A THEOREM ON k-SATURATED GRAPHS

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1. Introduction. In this paper we consider finite graphs without loops and multiple edges. A *graph* $\mathfrak{G}$ is considered to be an ordered pair $\langle G, \mathfrak{G}^* \rangle$ where G is a finite set the elements of which are called the *vertices* of $\mathfrak{G}$ while $\mathfrak{G}^*$ is a subset of $[G]^2$ (where $[G]^2$ is the set of all subsets of two elements of G). The elements of $\mathfrak{G}^*$ are called the *edges* of $\mathfrak{G}$. If $\{P, Q\} \in \mathfrak{G}^*$, we say that Q is *adjacent* to P. The *degree* of a vertex is the number of vertices adjacent to it. Let k be an integer. We say that $\mathfrak{G}$ is the complete *k-graph* if G has k elements and $\mathfrak{G}^* = [G]^2$. If $G \subseteq H$ and $\mathfrak{G}^* \subseteq \mathfrak{H}^*$ we say that $\mathfrak{H}$ *contains* $\mathfrak{G}$ and we write $\mathfrak{G} \subseteq \mathfrak{H}$. The number of elements of a set H will be denoted by $|H|$.

Let k be an integer. The graph $\mathfrak{G}$ is said to be *k-saturated* if it does not contain a complete $(k+1)$-graph, but every graph $\mathfrak{G}'$ obtained from it with the addition of a new edge contains a complete $(k+1)$-graph. (This concept was first defined by Zykov **(5)**.) The vertex P is said to be a *conical vertex* of $\mathfrak{G}$ if it is adjacent to all vertices of $\mathfrak{G}$ different from P.

The aim of this paper is to prove the following conjecture of T. Gallai.

THEOREM 1. *Assume $\mathfrak{G}$ is k-saturated. Then either $\mathfrak{G}$ has a conical vertex or the degree of every vertex of $\mathfrak{G}$ is at least* $2(k-1)$.

Let n denote the number of vertices of $\mathfrak{G}$ and assume that $2k - n > 0$, $k \geqslant 2$. Theorem 1 implies immediately that $\mathfrak{G}$ has a conical vertex provided $\mathfrak{G}$ is k-saturated. Instead of this we can give a short proof of the following slightly stronger result.

THEOREM 2. *Assume $\mathfrak{G}$ is k-saturated, $|G| = n$. Then $\mathfrak{G}$ contains at least $2k - n$ conical vertices.*

Theorem 2 is equivalent to a theorem of P. Erdös and T. Gallai **(1)**. To state this theorem we need some definitions. A set of vertices is said to *represent* the edges of a graph if each edge contains at least one of these vertices. A graph is said to be *edge p-critical* if the minimal number of vertices necessary to represent the edges of the graph is p, but if any edge is omitted, the remaining edges can be represented by $p - 1$ vertices. The following theorem is essentially the same as Theorem 3.10 of **(1)**:

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THEOREM 3. *Assume $\mathfrak{G}$ is edge p-critical, $|G| = n$. Then $\mathfrak{G}$ has at least $n - 2p$ isolated vertices.*

Theorem 3 follows trivially from Theorem 2 when one considers that the complementary graph of an edge p-critical graph is $(n - p)$-saturated and that the isolated vertices of a graph $\mathfrak{G}$ are just the conical vertices of the complementary graph of $\mathfrak{G}$.

In a joint paper with P. Erdös and J. W. Moon **(2)**, we recently proved that the minimal number of edges of a k-saturated graph $\mathfrak{G}$ of n vertices is $n(k - 1) - \binom{k}{2}$. This result also follows immediately from Theorem 1 by induction on k. (Our result remains valid if we replace the assumption that $\mathfrak{G}$ is k-saturated by the weaker assumption that the addition of a new edge increases the number of $(k + 1)$-graphs contained in the graph. It is to be remarked that Theorems 1 and 2 are no longer true under this weaker assumption.) Considering that our extreme graphs contain conical vertices, the following problem remains open.

Problem. Let $2 \leqslant k \leqslant n$ be integers. What is the minimal number of edges of the k-saturated graphs $\mathfrak{G}$ of n vertices which do not contain conical vertices?

