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AN EXTENSION OF M. RIESZ'S MEAN VALUE THEOREM FOR INFINITE INTEGRALS

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1. Introduction. Isaacs [**4**] has proved the following theorem.

THEOREM A. *If* $0 < \alpha < 1$ *and*

$$\int^{\infty} t^{\alpha-1} g(t)dt$$

is convergent, then for $u < w$,

$$(1.1) \qquad \left| \frac{1}{\Gamma(\alpha)} \int_w^{\infty} (t-u)^{\alpha-1} g(t)dt \right| \leqslant \operatorname*{ess.\,bound.}_{w \leqslant v < \infty} \left| \frac{1}{\Gamma(\alpha)} \int_v^{\infty} (t-v)^{\alpha-1} g(t)dt \right|.$$

In the case where $g(t) = 0$ for $t > c$, c finite, this becomes Riesz's Inequality.

The object of this note is to extend Theorem A (in the case of absolute convergence) by replacing the function $t^{\alpha-1}/\Gamma(\alpha)$ by a general function $G(t)$. The role of the related function $t^{-\alpha}/\Gamma(1-\alpha)$ is then played by a function $H(t)$ such that

$$(1.2) \qquad \int_0^y G(y-t)H(t)dt = 1, \quad \text{for } y > 0$$
$$= 0, \quad \text{for } y = 0.$$

A similar extension of Riesz's inequality has been given by Bosanquet [**1**].

In [**2**], Bosanquet has shown the existence of more than one pair of functions $G(t)$ and $H(t)$ which satisfy (1.2) as well as the conditions laid down in our theorem in section 3 below.

2. Lemmas. In order to prove our theorem we need a few lemmas. The proof of Lemmas 1–5 is given in [**1**].

LEMMA 1. *If* $G(t)$ *and* $H(t)$ *are positive for* $t > 0$, *and satisfy* (1.2), *and if* $R(t) \in L(c, x)$, *where* $x > c$, *then*

$$\int_c^x R(t)dt = \int_c^x G(x-u)du \int_c^u H(u-t)R(t)dt,$$

the inner integral existing for almost every u *in* (c, x).

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LEMMA 2. *If $G(t)$ is positive for $t > 0$, $G(t) \in L(0, x - c)$ and $R(t) \in L(c, x)$, where $x > c$, then*

$$\int_c^x G(x - t)dt \int_c^t R(w)dw = \int_c^x du \int_c^u G(u - w)R(w)dw,$$

the inner integral on the right existing for almost every u in (c, x).

LEMMA 3. *If $G(t)$ is continuous for $t > 0$, and $R(t) \in L(b, c)$, where $b < c$, then the function*

$$f(w) = \int_b^c G(t - w)R(t)dt$$

is continuous for $w < b$.

LEMMA 4. *If $G(t)$ is continuous for $t > 0$, $G(t) \in L(0, y)$, $y > 0$, and $R(t)$ is bounded in every finite interval (c, x) then the function*

$$h(x) = \int_c^x G(x - t)R(t)dt$$

is continuous for $x > c$.

LEMMA 5. *If $G(t)$ is decreasing and positive for $t > 0$, and $G(t - x)R(t) \in L(a, b)$, where $x < a$, then $R(t) \in L(a, b)$.*

LEMMA 6. *Let $G(t)$ and $H(t)$ be decreasing and positive for $t > 0$, and let (1.2) hold. Then if $H(t) \to A > 0$ as $t \to \infty$ we have*

$$\lim_{x\to\infty} \int_0^x G(u)du = \frac{1}{A}, \tag{2.1}$$

and conversely.

Proof. If $0 < \epsilon < A/2$, choose $k > 0$ so that $A - \epsilon < H(t) < A + \epsilon$ for $t > k$.

Write, for $x > k$,

$$1 = \int_0^x G(x - t)H(t)dt = \left(\int_0^k + \int_k^x\right) G(x - t)H(t)dt \tag{2.2}$$
$$= I_1 + I_2.$$

Then, first

$$I_2 > (A - \epsilon) \int_0^{x-k} G(u)du \quad \text{and} \quad I_1 > 0.$$

Therefore, from (2.2), it follows that

$$\int_0^{x-k} G(u)du \leqq 2/A, \quad \text{for } x > k,$$

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and hence

$$\int_0^\infty G(u)du$$

is convergent.

