.9))

$$\{Z \in [0,+\infty) : \Delta_0(Z) = 0\} = \{0, Z_1\}. \tag{A.11}$$

Now we prove that $\Delta_0'(Z_1) \neq 0$. By the proof of Lemma A.2, we have $F_0(Z_1) = 0$. Then by (A.7) and (A.10), we have $\widetilde{F}_0(Z_1) = 2(1 - Z_1 v(Z_1)) > 0$. Thus (using (A.9)), $\Delta_0'(Z_1) = Z_1 F_0'(Z_1)\widetilde{F}_0(Z_1)$. Assume on the contrary that $\Delta_0'(Z_1) = 0$, then $F_0'(Z_1) = 0$.

Let $\Delta_1(Z) := \Delta_v(Z, v(Z))$. Then (2.8) becomes $\Delta_0(Z)v'(Z) = \Delta_1(Z)$. Taking derivative with respect to Z at $Z = Z_1$ on both sides, we obtain(using $\Delta_0(Z_1) = 0$) $\Delta_1'(Z_1) = \Delta_0'(Z_1)v'(Z_1) = 0$ and $\Delta_1(Z_1) = 0$. By (2.8), we have $\Delta_1(Z) = F_1(Z)\widetilde{F}_1(Z)$ with $F_1(Z) := Z - g_1(v(Z))$, $\widetilde{F}_1(Z) := (1 - v(Z)^2)(\beta\ell + (k - \beta\ell)v(Z)^2)$, $g_1(v) := kv/(\beta\ell + (k - \beta\ell)v^2)$ (note that $0 < \beta\ell < k$ using (2.9)). As $v(Z_1) \in (-1, 1)$, we have $\widetilde{F}_1(Z_1) > 0$, then by $\Delta_1'(Z_1) = 0$ and $\Delta_1(Z_1) = 0$ we have $F_1(Z_1) = 0$, $\Delta_1'(Z_1) = F_1'(Z_1)\widetilde{F}_1(Z_1) = 0$ and $F_1'(Z_1) = 0$. Thus, $0 = 1 - g_1'(v(Z_1))v'(Z_1) = 1 - g_1'(v_1)v'(Z_1)$ (using Lemma A.2).

Similarly, by (A.7), we have $F_0(Z) = F_2(Z)\widetilde{F}_2(Z)$ with $F_2(Z) := g_2(v(Z)) - Z$, $\widetilde{F}_2(Z) := v(Z) + \ell$, $g_2(v) := (1 + \sqrt{\ell}v)/(v + \sqrt{\ell})$, and $\widetilde{F}_2(Z_1) > 0$. Thus, $F_2(Z_1) = 0$, $F_2'(Z_1) = 0$ (using $F_0(Z_1) = F_0'(Z_1) = 0$) and $0 = g_2'(v_1)v'(Z_1) - 1$.

Now we have $1 = g_1'(v_1)v'(Z_1) = g_2'(v_1)v'(Z_1)$ and $g_1'(v_1) = g_2'(v_1)$. On the other hand,

$$
\begin{aligned}
g_1(v) - g_2(v) &= \frac{kv}{\beta\ell + (k - \beta\ell)v^2} - \frac{1 + \sqrt{\ell}v}{v + \sqrt{\ell}} = \frac{(1 - v^2)(kv\sqrt{\ell} - \beta\ell(1 + \sqrt{\ell}v)}{(\beta\ell + (k - \beta\ell)v^2)(v + \sqrt{\ell})} \\
&= \frac{(1 - v^2)(k - \beta\ell)\sqrt{\ell}(v - v_1)}{(\beta\ell + (k - \beta\ell)v^2)(v + \sqrt{\ell})},
\end{aligned}
$$

here we used (A.6), thus

$$
g_1'(v_1) - g_2'(v_1) = \frac{(1 - v_1^2)(k - \beta\ell)\sqrt{\ell}}{(\beta\ell + (k - \beta\ell)v_1^2)(v_1 + \sqrt{\ell})} > 0,
$$

which is a contradiction. Therefore, $\Delta_0'(Z_1) \neq 0$.

By $F_0(0) = \widetilde{F}_0(0) = 1 > 0$, (A.9), and (A.11), we have $\Delta_0(Z) > 0$ for $Z \in (0, Z_1)$. Finally, using $\Delta_0'(Z_1) \neq 0$ and (A.11), we have $\Delta_0'(Z_1) < 0$ and $\Delta_0(Z) < 0$ for all $Z \in (Z_1, +\infty)$. □

To finish the proof of Remark 2.2, it remains to show that $v(Z) < Z$ and $Zv(Z) < 1$ for all $Z \in (0, +\infty)$. We use the barrier function method. For any $V \in C^1((0, +\infty))$, we define

$$
(\mathscr{A}V)(Z) := -\Delta_Z(Z, V(Z))V'(Z) + \Delta_v(Z, V(Z)), \quad \forall\, Z \in (0, +\infty). \tag{A.12}
$$

Then $\mathscr{A}v = 0$ if v is given by Assumption 1.

Lemma A.4. *Under Assumption 1, we have $v(Z) < Z$ for all $Z \in (0, +\infty)$.*

Proof. Since $v(Z) \in (-1, 1)$ for all $Z \in (0, +\infty)$ by Assumption 1, it suffices to prove $v(Z) < Z$ for all $Z \in (0, 1)$. We first show that $v(Z) < Z$ for all $Z \in (Z_1, 1)$. By (A.9), $F_0(0) = 1, F_0(Z_1) = 0$ and (A.11), we have

$$
\{Z \in [0, +\infty) : F_0(Z) = 0\} = \{Z_1\}.
$$

As $F_0(1) < 0$ and $Z_1 \in (0, 1)$, we have $F_0(Z) < 0$ for all $Z \in (Z_1, 1)$, hence

$$
v(Z) < \frac{\sqrt{\ell}Z - 1}{\sqrt{\ell} - Z} < Z, \quad \forall\, Z \in (Z_1, 1).
$$

Next we prove that $v(Z) < Z$ for all $Z \in (0, Z_1)$. Let $V_1(Z) := Z$ for all $Z \in [0, +\infty)$, then we have

$$
(\mathscr{A}V_1)(Z) = (\beta\ell - k - 1)Z(1 - Z^2)^2 < 0, \quad \forall\, Z \in (0, 1), \tag{A.13}
$$

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where we have used $\beta\ell - k - 1 < 0$, which follows from (2.9). On the other hand, letting $Z \to 0+$ in $\mathrm{d}v/\mathrm{d}Z = \Delta_v(Z, v(Z))/\Delta_Z(Z, v(Z))$, by L'Hôpital's rule, we have

$$v'(0) = \frac{\partial_Z \Delta_v(0,0) + \partial_v \Delta_v(0,0)v'(0)}{\partial_Z \Delta_Z(0,0) + \partial_v \Delta_Z(0,0)v'(0)} = \beta\ell - kv'(0),$$

hence $v'(0) = \beta\ell/(k+1) < 1 = V_1'(0)$. As $v(0) = 0 = V_1(0)$, there exists $\delta \in (0, Z_1)$ such that $v(Z) < V_1(Z)$ for all $Z \in (0, \delta)$. Assume for contradiction that $Z_* \in (0, Z_1) \subset (0,1)$ satisfies $v(Z) < V_1(Z)$ for all $Z \in (0, Z_*)$ and $v(Z_*) = V_1(Z_*) = Z_*$, then $v'(Z_*) \geq V_1'(Z_*)$. Thus, by $\Delta_Z(Z_*, V_1(Z_*)) = \Delta_Z(Z_*, v(Z_*)) = Z_*(1 - Z_*^2)^2 > 0$, we have

$$\begin{aligned}(\mathscr{A}V_1)(Z_*) &= -\Delta_Z(Z_*, V_1(Z_*))V_1'(Z_*) + \Delta_v(Z_*, V_1(Z_*))\\ &\geq -\Delta_Z(Z_*, v(Z_*))v'(Z_*) + \Delta_v(Z_*, v(Z_*)) = (\mathscr{A}v)(Z_*) = 0,\end{aligned}$$

which contradicts with (A.13). Therefore, we have $v(Z) < V_1(Z) = Z$ for all $Z \in (0, Z_1)$.