A conical vertex has degree $n - 1$. More generally, we can ask: What is the minimal number of edges of k-saturated graphs $\mathfrak{G}$ of n vertices which do not contain vertices of degree $\geqslant n - t$ for $t = 1, 2, \ldots$? The special case $k = 2$ of this problem is treated in a paper of P. Erdös and A. Rényi **(3)**, but the answer in the general case seems to be very complicated.

2. Proof of the theorems.

We need the following lemma.

LEMMA. *Let $\mathfrak{G}$ be a graph, k an integer. Assume $\mathfrak{G}$ does not contain a complete $(k + 1)$-graph. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_v$ be a system of complete k-graphs contained in $\mathfrak{G}$. Let n_v denote the number of elements of the set $\bigcup^{v}_{m=1} A_m$. Then this set has at least $2k - n_v$ elements.*

Proof (by induction on v): We can assume that $k \geqslant 2$, $n_v < 2k$. For $v = 1$ the statement is trivial. Assume that it is true for $v - 1$ $(v > 1)$. Put

$$A = \bigcup_{m=1}^{v-1} A_m, \quad B = \bigcup_{m=1}^{v} A_m, \quad C = \bigcap_{m=1}^{v-1} A_m, \quad D = \bigcap_{m=1}^{v} A_m,$$
$$|A| = n_{v-1}, \quad |B| = n_v\,.$$

By the induction hypothesis we have

$$|C| \geqslant 2k - n_{v-1} > 0. \tag{1}$$

Each vertex of C is adjacent to each vertex of A. Hence the vertices of the set $(A \cap A_v) \cup C$ are all adjacent to each other. Since $\mathfrak{G}$ does not contain a complete $(k + 1)$-graph, we therefore have

$$|(A \cap A_v) \cup C| \leqslant k. \tag{2}$$

Considering that $|A \cap A_v| = k - |B - A| = k - (n_v - n_{v-1})$, it follows from (2) that

(3) $$|C - A_v| \leqslant n_v - n_{v-1}.$$

Comparing (1) and (3), we obtain the desired result

$$|D| = |A_v \cap C| \geqslant 2k - n_{v-1} - (n_v - n_{v-1}) = 2k - n_v.$$

Proof of Theorem 2. We may assume that $n < 2k$, $k \geqslant 2$. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_v$ be the system of all complete k-graphs contained in $\mathfrak{G}$. Put

$$A = \bigcap_{m=1}^{v} A_m.$$

Since the union of the sets A_m has at most n elements, it follows from the lemma that $|A| \geqslant 2k - n$.

We prove that all the vertices in A are conical vertices of the graph $\mathfrak{G}$. Let $P \in A$ and $Q \in G$, $P \neq Q$. Suppose P is not adjacent to Q. Then, by the assumption that $\mathfrak{G}$ is k-saturated, if we join the edge $\{P, Q\}$ to $\mathfrak{G}$, the new graph thus obtained contains a complete $(k+1)$-graph. This means that there exists a complete $(k-1)$-graph $\mathfrak{B}$ contained in $\mathfrak{G}$ all the vertices of which are adjacent to both P and Q. But then adding Q to $\mathfrak{B}$ we obtain a complete k-graph contained in $\mathfrak{G}$ which does not contain P. This contradicts the definition of A. Hence P must be adjacent to Q. This proves Theorem 2.

Proof of Theorem 1. Let $\mathfrak{G}$ be a k-saturated graph which has no conical vertices. We assume that there exists a vertex P_0 of degree $\leqslant 2k - 3$. This will yield a contradiction.

Let H denote the set of vertices of $\mathfrak{G}$ adjacent to P_0 and let K denote the set of the remaining vertices different from P_0. Thus

(4) $$G = \{P_0\} \cup H \cup K.$$

We can assume that $n > 1$. Then H and K are non-empty and, by our assumption,

(5) $$|H| \leqslant 2k - 3.$$

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_v$ be the system of all those complete k-graphs contained in $\mathfrak{G}$ which contain P_0. Put

$$A = \bigcup_{m=1}^{v} A_m.$$

Obviously

(6) $$A \subseteq H \cup \{P_0\}.$$

Put $u = |H - A|$. Then by (5) and (6), $|A| \leqslant 2k - 2 - u$. It follows from the lemma that the set $\bigcap_{m=1}^{v} A_m$ has at least $u + 2$ elements. Since P_0 belongs to it, we can write it in the form