Then, since $G(u)$ is positive and decreasing, it follows that $G(u)$ must $\to 0$ as $u \to \infty$. Hence

$$I_1 \to 0 \quad \text{and} \quad I_2 \geqq (A - \epsilon) \int_0^{x-k} G(u)du, \quad \text{for } x > k.$$

Thus

$$\overline{\lim_{x\to\infty}} \int_0^x G(u)du \leqq 1/(A - \epsilon), \quad \text{whenever } 0 < \epsilon < A/2.$$

Since ϵ is arbitrary, we have

$$\overline{\lim_{x\to\infty}} \int_0^x G(u)du \leqq 1/A.$$

Again, since

$$I_1 \to 0 \quad \text{and} \quad I_2 \leqq (A + \epsilon) \int_0^{x-k} G(u)du,$$

it follows that

$$\underline{\lim}_{x\to\infty} \int_0^x G(u)du \geqq 1/A,$$

which completes the proof of (2.1).

Conversely, if (2.1) holds, then $H(t)$ can only tend to A.

3. The main theorem.

THEOREM. *Let $G(t)$ and $H(t)$ be decreasing and positive for $t > 0$, and satisfy the relation* (1.2). *Let $G(t)$, $H(t)$ and $H'(t)$ be continuous. If $g(t) \in L(\xi, T)$, for every $T > \xi$, and the integral on the left of* (3.1) *converges absolutely at the upper limit, then for $x < \xi$,*

$$\left| \int_\xi^\infty G(t - x)g(t)dt \right| \leqq \operatorname*{ess.\,sup.}_{y\in(\xi,\infty)} \left| \int_y^\infty G(t - y)g(t)dt \right|. \tag{3.1}$$

Proof. We first establish the formula

$$\int_\xi^\infty G(t - x)g(t)dt = \int_\xi^\infty K(x, y)dy \int_y^\infty G(t - y)g(t)dt \tag{3.2}$$

for $x < \xi$, where $K(x, y)$ is a certain function of x and y.

This will be true if and only if

$$(3.3)\qquad \int_{\xi}^{\infty} G(t-x)g(t)dt = \int_{\xi}^{\infty} g(t)dt \int_{\xi}^{t} G(t-y)K(x,y)dy,$$

provided the inversion of the repeated integral is justified. Again (3.3) will be established if we show that $G(t-x)$ can be expressed in the form

$$(3.4)\qquad G(t-x) = \int_{\xi}^{t} G(t-y)K(x,y)dy$$

for every $x < \xi$, provided at least one side of (3.3) exists. But the left-hand side of (3.3) exists by hypothesis.

To find the necessary form of $K(x, y)$, we *assume first that* (3.4) *does hold.* It follows from (3.4) and Lemma 5 that $K(x, y)$ is integrable with respect to y in (ξ, t), whenever $x < \xi < t$. Therefore, by Lemma 1 and (3.4), we obtain

$$(3.5)\qquad \int_{\xi}^{w} H(w-t)G(t-x)dt = \int_{\xi}^{w} K(x,y)dy$$

for every $x < \xi < w$.

For each $x < \xi$, (3.5) is differentiable with respect to w for almost every $w > \xi$ (the exceptional w's depending on x). Thus, by (1.2),

$$K(x,w) = -\int_{x}^{\xi} H'(w-t)G(t-x)dt.$$

Now *define* $K(x, w)$ by the equation

$$(3.6)\qquad K(x,w) = -\int_{x}^{\xi} H'(w-t)G(t-x)dt \quad (x < \xi < w).$$

With this choice of $K(x, w)$ the exceptional sets disappear, since the last integral is continuous with respect to w for $w > \xi$, by Lemma 3. It also follows from (3.6) and the hypotheses of the theorem that $K(x, w) \geqq 0$ for $x < \xi < w$. Further $K(x, w)$ is continuous with respect to x, by Lemma 4, since we know that $H'(w - t)$ is continuous for $t \leqq \xi$, if $w > \xi$.

