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# CENTER POINTS OF NETS

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**1. Introduction.** Suppose $x = (x_\alpha)$ is a net with values in a metric space $X$ having metric $\rho$. If a point $z$ in $X$ can be found to minimize

$$(1) \quad R(z) = \limsup_\alpha \rho(x_\alpha, z)$$

then $z$ is called a center point (c.p.) of $x$. The space $X$ is (netwise) c.p. complete if every bounded net has at least one c.p.; it is sequentially c.p. complete if every bounded sequence has a c.p. Netwise c.p. completeness implies sequential c.p. completeness, and the latter implies completeness since any c.p. of a Cauchy sequence will necessarily be a limit point of that sequence.

These notions are related to the set centers of Calder *et al.* [**2**]. Let $M$ be a bounded infinite subset of $X$ and consider the directed set $D$ consisting of pairs $\alpha = (A_\alpha, x_\alpha)$, where $A_\alpha$ is a finite subset of $M$ and $x_\alpha$ is any point of $M - A_\alpha$. The set $D$ is directed so $\beta$ follows $\alpha$ if $A_\beta \supseteq A_\alpha$. Then a c.p. of the set $M$, in the sense of Calder *et al.*, is precisely a c.p. of the net $(x_\alpha : \alpha \in D)$. We say $X$ is setwise c.p. complete if every bounded infinite subset has a c.p. One of the purposes of this paper is to settle some questions concerning set centers which were left open in [**2**]. Our other goal is to develop some basic theory of centers of nets in Banach spaces. Some of these concepts have proven useful in the area of fixed point theory [**1**; **3**].

THEOREM 1. *For a metric space $X$,*

(1) *netwise c.p. completeness implies setwise c.p. completeness;*

(2) *setwise c.p. completeness implies sequential c.p. completeness, provided $X$ has no isolated points;*

(3) *separability and sequential c.p. completeness together imply setwise c.p. completeness;*

(4) *sequential c.p. completeness implies completeness.*

*Proof.* Statements (1) and (4) are obvious. Statements (2) and (3) follow easily from the following two lemmas.

LEMMA 1. *Suppose $x = (x_\alpha)$ and $y = (y_\alpha)$ are nets of points in $X$ and $\rho(x_\alpha, y_\alpha) \to 0$. Then any c.p. of $x$ is a c.p. of $y$.*

LEMMA 2. *Suppose $x = (x_n)$ is a sequence of distinct points in $X$. Then a point of $X$ is a c.p. of $x$ if and only if it is a c.p. of the set of values of $x$.*

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**2. c.p. complete Banach spaces.** In this section we give sufficient conditions for a Banach space to be (netwise) c.p. complete. Our first two theorems in this section are straightforward generalizations of theorems in [**2**]. The third theorem is presented *ad hoc* to show that the sequence space $l^1$ is netwise c.p. complete, thus settling a question left open in [**2**].

THEOREM 2. *Let $X$ be a reflexive Banach space with distance defined by the norm $N$. Then $X$ is netwise c.p. complete.*

*Proof.* Let $x = (x_\alpha)$ be a bounded net in $X$ and consider the function $R : X \rightarrow [0, \infty]$ defined by equation (1). For any $t \in [0, \infty)$ the set $K_t$ of $z \in X$ satisfying $R(z) \leqq t$ is convex, bounded, and closed in the norm topology, hence also in the weak topology. Thus, with respect to the weak topology, $R$ is lower semi-continuous, and hence it attains a minimum on the compact set $K_t$, which is non-empty if $t$ is sufficiently large.

The next theorem must be phrased in terms of a property called "property (H)" in [**2**]. Since the term "property (H)" is widely used in a different sense, we will refer instead to the "chained exchangeability property". The property is meaningful for an arbitrary metric space $X$. If $B$ and $b$ are (open) balls in $X$, of radius $R$ and $r \leqq R$ respectively, we will say that these balls are $\epsilon$-exchangeable, where $\epsilon \geqq 0$, if there exists a ball $B'$ of radius $r$ such that $B' \supseteq B \cap b$ and such that the center of $B'$ is within a distance $\epsilon$ of the center of $B$.

*Definition.* A metric space $X$ is said to have the chained exchangeability property if for every $r > 0$ there exist sequences $(r_n)$, $(h_n)$ of positive real numbers such that

(1) $r_n \rightarrow r$ monotonically from above;

(2) $h_n \rightarrow 0$ monotonically from above and $\sum_n h_n < \infty$;

(3) every ball of radius $r_{n+1}$ is $h_n$-exchangeable with every ball of radius $r_n$.

Note that in a Banach space, the chained exchangeability property is equivalent to saying as in [**2**] that for every $h > 0$ there exists $d > 0$ with $d < 1$ such that every ball of radius $1 - d$ is $h$-exchangeable with every ball of radius 1.