Finally, by $v(Z_1) = v_1 < Z_1 < 1$, we obtain $v(Z) < Z$ for all $Z \in (0,1)$. □

Lemma A.5. *Under Assumption 1, we have $v(Z) < 1/Z$ for all $Z \in (0, +\infty)$.*

Proof. Since $v(Z) \in (-1, 1)$ for all $Z \in (0, +\infty)$ by Assumption 1, it suffices to prove $v(Z) < 1/Z$ for all $Z \in (1, +\infty)$. Let $V_2(Z) := 1/Z$ for $Z \in (0, +\infty)$, then we have

$$(\mathscr{A}V_2)(Z) = (\beta - 1)\ell Z\left(1 - 1/Z^2\right)^2 > 0, \quad \forall\, Z \in (1, +\infty), \tag{A.14}$$

where we have used $\beta > 1$ by (2.9). As $v(Z_1) = v_1 < 1 = V_2(Z_1)$, there exists $\delta > 0$ such that $v(Z) < V_2(Z)$ for $Z \in [Z_1, Z_1 + \delta)$. Assume for contradiction that $Z^* \in (1, +\infty)$ satisfies $v(Z) < V_2(Z)$ for all $Z \in (Z_1, Z^*)$ and $v(Z^*) = V_2(Z^*)$, then $v'(Z^*) \geq V_2'(Z^*)$. Thus, by $\Delta_Z(Z^*, V_2(Z^*)) = \Delta_Z(Z^*, v(Z^*)) = -\ell(1 - Z_*^2)^2/Z_* < 0$, we have

$$\begin{aligned}(\mathscr{A}V_2)(Z^*) &= -\Delta_Z(Z^*, V_2(Z^*))V_2'(Z^*) + \Delta_v(Z^*, V_2(Z^*))\\ &\leq -\Delta_Z(Z^*, v(Z^*))v'(Z^*) + \Delta_v(Z^*, v(Z^*)) = (\mathscr{A}v)(Z^*) = 0,\end{aligned}$$

which contradicts with (A.14). Therefore, $v(Z) < V_2(Z) = 1/Z$ for all $Z \in (1, +\infty)$. □

The proof of Remark 2.2 is completed now. To conclude this appendix, we prove that $\widehat{\phi}_0, \widehat{\rho}_0 \in C_{\mathrm{e}}^\infty([0, +\infty))$, where $\widehat{\phi}_0$ and $\widehat{\rho}_0$ are defined by (2.12).

Lemma A.6. *Let $v \in C_{\mathrm{o}}^\infty([0, +\infty))$ be given by Assumption 1, and define $\widehat{\phi}_0, \widehat{\rho}_0$ by (2.12). Then we have $\widehat{\phi}_0, \widehat{\rho}_0 \in C_{\mathrm{e}}^\infty([0, +\infty))$.*

Proof. We first claim that

$$f \in C_{\mathrm{o}}^\infty([0, +\infty)) \Longrightarrow F(Z) := \int_0^Z f(s)\,\mathrm{d}s \in C_{\mathrm{e}}^\infty([0, +\infty)). \tag{A.15}$$

Now we prove that $\widehat{\phi}_0 \in C_{\mathrm{e}}^\infty([0, +\infty))$. By $v \in C_{\mathrm{o}}^\infty([0, +\infty))$, we have $1 - Zv(Z) \in C_{\mathrm{e}}^\infty([0, +\infty))$. Since $Zv(Z) < 1$ for all $Z \in [0, +\infty)$ by Lemma A.5, it follows from (5.8) that $0 < \frac{1}{1 - Zv(Z)} \in C_{\mathrm{e}}^\infty([0, +\infty))$, hence by $v \in C_{\mathrm{o}}^\infty([0, +\infty))$ and (5.7) we have $\frac{v(Z)}{1 - Zv(Z)} \in C_{\mathrm{o}}^\infty([0, +\infty))$, then by (A.15) we obtain

$$(\beta - 1)\int_0^Z \frac{v(s)}{1 - sv(s)}\,\mathrm{d}s \in C_{\mathrm{e}}^\infty([0, +\infty)).$$

Thus, by (5.9) and (2.12) we have $\widehat{\phi}_0 \in C_{\mathrm{e}}^\infty([0, +\infty))$.

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As for $\widehat{\rho}_0$, by $0 < \frac{1}{1-Zv(Z)} \in C_{\mathrm{e}}^{\infty}([0,+\infty))$, and (5.10), we have $1/(1-Zv(Z))^{\frac{2}{p-1}} \in C_{\mathrm{e}}^{\infty}([0,+\infty))$. Similarly, using $\widehat{\phi}_0 \in C_{\mathrm{e}}^{\infty}([0,+\infty))$ and $\widehat{\phi}_0(Z) > 0$ for all $Z \in [0,+\infty)$, we get $\widehat{\phi}_0(Z)^{2/(p-1)} \in C_{\mathrm{e}}^{\infty}([0,+\infty))$. It follows from $1-v(Z)^2 \in C_{\mathrm{e}}^{\infty}([0,+\infty))$, $v(Z) \in (-1,1)$ for all $Z \in [0,+\infty)$ and (5.10) that $(1-v(Z)^2)^{\frac{1}{p-1}} \in C_{\mathrm{e}}^{\infty}([0,+\infty))$. Therefore, by (2.12) and (5.5), we have $\widehat{\rho}_0 \in C_{\mathrm{e}}^{\infty}([0,+\infty))$.

Finally, it suffices to show the claim (A.15). By (2.11), there exists $\widetilde{f} \in C^{\infty}([0,+\infty))$ such that $f(Z) = Z\widetilde{f}(Z^2)$ for all $Z \in [0,+\infty)$. Let

$$\widetilde{F}(Z) := \frac{1}{2}\int_0^Z \widetilde{f}(s)\,\mathrm{d}s, \quad \forall\, Z \in [0,+\infty),$$

then $\widetilde{F} \in C^{\infty}([0,+\infty))$. Moreover, we have

$$F(Z) = \int_0^Z f(s)\,\mathrm{d}s = \int_0^Z s\widetilde{f}(s^2)\,\mathrm{d}s = \frac{1}{2}\int_0^{Z^2} \widetilde{f}(s)\,\mathrm{d}s = \widetilde{F}(Z^2), \quad \forall\, Z \in [0,+\infty).$$

Hence by (2.10), we have $F \in C_{\mathrm{e}}^{\infty}([0,+\infty))$. □

B. Linear ODEs with singular points

In this appendix, we establish the well-posedness theory for a class of second order linear ODEs with singular points.