We must now show that (3.4) does hold, with our definition of $K(x, w)$. For $x < \xi < w$, since $K(x, y) \geqq 0$ in $\xi < y \leqq w$,

$$\begin{aligned}\int_{\xi}^{w} K(x,y)dy &= -\int_{\xi}^{w} dy \int_{x}^{\xi} H'(y-t)G(t-x)dt \\ &= -\int_{x}^{\xi} G(t-x)dt \int_{\xi}^{w} H'(y-t)dy \\ &= -\int_{x}^{\xi} G(t-x)\{H(w-t) - H(\xi-t)\}dt,\end{aligned}$$

which is finite. Hence $K(x, y)$ is integrable over the interval $\xi \leqq y \leqq w$, and

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we obtain, for $x < \xi < w$,

$$\int_\xi^w K(x, y)dy = 1 - \int_x^\xi H(w - t)G(t - x)dt$$
$$= \int_\xi^w H(w - t)G(t - x)dt$$

which is (3.5).

It follows, for $x < \xi < w$, that

$$\int_\xi^w G(w - u)du \int_\xi^u H(u - t)G(t - x)dt$$
$$= \int_\xi^w G(w - u)du \int_\xi^u K(x, y)dy.$$

Inverting the order of integration on the left-hand side, and applying Lemma 2 to the right-hand side, we get

$$\int_\xi^w G(t - x)dt \int_t^w G(w - u)H(u - t)dt$$
$$= \int_\xi^w dv \int_\xi^v G(v - y)K(x, y)dy,$$

i.e.,

$$\int_\xi^w G(t - x)dt = \int_\xi^w du \int_\xi^u G(u - y)K(x, y)dy. \tag{3.7}$$

Hence

$$G(w - x) = \int_\xi^w G(w - y)K(x, y)dy, \tag{3.8}$$

since the left-hand side is continuous with respect to w, for $w > x$, by hypothesis, and the right-hand side is continuous for $w > x$, by Lemma 4.

Thus (3.3) is established. The formula also holds with g replaced by $|g|$, since g may be replaced by $|g|$ in our hypothesis. Since the functions G and K are non-negative, it follows that the right-hand side of (3.3) is absolutely convergent. This justifies the inversion of the repeated integral and hence (3.2) is proved.

Finally, (3.1) follows from (3.2). For (3.2) implies that, if $x < \xi$,

$$\left| \int_\xi^\infty G(t - x)g(t)dt \right| \leqq W_{\xi,x} \operatorname*{ess.\,sup.}_{y \in (\xi, \infty)} \left| \int_y^\infty G(t - y)g(t)dt \right|, \tag{3.9}$$

where

$$W_{\xi,x} = \int_\xi^\infty |K(x, y)|dy = \int_\xi^\infty K(x, y)dy = \lim_{w \to \infty} \int_\xi^w K(x, y)dy. \tag{3.10}$$

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Since, by (3.5),

$$(3.11)\qquad \int_\xi^w K(x, y)dy = \int_\xi^w H(w - t)G(t - x)dt < 1,$$

(3.1) follows from (3.9)–(3.11).

4. The factor $W_{\xi,x}$**.** Since $H(t)$ is positive and decreasing, it either tends to zero or to a positive limit as $t \to \infty$.

(i) Suppose that $H(t) \to 0$ as $t \to \infty$. Then, if $x < \xi < w$,

$$(4.1)\qquad \int_\xi^w K(x, y)dy = 1 - \int_x^\xi H(w - t)G(t - x)dt \to 1 \quad \text{as } w \to \infty,$$

so that

$$W_{\xi,x} = 1.$$

(ii) Suppose that $H(t) \to A > 0$ as $t \to \infty$. Then (4.1) implies that

$$(4.2)\qquad W_{\xi,x} = 1 - A \int_0^{\xi - x} G(u)du.$$

It follows from Lemma 6 that, in this case,

$$1 > W_{\xi,x} = 1 - A \int_0^{\xi - x} G(u)du > 0.$$

We now give two examples, the purpose of which is to show that case (ii) can occur. In the following examples we shall write $k(s)$ for the Laplace transform of $K(t)$.