THEOREM 3. *If a complete metric space $X$ has the chained exchangeability property, it is netwise c.p. complete.*

*Proof.* Let $(x_\alpha)$ be a bounded net in $X$ and let $r = \inf \{R(z) : z \in X\}$. Take $r_n$ and $h_n$ as in the above definition and define $z_n \in X$ inductively so $R(z_n) < r_n$ and $\rho(z_{n+1}, z_n) < h_n$. Then observe that $z = \lim_n z_n$ is a c.p. of $(x_\alpha)$.

The last theorem in this section concerns a certain class of Banach sequence spaces of which $l^1$ is an important example. Let $s$ be the space of sequences $x = (x(n))$ of real numbers. If $x \in s$ and $0 \leqq a \leqq b < \infty$, denote by $(a|x|b)$ the element of $s$ whose $n$-th term is $x(n)$ when $a < n \leqq b$ and 0 otherwise. The special cases $a = 0$ and $b = \infty$ are denoted by $(x|b) = (0|x|b)$ and

$(a|x) = (a|x|\infty)$ respectively. A sequential norm [**7**] is a function $N$ from $s$ into $[0, \infty]$ such that (1) $N$ is an extended norm, i.e., $N$ has the usual formal properties of a norm but can be infinite; and (2) for every $x \in s$, $N(x|n) < \infty$ for every $n$ and $N(x) = \sup_n N(x|n)$. Then $B(N) = \{x \in s : N(x) < \infty\}$ is regarded as a Banach space with norm $N$, and

$$C(N) = \{x \in B(N) : N(n|x) \to 0 \text{ as } n \to \infty\}$$

is a subspace. A sequential norm $N$ is said to be balanced if for any $x, y \in s$ the inequalities $|x(n)| \leqq |y(n)|$, $n = 1, 2, \ldots$, jointly imply $N(x) \leqq N(y)$.

*Definition.* A balanced sequential norm $N$ will be said to have the Archimedean property if, for all positive real numbers $h$, $K$, there exists an integer $m$ such that $N(x) > K$ for every $x \in s$ which can be written as a sum $x = x_1 + \ldots + x_m$ where (1) each $N(x_i) > h$, and (2) the $x_i$ have pairwise disjoint supports. Here, by the support of $x_i$, we mean $\{n : x_i(n) \neq 0\}$.

It should be clear that the Archimedean property implies $B(N) = C(N)$. Note also that the usual $l^1$ norm has this property.

THEOREM 4. *If $N$ is a balanced sequential norm having the Archimedean property, then $B(N)$ is netwise c.p. complete.*

*Proof.* Let $x = (x_\alpha)$ be a bounded net in $B(N)$ and let $r = \inf\{R(z) : z \in B(N)\}$. Let $(z_n)$ be a sequence in $B(N)$ such that $R(z_n) \to r$. Since $(z_n)$ could be replaced by a subsequence, we can assume that $(z_n)$ converges in each coordinate to some $w \in s$. Since $N(w|m) = \lim_n N(z_n|m)$ for each $m$, and since $(z_n)$ is bounded, we know $w \in B(N)$. We shall show $w$ is a c.p. of $(x_n)$, assuming without loss of generality that $w = 0$. By induction, define a sequence $(w_n)$ of points in $B(N)$ and an increasing sequence $(a_n)$ of positive integers such that (1) $w_n = (a_n|w_n|a_{n+1})$, and (2) there exists some $m(n)$ so $N(w_n - z_{m(n)}) < 2^{-n}$ and $R(z_{m(n)}) < r + 2^{-n}$. This is easily done since for given $a_n$, taking $m(n)$ large enough will ensure $N(z_{m(n)}|a_n) < 4^{-n}$, and then, since $B(N) = C(N)$, there exists $a_{n+1} > a_n$ such that $N(a_{n+1}|z_{m(n)}) < 4^{-n}$. Thus we may take $w_n = (a_n|z_{m(n)}|a_{n+1})$. Note that $\lim_n R(w_n) = r$, because of (2), and that, because of (1) and the fact that $N$ is balanced,

$$N(x_\alpha - w_n) \geqq N((x_\alpha|a_n) + (a_{n+1}|x_\alpha)) \geqq N(x_\alpha) - N(a_n|x_\alpha|a_{n+1}).$$

Thus if there were to exist a positive number $h < R(0) - r$, then for any positive number $K$, however large, we could find $m$ as in the above definition and assert that frequently one finds $x_\alpha$ satisfying each inequality.

$$N(a_i|x_\alpha|a_{i+1}) \geqq N(x_\alpha) - N(x_\alpha - w_i)$$

for $m$ consecutive values of $i$, and hence satisfying $N(x_\alpha) > K$. Since $(x_\alpha)$ is assumed bounded, we must therefore conclude that $R(0) = r$ and so $(x_\alpha)$ has $w = 0$ as a c.p.