(7) $$\bigcap_{m=1}^{v} A_m = \{P_0\} \cup B,$$

where $B \subseteq H$ and $|B| \geqslant u + 1$. For an arbitrary $X \subseteq G$ we denote by $\phi(X)$ the set of those $P \in G$ for which there exists a $Q \in X$ not adjacent to P. Now we prove that

$$(8) \qquad \phi(B) \subseteq H - A.$$

We have to prove that if $P \notin H - A$, then P is adjacent to all vertices of B. If $P \notin K$, this is trivial by the definition of A and B. If $P \in K$, then by the definition of K, P_0 is not adjacent to P. Since $\mathfrak{G}$ is k-saturated, there exists a complete $(k-1)$-graph $\mathfrak{C}$ contained in $\mathfrak{G}$ and such that all the vertices of $\mathfrak{C}$ are adjacent to both P_0 and P. Now if we add P_0 to $\mathfrak{C}$, we obtain a complete k-graph contained in $\mathfrak{G}$ which contains P_0. Thus, by (7), C contains B and P is adjacent to all the vertices in B in this case too.

Comparing (6), (7), and (8), we obtain

$$(9) \qquad \phi(B) \cap B = \emptyset, \qquad |\phi(B)| < |B|.$$

On the other hand we prove that whenever $X \subseteq G$, $\phi(X) \cap X = \emptyset$, then

$$(10) \qquad |\phi(X)| \geqslant |X|.$$

This obviously contradicts (9) and proves our theorem.

To prove (10) we put $|X| = v$ and proceed by induction on v. If $v = 1$, (10) follows directly from the assumption that $\mathfrak{G}$ has no conical vertices. Assume that (10) is true for all sets Y with $|Y| < v + 1$. Let X be a set of $v + 1$ elements such that $X \cap \phi(X) = \emptyset$. Put

$$X = \{P_1, \ldots, P_v, P_{v+1}\}, \qquad X_0 = \{P_1, \ldots, P_v\}.$$

We are going to prove that the assumption $|\phi(X)| < v + 1$ leads to a contradiction. By our induction hypothesis $|\phi(X_0)| \geqslant v$. Hence

$$(11) \qquad |\phi(X_0)| = |\phi(X)| = v \quad \text{and} \quad |\phi(Y)| \geqslant |Y|, \phi(Y) \subseteq \phi(X_0)$$

for an arbitrary subset Y of X_0.

Using a well-known theorem of König, or more precisely a formulation of it given by Ore **(4)**, (11) implies that there exists an ordering

$$\phi(X_0) = \{Q_1, \ldots, Q_v\}$$

of $\phi(X_0)$ such that

$$(12) \qquad P_i \text{ is not adjacent to } Q_i \text{ for } i = 1, \ldots, v.$$

Since P_{v+1} is not a conical vertex, there is a vertex Q not adjacent to it. Q must be one of the vertices Q_i, since $\phi(X) = \phi(X_0)$ by (11). We may assume that P_{v+1} is not adjacent to Q_1.

Because $\mathfrak{G}$ is k-saturated, there exists a complete $(k-1)$-graph $\mathfrak{D} \subseteq \mathfrak{G}$ all of whose vertices are adjacent to both P_{v+1} and Q_1. By (12), D does not

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contain P_1 and Q_1 and for each i, D contains at most one of the vertices P_i, Q_i; $i = 2, \ldots, v$. Hence

$$|D \cap (X \cup \phi(X))| \leqslant v - 1.$$

Put $E = D - (X \cup \phi(X))$. Then

$$|E| \geqslant (k - 1) - (v - 1) = k - v \quad \text{and} \quad |E \cup X| \geqslant k + 1.$$

Any two distinct vertices of $E \cup X$ are adjacent. If both belong to E, this follows from $E \subseteq D$; if both belong to X, it is a consequence of $X \cap \phi(X) = \emptyset$; finally if one belongs to E and the other to X, it follows from $E \cap \phi(X) = \emptyset$. This contradicts the assumption that $\mathfrak{G}$ is k-saturated and thus does not contain a complete $(k + 1)$-graph.

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