First of all, we introduce a preliminary lemma, which ensures that the functions we are considering are smooth in the sense of multi-variable functions. Let $I \subset \mathbb{R}$ be an interval and let $\Omega \subset \mathbb{C}$ be an open subset. We define

$$\begin{aligned} \mathrm{Hol}(\Omega) &:= \{x = x(\lambda) \text{ is holomorphic (or equivalently, analytic) on } \Omega\}, && \text{(B.1)}\\ \mathcal{H}_I^0(\Omega) &:= \big\{x = x(t;\lambda) \in C(I\times\Omega;\mathbb{C}) : x(\cdot;\lambda) \in C^{\infty}(I) \text{ for all } \lambda \in \Omega,\\ &\qquad x(t;\cdot) \in \mathrm{Hol}(\Omega) \text{ for all } t \in I \text{ and } \partial_t^j x \in L^{\infty}(I\times\Omega) \text{ for all } j \in \mathbb{Z}_{\geq 0}\big\}, && \text{(B.2)}\\ \mathcal{H}_I(\Omega) &:= \{x = x(t;\lambda) \in C^{\infty}(I\times\Omega;\mathbb{C}) : x(t;\cdot) \in \mathrm{Hol}(\Omega) \text{ for all } t \in I\}. && \text{(B.3)} \end{aligned}$$

Then $\mathrm{Hol}(\Omega)$, $\mathcal{H}_I^0(\Omega)$, $\mathcal{H}_I(\Omega)$ are rings and the definitions in (B.1), (B.3) are the same as in section 5.3. This appendix is only used in the proof of Lemma 5.6, which does not require the definition of Z in $Z = |x|/(T-t)$.

So, with abuse of notation, we replace Z by t and use x to denote a general function of $(t;\lambda)$. We stress that here (t,x) has nothing to do with the coordinates in $\mathbb{R}^{1+d}$.

Lemma B.1. *Let $I \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{C}$ be an open subset. Then $\mathcal{H}_I^0(\Omega) \subset \mathcal{H}_I(\Omega)$.*

Proof. Let $x = x(t;\lambda) \in \mathcal{H}_I^0(\Omega)$. Pick $\lambda_0 \in \Omega$ and let $r \in (0,1)$ be such that $B_r(\lambda_0) := \{\lambda \in \mathbb{C} : |\lambda-\lambda_0| < r\} \subset \Omega$. By Cauchy's integration formula (Theorem 4.4 in Chapter 2 of [69]), for any $t \in I, \lambda \in \Omega$ we have

$$x(t;\lambda) = \sum_{k=0}^{\infty} x_k(t)(\lambda-\lambda_0)^k, \tag{B.4}$$

where

$$x_k(t) = \frac{1}{2\pi\mathrm{i}}\int_{|\lambda-\lambda_0|=r} \frac{x(t;\lambda)}{(\lambda-\lambda_0)^{k+1}}\,\mathrm{d}\lambda = \frac{r^{-k}}{2\pi}\int_0^{2\pi} x\left(t;\lambda_0+re^{\mathrm{i}\theta}\right)e^{-\mathrm{i}k\theta}\,\mathrm{d}\theta \tag{B.5}$$

for all $t \in I, k \in \mathbb{Z}_{\geq 0}$. Since $x(\cdot;\lambda) \in C^\infty(I)$ for all $\lambda \in \Omega$ and $\partial_t^j x \in L^\infty(I\times\Omega)$, by (B.5) and the dominated convergence theorem, we have $x_k \in C^\infty(I)$ and

$$\left\|x_k^{(j)}\right\|_{L^\infty(I)} \leq \|\partial_t^j x\|_{L^\infty(I\times\Omega)} r^{-k}, \quad \forall\, j \in \mathbb{Z}_{\geq 0},\ \forall\, k \in \mathbb{Z}_{\geq 0}. \tag{B.6}$$

Using (B.6), we know that $\sum_{k=0}^\infty x_k^{(j)}(t)\partial_\lambda^\alpha((\lambda-\lambda_0)^k)$ is uniformly absolutely convergent on $I\times B_{r/2}(\lambda_0)$ for all $j \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (\mathbb{Z}_{\geq 0})^2$, hence (B.4) implies that $x \in C^\infty(I\times B_{r/2}(\lambda_0))$. Since $\lambda_0 \in \Omega$ is arbitrary, we have $x \in C^\infty(I\times\Omega)$. Hence $x \in \mathcal{H}_I(\Omega)$. □

Remark B.2. As smoothness is a local property, we have $\mathcal{H}^0_{I,\mathrm{loc}}(\Omega) \subset \mathcal{H}_I(\Omega)$, where

$$\mathcal{H}^0_{I,\mathrm{loc}}(\Omega) := \left\{x = x(t;\lambda) : I\times\Omega \to \mathbb{C}\middle| x \in \mathcal{H}^0_J(\Omega) \text{ for any compact sub-interval } J \subset I\right\}.$$

Moreover, we have $\mathcal{H}_I(\Omega) \subset \mathcal{H}^0_{I,\mathrm{loc}}(\Omega')$ for any open subset $\Omega' \subset\subset \Omega$ (i.e. there exists a compact set K such that $\Omega' \subset K \subset \Omega$).

Lemma B.3. *Let $I \subset \mathbb{R}$ be an interval and let $A(t) \in C^\infty(I;\mathbb{C})$ be such that $A(t) = 0$ has a unique solution $t = t_0$ in I[12] with $A'(t_0) \neq 0$. Let $\Omega \subset \mathbb{C}$ be an open subset and let $B(t;\lambda), D(t;\lambda) \in C^\infty(I\times\Omega;\mathbb{C})$ be such that $B, D \in \mathcal{H}^0_{I,\mathrm{loc}}(\Omega)$, which implies*

$$N_0^* := \sup_{\lambda\in\Omega}\left(\max\left\{3, -\mathrm{Re}\left(\frac{B(t_0;\lambda)}{A'(t_0)}\right)+1\right\}\right) < +\infty.$$

Then there exists $N_0 > N_0^*$ such that for all $N \in \mathbb{Z}\cap(N_0,+\infty)$, if $f \in \mathcal{H}^0_{I,\mathrm{loc}}(\Omega)$ is such that

$$\frac{|f(t;\lambda)|}{|t-t_0|^N} \in L^\infty(J\times\Omega) \quad \textit{for any compact sub-interval} \quad J \subset I, \tag{B.7}$$

then the linear ODE (here the prime ′ refers to the derivative with respect to t)

$$A(t)x''(t;\lambda) + B(t;\lambda)x'(t;\lambda) + D(t;\lambda)x(t;\lambda) = f(t;\lambda) \tag{B.8}$$

has a (complex-valued) smooth solution $x(t;\lambda)$ on $I\times\Omega$ such that $x \in \mathcal{H}_I(\Omega)$.

Proof. Without loss of generality, we assume that $t_0 = 0 \in I$, and there exists $\delta_0 \in (0,1)$ such that $I_0 = [-\delta_0,\delta_0] \subset I$ or $I_0 = [0,\delta_0] = I\cap[-\delta_0,\delta_0]$.

Step 1. Existence of a C^2 local solution. We define the Banach space

$$Y_N := \left\{y \in C(I_0\times\Omega;\mathbb{C}) : y(t;\lambda)/|t|^N \in L^\infty(I_0\times\Omega) \text{ and } y(t;\cdot) \in \mathrm{Hol}(\Omega)\ \forall\, t \in I_0\right\},$$

where $N \geq 3$ is an integer, with the norm $\|y\|_{Y_N} := \left\|y(t;\lambda)/|t|^N\right\|_{L^\infty(I_0\times\Omega)}$.

We define a linear operator $\mathcal{T}_N : Y_N \to Y_N$ by

$$(\mathcal{T}_N y)(t;\lambda) := \int_0^t \left(\frac{B(s;\lambda)}{A(s)}y(s;\lambda) + \frac{D(s;\lambda)}{A(s)}\int_0^s y(\tau;\lambda)\,\mathrm{d}\tau\right)\mathrm{d}s, \qquad \forall\, t \in I_0,\ \forall\, \lambda \in \Omega.$$

By the hypotheses on the coefficients A, B, D, we have

$$M := \sup_{s\in I_0}\left|\frac{s}{A(s)}\right| + \sup_{s\in I_0,\lambda\in\Omega}\left|\frac{sB(s;\lambda)}{A(s)}\right| + \sup_{s\in I_0,\lambda\in\Omega}\left|\frac{sD(s;\lambda)}{A(s)}\right| \in (0,+\infty). \tag{B.9}$$

[12]It means that $\{t \in I : A(t) = 0\} = \{t_0\}$, and also for Proposition B.4.