Example 1. If $H(t) = A(1 + \pi^{-\frac{1}{2}} t^{-\frac{1}{2}})$, $(A > 0)$, then $H(t) \to A$ as $t \to \infty$, $h(s) = A(s^{-1} + s^{-\frac{1}{2}})$, $sh(s) = A(1 + s^{\frac{1}{2}})$, and (formally)

$$g(s) = A^{-1}(1 + s^{\frac{1}{2}})^{-1} = A^{-1}\left[s \frac{1}{s^{\frac{1}{2}}(s - 1)} - \frac{1}{s - 1}\right].$$

We have, for $s > 1$,

$$\frac{1}{s^{\frac{1}{2}}(s - 1)} = \frac{1}{s\Gamma(\frac{1}{2})} \int_0^\infty \left\{\frac{d}{dt}\left(e^t \int_0^t u^{-\frac{1}{2}} e^{-u} du\right)\right\} e^{-st} dt.$$

Hence

$$G(t) = A^{-1}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(t^{-\frac{1}{2}} + e^t \int_0^t u^{-\frac{1}{2}} e^{-u} du\right) - e^t\right]$$

$$= \frac{A^{-1} e^t}{2\Gamma(\frac{1}{2})} \int_t^\infty u^{-3/2} e^{-u} du.$$

Clearly (put $t = u - x$), $G(t)$ is decreasing and $G(t)$ is $O(t^{-\frac{1}{2}})$ for small t and $O(t^{-3/2})$ for large t. Hence $g(s)$ exists for $s > 0$, and by analytic continuation $g(s) = 1/A(1 + s^{\frac{1}{2}})$ for $s > 0$.

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In this case (4.2) becomes

$$W_{\xi,x} = 1 - \frac{1}{2\Gamma(\frac{1}{2})} \int_0^{\xi - x} \left(e^t \int_t^\infty u^{-3/2} e^{-u} du \right) dt < 1 \tag{4.3}$$

since the integrand in (4.3) is positive.

Further, it can easily be verified that this pair of functions $G(t)$ and $H(t)$ indeed satisfy (1.2).

Example 2 [**2**, Example 5]. If $G(t) = t^{-\frac{1}{2}} e^{-t}/\Gamma(\frac{1}{2})$, then

$$g(s) = \frac{1}{(s+1)^{\frac{1}{2}}}, h(s) = (s+1)^{\frac{1}{2}} s^{-1} = \frac{1}{(s+1)^{\frac{1}{2}}} + \frac{1}{s(s+1)^{\frac{1}{2}}}.$$

Therefore

$$\begin{aligned} H(t) &= \frac{1}{\Gamma(\frac{1}{2})} \left(t^{-\frac{1}{2}} e^{-t} + \Gamma(\tfrac{1}{2}) - \int_t^\infty u^{-\frac{1}{2}} e^{-u} du \right) \\ &= 1 + \frac{1}{2\Gamma(\frac{1}{2})} \int_t^\infty u^{-3/2} e^{-u} du. \end{aligned} \tag{4.4}$$

It is clear from (4.4) that $H(t)$ decreases to 1.

In this case also (1.2) is satisfied and $W_{\xi,x} < 1$.

5. The best possible factor. Next, we consider whether the factor $W_{\xi,x}$ is best possible. We have obtained the inequality

$$\left| \int_\xi^\infty G(t-x) g(t) dt \right| \leqq W_{\xi,x} \operatorname*{ess.\,sup.}_{y \in (\xi, \infty)} \left| \int_y^\infty G(t-y) g(t) dt \right|. \tag{5.1}$$

By taking $g(t) = 0$ in (Y, ∞), (5.1) becomes

$$\left| \int_\xi^Y G(t-x) g(t) dt \right| \leqq W_{\xi,x} \operatorname*{ess.\,sup.}_{\xi \leqq y \leqq Y} \left| \int_y^Y G(t-y) g(t) dt \right|.$$

From Theorem 7(b) [**1**, with $R(u) = 1$], we have

$$\left| \int_\xi^Y G(t-x) g(t) dt \right| \leqq W_{\xi,x,Y} \operatorname*{ess.\,sup.}_{\xi \leqq y \leqq Y} \left| \int_y^Y G(t-y) g(t) dt \right|, \tag{5.2}$$

where

$$W_{\xi,x,Y} = \int_\xi^Y G(t-x) H(Y-t) dt,$$

and equality occurs in (5.2) if and only if

$$g(t) = H(Y-t) \text{ in } (\xi, Y).$$

But

$$W_{\xi,x,Y} \to W_{\xi,x} \quad \text{as } Y \to \infty.$$

Hence $W_{\xi,x}$ is best possible, i.e., cannot be replaced by a smaller number.