COROLLARY. *The sequence space $l^1$ is netwise and hence setwise c.p. complete.*

The corollary answers a question left open in [**2**]. We have not settled the question, by the way, as to whether $l^1$ has the chained exchangeability property. Certain related questions can be answered, however, by considering the balanced sequential norm $N$, defined by

$$N(x) = (||x||_1^2 + ||x||_2^2)^{1/2}.$$

This sequential norm $N$ is equivalent to the $l^1$ norm on $B(N) = l^1$ and, because $N$ also has the Archimedean property, $B(N)$ is still c.p. complete. However, Calder, *et al.* [**2**] prove that, for Banach spaces having the chained exchangeability property, strict convexity implies uniform convexity. Hence $B(N)$ lacks the chained exchangeability property, since it is obviously strictly convex. As a matter of fact, $B(N)$ is uniformly convex in every direction, as defined in [**2**]. Hence the example also shows that uniform convexity in every direction and setwise c.p. completeness do not jointly imply reflexivity. This question was also raised in [**2**].

**3. Banach spaces lacking c.p. completeness.** A rich source of examples is provided by the following.

THEOREM 5. *Let $N$ be a balanced sequential norm for which $B(N) \neq C(N)$. Suppose also that for each non-zero $x \in B(N)$, $N(x)$ is strictly greater than $T(x) = \lim_n N(n|x)$, which is the distance of $x$ from $C(N)$. Then neither $B(N)$ nor $C(N)$ is sequentially c.p. complete.*

*Proof.* Take any point $w \in B(N)$ such that $w \notin C(N)$, i.e., $T(w) > 0$, and set $x_n = 2(w|n) \in B(N)$. Suppose $z \in B(N)$ is a c.p. of the sequence $(x_n)$ and note that $R(z)$ cannot be greater than $T(w) = \lim_m N(m|w)$ because

$$N(m|w) = \limsup_n N(x_n - x_m - (m|w)) = R(x_m + (m|w)).$$

Since $R(2w) = 2T(w)$, we know $z \neq 2w$ and therefore

$$\begin{aligned} T(z - 2w) < N(z - 2w) &= \lim_n N((z - 2w)|n) \\ &\leqq \limsup_n N(z - x_n) = R(z) \leqq T(w). \end{aligned}$$

Since $T$ inherits the homogeneity of $N$ and the triangle inequality, it follows that

$$T(z) \geqq T(2w) - T(2w - z) > T(w).$$

On the other hand

$$T(z) = T(z - x_n) \leqq N(z - x_n) \leqq T(w) + \epsilon_n$$

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where $\epsilon_n \to 0$, so we have a contradiction. Thus $B(N)$ is not sequentially c.p. complete. To show that $C(N)$ is not sequentially c.p. complete, consider the sequence whose $n$th term is $x_n = (w|n) \in C(N)$. For any $z \in C(N)$, $R(z)$ is no less than $\limsup_n N((z - w)|n) = N(z - w)$, which is strictly greater, since $z \neq w$, than $T(z - w) = T(w)$. Since a proper choice of $m$ will make $\limsup_n N(x_n - (w|m))$ arbitrarily close to $T(x)$, we see that $T(x)$ is an unattainable greatest lower bound for $\{R(z) : z \in C(N)\}$.

COROLLARY. *There exists a Banach space whose conjugate is not sequentially c.p. complete.*

*Proof.* The Banach sequence space given in the example of [**6**, p. 69] is a conjugate space, by other results in [**6**], and the norm is clearly of the type described in the theorem.

Some related questions are still open. Can the conjugate of a Banach space always be renormed to attain c.p. completeness? If the conjugate of a Banach space is separable, is it then c.p. complete?

**4. Miscellany.** (a) For applications to fixed point theory, as in [**1**], it is important to consider more generally a subset $K$ of a metric space $X$, and a net $x = (x_\alpha)$ with values in $X$. *A* $K$-center of $x$ is a point $s \in K$ which minimizes $R(z)$ subject to $z \in K$. See [**2**] and [**5**] for some extensions in this direction.

(b) It may be recalled that a Banach space $X$ is Chebychev c.p. complete if every bounded subset $M$ of $X$ has a Chebychev c.p., i.e., a point $z \in X$ for which $\sup \{\rho(m, z) : m \in M\}$ is minimal. It was shown in [**2**] that setwise c.p. completeness implies Chebychev c.p. completeness. We do not know whether sequential c.p. completeness implies Chebychev c.p. completeness or whether setwise c.p. completeness implies netwise c.p. completeness. Garkavi [**4**] showed that the conjugate of a Banach space is Chebychev c.p. complete. Compare this result to the Corollary of Theorem 5.

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