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Hence, for all $t \in I_0 \subset [-1, 1]$ and for all $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}|(\mathcal{T}_N y)(t;\lambda)| &= \left|\int_0^t \left(\frac{sB(s;\lambda)}{A(s)}\frac{y(s;\lambda)}{s^N}s^{N-1} + \frac{sD(s;\lambda)}{A(s)}\frac{1}{s}\int_0^s \frac{y(\tau;\lambda)}{\tau^N}\tau^N\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\ &\le \frac{M}{N}|t|^N \|y\|_{Y_N},\end{aligned}$$

which gives

$$\|\mathcal{T}_N\|_{Y_N\to Y_N} \le \frac{M}{N}, \qquad \forall\, N \in \mathbb{Z}\cap[3,+\infty). \tag{B.10}$$

We also define an operator $\mathcal{F}_N : Y_N \to Y_N$ by

$$(\mathcal{F}_N f)(t;\lambda) := \int_0^t \frac{f(s;\lambda)}{A(s)}\,\mathrm{d}s, \qquad \forall\, t \in I_0,\ \forall\, \lambda \in \Omega.$$

Using (B.9), we know that $\mathcal{F}_N : Y_N \to Y_N$ is a bounded linear operator with $\|\mathcal{F}_N f\|_{Y_N} \le \frac{M}{N}\|f\|_{Y_N}$ for all $f \in Y_N$. Now we take $N_0 \in \mathbb{Z}$ such that $N_0 > N_0^* + 2M$. For any $N \in \mathbb{Z}\cap(N_0,+\infty)$, by (B.10) we know that $\|\mathcal{T}_N\|_{Y_N\to Y_N} \le 1/2$, hence $\mathrm{id}+\mathcal{T}_N : Y_N \to Y_N$ is invertible, then $(\mathrm{id}+\mathcal{T}_N)^{-1}\mathcal{F}_N : Y_N \to Y_N$ is a bounded linear operator with

$$\left\|(\mathrm{id}+\mathcal{T}_N)^{-1}\mathcal{F}_N\right\|_{Y_N\to Y_N} \le \left\|(\mathrm{id}+\mathcal{T}_N)^{-1}\right\|_{Y_N\to Y_N}\|\mathcal{F}_N\|_{Y_N\to Y_N} \le \frac{2M}{N}.$$

For any $N \in \mathbb{Z}\cap(N_0,+\infty)$, given $f \in \mathcal{H}^0_{I,\mathrm{loc}}(\Omega)$ satisfying (B.7) (then $f \in Y_N$), we define

$$y = (\mathrm{id}+\mathcal{T}_N)^{-1}\mathcal{F}_N f \in Y_N, \quad x(t;\lambda) = \int_0^t y(s;\lambda)\,\mathrm{d}s,\ \forall\, t \in I_0,\ \forall\, \lambda \in \Omega,$$

then $x \in C(I_0\times\Omega)$, $x(\cdot;\lambda) \in C^1(I_0)$ for all $\lambda \in \Omega$, $x(t;\cdot) \in \mathrm{Hol}(\Omega)$ for all $t \in I_0$,

$$x'(t;\lambda) = \int_0^t \left(-\frac{B(\tau;\lambda)}{A(\tau)}x'(\tau;\lambda) - \frac{D(\tau;\lambda)}{A(\tau)}x(\tau;\lambda) + \frac{f(\tau;\lambda)}{A(\tau)}\right)\mathrm{d}\tau, \quad \forall\, t \in I_0,\ \forall\, \lambda \in \Omega,$$

and $x'(t;\lambda) = y(t;\lambda)$, $x(t;\lambda) = \int_0^t x'(s;\lambda)\,\mathrm{d}s$ for $t \in I_0$ and $\lambda \in \Omega$. Moreover, we have (recalling $t_0 = 0$)

$$\frac{x(t;\lambda)}{|t|^{N+1}} \in L^\infty(I_0\times\Omega), \qquad \frac{x'(t;\lambda)}{|t|^N} \in L^\infty(I_0\times\Omega). \tag{B.11}$$

On the other hand, since $x'(t;\lambda) = \int_0^t X(s;\lambda)\,\mathrm{d}s$ for $t \in I_0, \lambda \in \Omega$, where

$$\begin{aligned}X(s;\lambda) &:= -\frac{B(s;\lambda)}{A(s)}x'(s;\lambda) - \frac{D(s;\lambda)}{A(s)}x(s;\lambda) + \frac{f(s;\lambda)}{A(s)} \\ &= -\frac{sB(s;\lambda)}{A(s)}\frac{x'(s;\lambda)}{s^N}s^{N-1} - \frac{sD(s;\lambda)}{A(s)}\frac{x(s;\lambda)}{s^{N+1}}s^N + \frac{s}{A(s)}\frac{f(s;\lambda)}{s^N}s^{N-1}\end{aligned}$$

for $s \in I_0\setminus\{0\}$ and $\lambda \in \Omega$, thus there exists a constant $C > 0$ such that we have $|X(s;\lambda)| \le C|s|^{N-1}$ for $s \in I_0\setminus\{0\}, \lambda \in \Omega$. As $N > 1$, we know that $x'(\cdot;\lambda) \in C^1(I_0)$ (thus $x(\cdot;\lambda) \in C^2(I_0)$) and $x''(0;\lambda) = 0$ for $\lambda \in \Omega$. Hence, x solves (B.8) on $(t,\lambda) \in I_0\times\mathbb{C}$. We also have

$$\frac{x''(t;\lambda)}{|t|^{N-1}} \in L^\infty(I_0\times\Omega). \tag{B.12}$$

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Step 2. Smoothness of the C^2 local solution. In this step, we show that $x(\cdot;\lambda) \in C^\infty(I_0)$ for any $\lambda \in \Omega$. By standard ODE theory, we have $x(\cdot;\lambda) \in C^\infty(I_0 \setminus \{0\})$ for all $\lambda \in \Omega$. We claim that for any $k \in \mathbb{Z} \cap [0, N]$, there exists a constant $C_k > 0$ such that

$$x^{(k)}(0;\lambda) = 0 \qquad \text{and} \qquad |x^{(k)}(t;\lambda)| \le C_k |t|^{N+1-k}, \qquad \forall\, t \in I_0,\ \forall\, \lambda \in \Omega. \tag{B.13}$$

We use the induction. By (B.11) and (B.12), we know that (B.13) holds for $k \in \{0, 1, 2\}$. Assume that for some $K \in \mathbb{Z} \cap [1, N-1]$, (B.13) holds for all $k \in \mathbb{Z} \cap [0, K]$. Now we prove that (B.13) holds for $k = K+1$. By our induction hypotheses, $x(\cdot;\lambda) \in C^{(K)}(I_0)$ and $x^{(K+1)}(0;\lambda) = \lim_{t\to 0}(x^{(K)}(t;\lambda)/t) = 0$ for all $\lambda \in \Omega$. For $t \in I_0 \setminus \{0\}$, taking derivative $K-1$ times on both sides of (B.8) with respect to t, we obtain

$$A(t)x^{(K+1)}(t;\lambda) + \sum_{j=0}^{K} A_{j,K}(t;\lambda)x^{(j)}(t;\lambda) = f^{(K-1)}(t;\lambda), \quad \forall\, t \in I_0 \setminus \{0\},\ \forall\, \lambda \in \Omega, \tag{B.14}$$

where $A_{j,K}$'s are linear combinations of A, B, D and their derivatives, hence $A_{j,K}(t;\lambda) \in C^\infty(I \times \Omega) \cap L^\infty(I_0 \times \Omega)$ for all $j \in \mathbb{Z} \cap [0, K]$.[13] As $f \in \mathcal{H}^0_{I,\mathrm{loc}}(\Omega)$ satisfies (B.7), we have[14] $f^{(k)}(t;\lambda)/|t|^{N-k} \in L^\infty(I_0 \times \Omega)$ for all $k \in \mathbb{Z} \cap [0, N]$. Therefore,

$$\frac{\left|x^{(K+1)}(t;\lambda)\right|}{|t|^{N-K}} = \frac{|t|}{|A(t)|} \frac{\left|f^{(K-1)}(t;\lambda) - \sum_{j=0}^{K} A_{j,K}(t;\lambda)x^{(j)}(t;\lambda)\right|}{|t|^{N-(K-1)}} \in L^\infty(I_0 \times \Omega).$$