6. Equality. We deduced from (3.2), since $K(x, y) \geqq 0$, that

$$\left| \int_\xi^\infty G(t-x)g(t)dt \right| \leqq \int_\xi^\infty K(x,y) \left| \int_y^\infty G(t-y)g(t)dt \right| dy$$
$$\leqq \int_\xi^\infty K(x,y)dy$$
$$\times \operatorname*{ess.\,sup.}_{y\in(\xi,\infty)} \left| \int_y^\infty G(t-y)g(t)dt \right| .$$

Therefore

$$\left| \int_\xi^\infty G(t-x)g(t)dt \right| \leqq W_{\xi,x} \operatorname*{ess.\,sup.}_{y\in(\xi,\infty)} \left| \int_y^\infty G(t-y)g(t)dt \right| . \tag{6.1}$$

Now equality occurs in (6.1) if and only if

$$\int_y^\infty G(t-y)g(t)dt$$

is of constant amplitude for almost every $y > \xi$, and

$$\left| \int_y^\infty G(t-y)g(t)dt \right|$$

equals its ess. sup. in (ξ, ∞), i.e., if and only if

$$f(y) = \int_y^\infty G(t-y)g(t)dt = C \quad \text{p.p. in } (\xi, \infty), \tag{6.2}$$

where C is a complex constant.

If $H(t) \to A > 0$ as $t \to \infty$, and $g(t) = AC$, then (6.2) is satisfied, since

$$AC \int_y^\infty G(t-y)dt = C, \quad \text{i.e.,} \quad AC \int_0^\infty G(u)du = C,$$

by Lemma 6.

Hence $g(t) = AC$ is *sufficient* for equality in (6.1), if $\lim H(t) = A > 0$. We have not been able to settle whether $g(t) = AC$ is also *necessary* for equality in (6.1) under the hypotheses of our theorem. However we get the desired result under the additional assumptions in the following theorem [**3**].

THEOREM B. *Assume that*

(i) $G(t), -G'(t), H(t) -H'(t)$ *are positive and continuous for* $t > 0$, *and satisfy*

$$\int_0^y G(y-t)H(t)dt = 1 \textit{ for } y > 0, = 0 \textit{ for } y = 0;$$

(ii) $|G'(t)|/G[t)$ *is non-increasing and is* $O(t^{-1})$ *as* $t \to \infty$;

(iii) $|H'(t)|/H(t)$ *is non-increasing and is* $O(t^{-1})$ *as* $t \to \infty$.

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If $f(x)$ is defined for almost all $x > 0$, and there is a function $g(x)$ such that

$$f(x) = \int_x^\infty G(t - x)g(t)dt \quad \text{p.p. } for\ x > 0,$$

then

$$g(x) = \lim_{w\to\infty}\left(-\frac{d}{dx}\int_x^w H(u - x)f(u)du\right) \quad p.p.\ for\ x > 0. \tag{6.3}$$

From (6.2) and (6.3), we get

$$\begin{aligned} g(t) &= C\lim_{w\to\infty}\left(-\frac{d}{dx}\int_x^w H(y - x)dy\right) \\ &= C\lim_{w\to\infty} H(w - x) \\ &= AC, \end{aligned} \tag{6.4}$$

which proves the necessity.

Now, when $H(t) \to 0$ as $t \to \infty$, (6.2) cannot hold unless $C = 0$. For, assume that (6.2) holds with $C \neq 0$. Then, from (6.4) we get $g(t) = 0$ p.p. which contradicts (6.2), since $C \neq 0$.

Thus in this case, $g(t) = 0$ p.p. is *necessary* and *sufficient* for equality in (6.1).

Remark. Excluding the trivial case in which $g(t) = 0$ p.p., the argument given above shows that equality in (6.1) is possible only if

$$\lim H(t) > 0. \tag{6.5}$$

It is worth noting that (6.5) is consistent with the hypotheses of Theorem B. In Example 1, this is true.

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