This proves (B.13) for $k = K+1$. Hence, (B.13) holds by the induction and thus $x(\cdot;\lambda) \in C^N(I_0)$ for all $\lambda \in \Omega$ and

$$\sup_{t\in I_0, \lambda\in\Omega} |x^{(k)}(t;\lambda)| < +\infty, \qquad \forall\, k \in \mathbb{Z} \cap [0, N]. \tag{B.15}$$

Next we claim that for $k \in \mathbb{Z} \cap [N, +\infty)$ we have

$$\sup_{t\in I_0\setminus\{0\}, \lambda\in\Omega} |x^{(k)}(t;\lambda)| < +\infty. \tag{B.16}$$

By (B.13), we know that (B.16) holds for $k = N$. Assume that for some $k \in \mathbb{Z}_{\ge N}$ we have

$$\sup_{t\in I_0\setminus\{0\}, \lambda\in\Omega} |x^{(N)}(t;\lambda)| < +\infty, \cdots, \sup_{t\in I_0\setminus\{0\}, \lambda\in\Omega} |x^{(k)}(t;\lambda)| < +\infty. \tag{B.17}$$

For $t \in I_0 \setminus \{0\}$ and $\lambda \in \Omega$, by (B.14) for $K = k+1$ and footnote 13 we have

$$A(t)x^{(k+2)}(t;\lambda) + (kA'(t) + B(t;\lambda))x^{(k+1)}(t;\lambda) = F_k(t;\lambda), \tag{B.18}$$

$$F_k(t;\lambda) := f^{(k)}(t;\lambda) - \sum_{j=0}^{k} A_{j,k+1}(t;\lambda)x^{(j)}(t;\lambda). \tag{B.19}$$

Then by $f \in \mathcal{H}^0_{I_0}(\Omega)$, (B.15) and (B.17) we have $\sup_{t\in I_0\setminus\{0\}, \lambda\in\Omega} |F_k(t;\lambda)| < +\infty$. Let $\widetilde{A}(t) := \int_0^1 A'(ts)\,\mathrm{d}s$ and $\widetilde{B}(t;\lambda) := B(t;\lambda) - \frac{B(0;\lambda)}{A'(0)}\widetilde{A}(t)$ for $t \in I_0, \lambda \in \Omega$, then $\widetilde{B}(0;\lambda) = 0$, $\widetilde{A} \in C^\infty(I_0)$,

[13](B.14) and the properties of $A_{j,K}$'s holds for all $K \in \mathbb{Z}_+$ (not merely for $K \in \mathbb{Z} \cap [1, N-1]$), and we also have $A_{K,K}(t;\lambda) = (K-1)A'(t) + B(t;\lambda)$ for all $t \in I_0, \lambda \in \Omega$.

[14]Indeed, (B.7) implies that $f^{(k)}(0;\lambda) = 0$ for any $k \in \mathbb{Z} \cap [0, N-1]$ and any $\lambda \in \Omega$. As a consequence, we have $|f^{(N-1)}(t;\lambda)| = \left|\int_0^t f^{(N)}(s;\lambda)\,\mathrm{d}s\right| \le \left(\sup_{s\in I_0,\lambda\in\Omega} |f^{(N)}(s;\lambda)|\right) |t|$ for all $t \in I_0, \lambda \in \Omega$, where we have used $f \in \mathcal{H}^0_{I_0}(\Omega)$. Similarly one shows that $f^{(k)}(t;\lambda)/|t|^{N-k} \in L^\infty(I_0 \times \Omega)$ for all $k \in \mathbb{Z} \cap [0, N]$.

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$\widetilde{B} \in C(I_0 \times \Omega)$ for $\lambda \in \Omega$, $\partial_t \widetilde{B} \in L^\infty(I_0 \times \Omega)$ and $\widetilde{A}(t) = A(t)/t$ for $t \in I_0 \setminus \{0\}$. Thus, $\widetilde{B}(t;\lambda)/t \in L^\infty(I_0 \times \Omega)$ and $\widetilde{B}(t;\lambda)/A(t) \in L^\infty(I_0 \times \Omega)$ (using (B.9)).

Let $\eta(t;\lambda) := |t|^{\frac{B(0;\lambda)}{A'(0)}} \exp\left(\int_0^t \frac{\widetilde{B}(s;\lambda)}{A(s)}\,\mathrm{d}s\right)$ then (here η is different from the one in (2.23))

$$C_\eta^{-1}|t|^{\mathrm{Re}\,\frac{B(0;\lambda)}{A'(0)}} \leq |\eta(t;\lambda)| \leq C_\eta |t|^{\mathrm{Re}\,\frac{B(0;\lambda)}{A'(0)}}, \qquad \forall\, t \in I_0 \setminus \{0\},\ \forall\, \lambda \in \Omega \tag{B.20}$$

for some constant $C_\eta > 0$. We also have $\eta \in C^\infty((I_0 \setminus \{0\}) \times \Omega; \mathbb{C} \setminus \{0\})$ and

$$\frac{\eta'(t;\lambda)}{\eta(t;\lambda)} = \frac{\widetilde{B}(t;\lambda)}{A(t)} + \frac{B(0;\lambda)}{A'(0)t} = \frac{B(t;\lambda)}{A(t)} - \frac{B(0;\lambda)}{A'(0)}\frac{\widetilde{A}(t)}{A(t)} + \frac{B(0;\lambda)}{A'(0)t} = \frac{B(t;\lambda)}{A(t)}, \quad \forall\, t \in I_0 \setminus \{0\}.$$

Here we used $\widetilde{A}(t) = A(t)/t$. It follows from (B.18) that

$$\left(A(t)^k \eta(t;\lambda) x^{(k+1)}(t;\lambda)\right)' = A(t)^{k-1}\eta(t;\lambda)F_k(t;\lambda), \qquad \forall\, t \in I_0 \setminus \{0\},\ \forall\, \lambda \in \Omega.$$

By (B.14) for $K = k$ and (B.15), (B.17), we have $A(t)x^{(k+1)}(t;\lambda) \in L^\infty((I_0 \setminus \{0\}) \times \Omega)$; using (B.20), $|A(t)| \sim |t|$ as $t \to 0$ and

$$k \geq N > N_0 > -\inf_{\lambda\in\Omega} \mathrm{Re}\,(B(0;\lambda)/A'(0)) + 1, \tag{B.21}$$

we have

$$\lim_{t\to 0} A(t)^k \eta(t;\lambda) x^{(k+1)}(t;\lambda) = 0, \quad \forall\, \lambda \in \Omega,$$

hence

$$A(t)^k \eta(t;\lambda) x^{(k+1)}(t;\lambda) = \int_0^t A(s)^{k-1}\eta(s;\lambda)F_k(s;\lambda)\,\mathrm{d}s, \qquad \forall\, t \in I_0 \setminus \{0\},\ \forall\, \lambda \in \Omega.$$

As a consequence, we have

$$\left|x^{(k+1)}(t;\lambda)\right| = \frac{\left|\int_0^t A(s)^{k-1}\eta(s;\lambda)F_k(s;\lambda)\,\mathrm{d}s\right|}{|A(t)|^k|\eta(t;\lambda)|} \leq C_{k+1}\frac{\int_0^{|t|} s^{k-1}s^{\mathrm{Re}(B(0;\lambda)/A'(0))}\,\mathrm{d}s}{|t|^k|t|^{\mathrm{Re}(B(0;\lambda)/A'(0))}} \leq \widetilde{C}_{k+1}$$

for all $t \in I_0 \setminus \{0\}$ and $\lambda \in \Omega$, where $C_{k+1} > 0$ and $\widetilde{C}_{k+1} > 0$ are constants. Here we have used $k + \inf_{\lambda\in\Omega} \mathrm{Re}(B(0;\lambda)/A'(0)) > 0$, which follows from (B.21). This proves (B.16).

Next we use once again the induction to prove that

$$x^{(k)}(0;\lambda) \text{ exists and } \lim_{t\to 0} x^{(k)}(t;\lambda) = x^{(k)}(0;\lambda), \qquad \forall\, \lambda \in \Omega,\ \forall\, k \in \mathbb{Z} \cap [0,+\infty). \tag{B.22}$$

We know from (B.13) that (B.22) holds for $k \leq N$. Now we assume that for some $k \in \mathbb{Z}_{\geq N}$, (B.22) holds for $0, 1, \cdots, k$. Then by (B.19), we have $F_k(\cdot;\lambda) \in C(I_0)$, by (B.16) with k replaced by $k+2$ and $A(0) = 0$ we have $\lim_{t\to 0} A(t)x^{(k+2)}(t;\lambda) = 0$, and by (B.18) we have

$$\lim_{t\to 0} x^{(k+1)}(t;\lambda) = \frac{F_k(0;\lambda)}{kA'(0) + B(0;\lambda)} \in \mathbb{C}, \qquad \forall\, \lambda \in \Omega,$$

where we have used $kA'(0) + B(0;\lambda) \neq 0$, which follows from $k > -\inf_{\lambda\in\Omega} \mathrm{Re}(B(0;\lambda)/A'(0))$ (see (B.21)). Finally, we get by L'Hôpital's rule that

$$x^{(k+1)}(0;\lambda) = \lim_{t\to 0} \frac{x^{(k)}(t;\lambda) - x^{(k)}(0)}{t} = \lim_{t\to 0} x^{(k+1)}(t;\lambda), \qquad \forall \lambda \in \Omega.$$

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This proves (B.22) for $k+1$. Then (B.22) holds for all $k \in \mathbb{Z}_{\geq N}$ by the induction. Hence, $x(\cdot;\lambda) \in C^\infty(I_0)$ for all $\lambda \in \Omega$. Moreover, combining (B.15) and (B.16) gives that

$$\partial_t^j x \in L^\infty(I_0 \times \Omega), \qquad \forall\, j \in \mathbb{Z}_{\geq 0}. \tag{B.23}$$

Step 3. $x \in C^\infty(I_0 \times \Omega)$. Recall that $x \in C(I_0 \times \Omega)$ satisfies $x(t;\cdot) \in \mathrm{Hol}(\Omega)$ for all $t \in I_0$ (in **Step 1**) and $x(\cdot;\lambda) \in C^\infty(I_0)$ for all $\lambda \in \Omega$ (in **Step 2**). Using (B.23) we have $x \in \mathcal{H}^0_{I_0}(\Omega)$. Then Lemma B.1 implies that $x \in C^\infty(I_0 \times \Omega)$.

Step 4. Extension of the smooth local solution. For any fixed $\lambda \in \Omega$, we have constructed a local solution $x_L(\cdot;\lambda) \in C^\infty(I_0)$ of (B.8) on $I_0 \subset I$. Moreover, we have $x_L \in C^\infty(I_0 \times \Omega)$ and $x_L(t;\cdot) \in \mathrm{Hol}(\Omega)$ for all $t \in I_0$. By standard ODE theory, the initial value problem

$$\begin{cases} x''(t;\lambda) + \frac{B(t;\lambda)}{A(t)}x'(t;\lambda) + \frac{D(t;\lambda)}{A(t)}x(t;\lambda) = \frac{f(t;\lambda)}{A(t)}, \\ x(\delta_0/2;\lambda) = x_L(\delta_0/2;\lambda), x'(\delta_0/2;\lambda) = x_L'(\delta_0/2;\lambda) \end{cases}$$

has a unique solution $x = x(t;\lambda)$ on $((0,+\infty) \cap I) \times \Omega$ and $x \in C^\infty(((0,+\infty) \cap I) \times \Omega)$. Moreover, by the analytic dependence on parameters (Lemma B.5), we have $x(t;\cdot) \in \mathrm{Hol}(\Omega)$ for all $t \in (0,+\infty) \cap I$. Hence, x_L can be extended to be a smooth solution of (B.8) on $((0,+\infty) \cap I) \times \Omega$; Similarly we can extend x_L on the negative direction (for the case $I_0 = [-\delta_0, \delta_0]$). And for the extended solution x, we have $x \in \mathcal{H}_I(\Omega)$. □

Proposition B.4. *Let $I \subset \mathbb{R}$ be an interval. Let $A(t) \in C^\infty(I;\mathbb{C})$ be such that $A(t) = 0$ has a unique solution $t = t_0$ in I with $A'(t_0) \neq 0$. Let $B(t;\lambda), D(t;\lambda) \in \mathcal{H}_I(\mathbb{C})$. Assume that $B(t;\lambda) = \widetilde{B}(t) + \lambda\widehat{B}(t)$ for $t \in I$ and $\lambda \in \mathbb{C}$, where $\widetilde{B}, \widehat{B} \in C^\infty(I;\mathbb{C})$. Suppose that*

$$\text{either } \widehat{B}(t_0) \neq 0 \quad \text{or} \quad \widehat{B}(t_0) = 0 \text{ and } -\widetilde{B}(t_0)/A'(t_0) \notin \mathbb{Z}_{\geq 0}. \tag{B.24}$$

We define

$$\Lambda_* := \{\lambda \in \mathbb{C} : nA'(t_0) + B(t_0;\lambda) = 0 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}. \tag{B.25}$$

Then $\Lambda_* \subset \mathbb{C}$ is a (probably empty) discrete set. Let $R \in (0,+\infty)$. There exists a nonzero polynomial $\psi_1(\lambda)$ satisfying $\{\lambda \in B_R : \psi_1(\lambda) = 0\} = \Lambda_* \cap B_R$ such that for every $f(t;\lambda) \in \mathcal{H}_I(\mathbb{C})$, the inhomogeneous ODE

$$\begin{cases} A(t)x''(t;\lambda) + B(t;\lambda)x'(t;\lambda) + D(t;\lambda)x(t;\lambda) = \psi_1(\lambda)f(t;\lambda), \quad t \in I, \lambda \in B_R, \\ x(t_0;\lambda) = \psi_1(\lambda), \quad \lambda \in B_R, \end{cases} \tag{B.26}$$

where the prime $'$ refers to the derivative with respect to $t \in I$, has a solution $x = x(t;\lambda) \in \mathcal{H}_I(B_R)$. Moreover, if $\widehat{B}(t_0) = 0$, then $\psi_1(\lambda) = 1$.

Proof. We first show that Λ_* is a discrete set. If $\widehat{B}(t_0) \neq 0$, then $\Lambda_* = \{-nA'(t_0)/\widehat{B}(t_0) - \widetilde{B}(t_0)/\widehat{B}(t_0) : n \in \mathbb{Z}_{\geq 0}\}$, hence Λ_* is a discrete set. If $\widehat{B}(t_0) = 0$, then $\lambda_* \in \Lambda_*$ if and only if $0 = nA'(t_0) + B(t_0;\lambda_*) = nA'(t_0) + \widetilde{B}(t_0) = 0$ for some $n \in \mathbb{Z}_{\geq 0}$, which implies that $-\widetilde{B}(t_0)/A'(t_0) \in \mathbb{Z}_{\geq 0}$, and this is a contradiction with our assumption (B.24). As a consequence, if $\widehat{B}(t_0) = 0$ (and $-\widetilde{B}(t_0)/A'(t_0) \notin \mathbb{Z}_{\geq 0}$), then $\Lambda_* = \emptyset$.

Next, we construct $\psi_1(\lambda)$. Let N_0 be given by Lemma B.3 (for $\Omega = B_R$) and fix an integer $N > \max\{N_0 + 1, -\inf_{\lambda \in B_R} \mathrm{Re}\,(B(t_0;\lambda)/A'(t_0)) + 1\}$. Let $\psi_1(\lambda) := 1$ for the case $\widehat{B}(t_0) = 0$ and $\psi_1(\lambda) := \prod_{j=0}^{N-1}\left(jA'(t_0) + B(t_0,\lambda)\right)$ for the case $\widehat{B}(t_0) \neq 0$.

Claim 1. $\psi_1(\lambda)$ is a nonzero polynomial. If $\widehat{B}(t_0) = 0$, then $\psi_1(\lambda) \equiv 1$ is a polynomial of degree 0; if $\widehat{B}(t_0) \neq 0$, as $B(t_0,\lambda) = \widetilde{B}(t_0) + \lambda\widehat{B}(t_0)$, then $\psi_1(\lambda)$ is a polynomial of degree N.

Claim 2. $\{\lambda \in B_R : \psi_1(\lambda) = 0\} = \Lambda_* \cap B_R$. If $\widehat{B}(t_0) = 0$, then $\{\lambda \in B_R : \psi_1(\lambda) = 0\} = \emptyset = \Lambda_* = \Lambda_* \cap B_R$. For the case $\widehat{B}(t_0) \neq 0$, if $\psi_1(\lambda) = 0$ then $jA'(t_0) + B(t_0,\lambda) = 0$ for

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some $j \in \mathbb{Z} \cap [0, N-1]$ and $\lambda \in \Lambda_*$, thus $\{\lambda \in B_R : \psi_1(\lambda) = 0\} \subseteq \Lambda_* \cap B_R$. On the other hand, if $\lambda_0 \in \Lambda_* \cap B_R$ (and $\widehat{B}(t_0) \neq 0$), then $nA'(t_0) + B(t_0; \lambda_0) = 0$ for some $n \in \mathbb{Z}_{\geq 0}$, and $n = -B(t_0; \lambda_0)/A'(t_0) \leq -\inf_{\lambda \in B_R} \operatorname{Re}(B(t_0; \lambda)/A'(t_0)) < N-1$, thus $n \in \mathbb{Z} \cap [0, N-1]$ and $\psi_1(\lambda_0) = 0$. So $\Lambda_* \cap B_R \subseteq \{\lambda \in B_R : \psi_1(\lambda) = 0\}$.

It remains to construct $x(t; \lambda)$. For any $n \in \mathbb{Z} \cap [0, N]$ and $\lambda \in \mathbb{C}$, let $\psi_{1,n}(\lambda) := 1$ for the case $\widehat{B}(t_0) = 0$ and $\psi_{1,n}(\lambda) := \prod_{j=n}^{N-1} \left(jA'(t_0) + B(t_0, \lambda)\right)$ (here $\psi_{1,N}(\lambda) := 1$) for the case $\widehat{B}(t_0) \neq 0$. Then $\psi_1(\lambda) = \psi_{1,0}(\lambda)$ for all $\lambda \in \mathbb{C}$.

Claim 3. If $n \in \mathbb{Z} \cap [0, N]$, $g(t; \lambda) \in \mathcal{H}_I(\mathbb{C})$, $\partial_t^i g(t_0; \lambda) = 0$ for $i \in \mathbb{Z}$, $0 \leq i < n$. Then

$$A(t)y''(t; \lambda) + B(t; \lambda)y'(t; \lambda) + D(t; \lambda)y(t; \lambda) = \psi_{1,n}(\lambda)g(t; \lambda), \ y(t_0; \lambda) = 0, \ \lambda \in B_R, \tag{B.27}$$

has a solution $y = y(t; \lambda) \in \mathcal{H}_I(B_R)$.

Let $g(t; \lambda) = f(t; \lambda) - D(t; \lambda)$, $n = 0$, then by Claim 3, (B.27) has a solution $y = y(t; \lambda) \in \mathcal{H}_I(B_R)$ with $n = 0$. $x(t; \lambda) = y(t; \lambda) + \psi_1(\lambda) \in \mathcal{H}_I(B_R)$ solves (B.26) (using $\psi_1(\lambda) = \psi_{1,0}(\lambda)$).

It remains to prove Claim 3. We use the (backward) induction. We need to prove that:

(i) Claim 3 holds for $n = N$;
(ii) if $j \in \mathbb{Z} \cap [0, N-1]$, Claim 3 holds for $n = j+1$, then Claim 3 holds for $n = j$.

Proof of (i). As $g \in \mathcal{H}_I(\mathbb{C}) \subset \mathcal{H}^0_{I,\mathrm{loc}}(B_R)$, $n = N$, by Taylor's theorem with integral remainders, we have $g(t; \lambda)/|t - t_0|^N \in L^\infty_{\mathrm{loc}}(I \times \mathbb{C})$, and we also have $\psi_{1,n}(\lambda) = \psi_{1,N}(\lambda) = 1$. Then the result follows from Lemma B.3.

Proof of (ii). We fix $j \in \mathbb{Z} \cap [0, N-1]$ and assume $g(t; \lambda) \in \mathcal{H}_I(\mathbb{C})$, $\partial_t^i g(t_0; \lambda) = 0$ for $i \in \mathbb{Z}$, $0 \leq i < j$. For $t \in I, \lambda \in \mathbb{C}$, let $x_j(t) := (t - t_0)^{j+1}$, $y_j(t; \lambda) := A(t)x_j''(t) + B(t; \lambda)x_j'(t) + D(t; \lambda)x_j(t)$, then $x_j \in C^\infty(I)$, $x_j(t_0) = 0$, $y_j(t; \lambda) \in \mathcal{H}_I(\mathbb{C})$, and

$$y_j(t; \lambda) = A(t)j(j+1)(t - t_0)^{j-1} + B(t; \lambda)(j+1)(t - t_0)^j + D(t; \lambda)(t - t_0)^{j+1}.$$

By Taylor's formula, we have $\partial_t^i y_j(t_0; \lambda) = 0$ for $i \in \mathbb{Z}$, $0 \leq i < j$ and

$$\begin{aligned}\partial_t^j y_j(t_0; \lambda) &= j! \lim_{t \to t_0} \frac{y_j(t; \lambda)}{(t - t_0)^j} = j! \lim_{t \to t_0} j(j+1)\frac{A(t)}{t - t_0} + j!B(t_0; \lambda)(j+1) \\ &= j!j(j+1)A'(t_0) + (j+1)!B(t_0; \lambda) = (j+1)!(jA'(t_0) + B(t_0; \lambda)).\end{aligned}$$

For the case of $\widehat{B}(t_0) \neq 0$, let $a_j = (j+1)!$, $b_j(\lambda) = jA'(t_0) + B(t_0; \lambda)$ then $a_j \neq 0$, $b_j \in \mathrm{Hol}(\mathbb{C})$. As $\psi_{1,n}(\lambda) = \prod_{j=n}^{N-1} \left(jA'(t_0) + B(t_0; \lambda)\right)$ for $n \in \mathbb{Z} \cap [0, N]$ we have $\psi_{1,j}(\lambda) = b_j(\lambda)\psi_{1,j+1}(\lambda)$ and $\partial_t^j y_j(t_0; \lambda) = (j+1)!(jA'(t_0) + B(t_0; \lambda)) = a_j b_j(\lambda)$ for all $\lambda \in \mathbb{C}$.

For the case of $\widehat{B}(t_0) = 0$, we have $jA'(t_0) + B(t_0, \lambda) = jA'(t_0) + \widetilde{B}(t_0) \neq 0$ (using (B.24)). Let $a_j = (j+1)!(jA'(t_0) + \widetilde{B}(t_0))$, $b_j(\lambda) = 1$ then $a_j \neq 0$, $b_j \in \mathrm{Hol}(\mathbb{C})$, $\partial_t^j y_j(t_0; \lambda) = a_j = a_j b_j$. As $\psi_{1,n}(\lambda) = 1$ for $n \in \mathbb{Z} \cap [0, N]$ we have $\psi_{1,j}(\lambda) = b_j(\lambda)\psi_{1,j+1}(\lambda)$ for all $\lambda \in \mathbb{C}$.

Thus, we always have $a_j \neq 0$, $b_j \in \mathrm{Hol}(\mathbb{C})$, $\psi_{1,j}(\lambda) = b_j(\lambda)\psi_{1,j+1}(\lambda)$, $\partial_t^j y_j(t_0; \lambda) = a_j b_j(\lambda)$.

For $t \in I, \lambda \in \mathbb{C}$, let $\widetilde{g}(t; \lambda) := b_j(\lambda)g(t; \lambda) - \partial_t^j g(t_0; \lambda) \cdot y_j(t; \lambda)/a_j$ then $\widetilde{g} \in \mathcal{H}_I(\mathbb{C})$,[15] and $\partial_t^j \widetilde{g}(t_0; \lambda) = 0$. As $\partial_t^i g(t_0; \lambda) = 0$, $\partial_t^i y_j(t_0; \lambda) = 0$ for $i \in \mathbb{Z}$, $0 \leq i < j$, we have $\partial_t^i \widetilde{g}(t_0; \lambda) = 0$, for $i \in \mathbb{Z}$, $0 \leq i < j$. Thus, $\partial_t^i \widetilde{g}(t_0; \lambda) = 0$, for $i \in \mathbb{Z}$, $0 \leq i \leq j$.

By the induction assumption (for $n = j+1$), there exists $\widetilde{y}(t; \lambda) \in \mathcal{H}_I(B_R)$ such that

$$A(t)\widetilde{y}''(t; \lambda) + B(t; \lambda)\widetilde{y}'(t; \lambda) + D(t; \lambda)\widetilde{y}(t; \lambda) = \psi_{1,j+1}(\lambda)\widetilde{g}(t; \lambda), \quad \widetilde{y}(t_0; \lambda) = 0, \quad \lambda \in B_R.$$

[15] Here we use the fact that if $x \in \mathcal{H}_I(\Omega)$, then $x^{(n)} \in \mathcal{H}_I(\Omega)$ for any $n \in \mathbb{Z}_{\geq 0}$.

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For all $t \in I, \lambda \in B_R$, let $y(t;\lambda) := \widetilde{y}(t;\lambda) + \psi_{1,j+1}(\lambda)\partial_t^j g(t_0;\lambda) \cdot x_j(t)/a_j$, then $y \in \mathcal{H}_I(B_R)$, $y(t_0;\lambda) = 0$ and

$$\begin{aligned} &A(t)y''(t;\lambda) + B(t;\lambda)y'(t;\lambda) + D(t;\lambda)y(t;\lambda) \\ &\quad = \psi_{1,j+1}(\lambda)\widetilde{g}(t;\lambda) + \psi_{1,j+1}(\lambda)\partial_t^j g(t_0;\lambda) \cdot y_j(t;\lambda)/a_j = \psi_{1,j+1}(\lambda)b_j(\lambda)g(t;\lambda) = \psi_{1,j}(\lambda)g(t;\lambda), \end{aligned}$$

where we have used $y_j(t;\lambda) = A(t)x_j''(t) + B(t;\lambda)x_j'(t) + D(t;\lambda)x_j(t)$, $\widetilde{g}(t;\lambda) + \partial_t^j g(t_0;\lambda) \cdot y_j(t;\lambda)/a_j = b_j(\lambda)g(t;\lambda)$ and $\psi_{1,j}(\lambda) = b_j(\lambda)\psi_{1,j+1}(\lambda)$. Thus, $y \in \mathcal{H}_I(B_R)$ solves (B.27) for $n = j$. This completes the proof. □

In the end of this appendix, we prove the analytic dependence on parameters of solutions to linear *regular* ODEs. The following lemma has been used in **Step 4** of the proof of Lemma B.3, to show that the extended smooth solution is analytic with respect to the parameter λ.

Lemma B.5. *Let $\Omega \subset \mathbb{C}$ be an open set and $I \subset \mathbb{R}$. Let $p(t;\lambda), q(t;\lambda), f(t;\lambda) \in C^\infty(I \times \Omega; \mathbb{C})$ be such that $p(t;\cdot), q(t;\cdot), f(t;\cdot)$ are analytic on Ω for each $t \in I$. Let $x_0(\lambda), x_1(\lambda)$ be two analytic functions on Ω and let $t_0 \in I$. For each $\lambda \in \Omega$, let $x(t;\lambda)(t \in I)$ be the unique smooth solution to the initial value problem*

$$x''(t;\lambda) + p(t;\lambda)x'(t;\lambda) + q(t;\lambda)x(t;\lambda) = f(t;\lambda), \quad x(t_0;\lambda) = x_0(\lambda), x'(t_0;\lambda) = x_1(\lambda),$$

where the prime $'$ refers to the derivative with respect to $t \in I$. Then for each $t \in I$, the function $\lambda \in \Omega \mapsto x(t;\lambda)$ is analytic.

Proof. By the standard ODE theory, we know that $x \in C^\infty(I \times \Omega)$. For any complex function $\varphi = \varphi(\lambda) : \mathbb{C} \to \mathbb{C}$ of class C^1 seen as a function on $\mathbb{R}^2$, we can define the Wirtinger derivatives

$$\partial_{\bar{\lambda}}\varphi(\lambda) = \frac{1}{2}\big(\partial_1\varphi(\lambda) + \mathrm{i}\partial_2\varphi(\lambda)\big), \quad \partial_\lambda\varphi(\lambda) = \frac{1}{2}\big(\partial_1\varphi(\lambda) - \mathrm{i}\partial_2\varphi(\lambda)\big).$$

Now it suffices to show that $\partial_{\bar{\lambda}}x(t;\lambda) = 0$ for all $(t,\lambda) \in I \times \Omega$. Since $x \in C^\infty(I \times \Omega)$, the derivative with respect to t and $\partial_{\bar{\lambda}}$ are commutable. By the analyticity of coefficients and the initial data, we know that $\partial_{\bar{\lambda}}x$ satisfies

$$(\partial_{\bar{\lambda}}x)''(t;\lambda) + p(t;\lambda)(\partial_{\bar{\lambda}}x)'(t;\lambda) + q(t;\lambda)\partial_{\bar{\lambda}}x(t;\lambda) = 0, \quad \partial_{\bar{\lambda}}x(t_0;\lambda) = (\partial_{\bar{\lambda}}x)'(t_0;\lambda) = 0.$$

By the uniqueness, we have $\partial_{\bar{\lambda}}x(t;\lambda) = 0$ for all $(t,\lambda) \in I \times \Omega$. □

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