A NON-EMBEDDABLE COMPOSITE OF EMBEDDABLE FUNCTIONS

A. RAN [1]

(Received 20 April 1966, revised 16 January 1967)

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Let Ω be the group of the functions $f(z)$ of the complex variable z, analytic in some neighborhood of $z = 0$, with $f(0) = 0$, $f'(0) = 1$, where the group operation is the composition $g[f(z)]$ $(g(z), f(z) \in \Omega)$. For every function $f(z) \in \Omega$ there exists [4] a unique formal power series

$$f(z, s) = \sum_{q=1}^{\infty} f_q(s) z^q, \tag{1}$$

where the coefficients $f_q(s)$ are polynomials of the complex parameter s, with $f_1(s) \equiv 1$, such that

$$f(z, 1) = f(z) \tag{2}$$

and, for any two complex numbers s and t, the formal law of composition

$$f[f(z, s), t] = f(z, s+t) \tag{3}$$

is valid.

From (2) and (3) follows, that, whenever the value of s is an integer, the power series (1) has a positive radius of convergence. As for other values of s, it was shown ([1], [3]), that only two cases are possible:

(A) The series (1) has a positive radius of convergence for all the complex values of s. The function $f(z, s)$ is then an analytic function of s. The function $f(z)$ is thus embeddable in a one-parameter continuous subgroup of Ω.

(B) The radius of convergence of the series (1) is zero for almost every complex s (and for almost every real s as well). The function $f(z)$ is then non-embeddable in a continuous subgroup of Ω.

Thus Ω appears as the union of two disjoint classes: the class A of all the embeddable functions of Ω, and the class B of the non-embeddable ones. Both classes are not empty; we have, for example:

[1] The paper is based on a part of the author's work towards the D.Sc. degree, under the guidance of Professor E. Jabotinsky at the Technion, Israel Institute of Technology, Haifa.

(4) $$H(z) = z\left(1+\frac{z}{2}\right)^{-1} \in A \text{ and also } G(z) = z\left(1-\frac{z^2}{4}\right)^{-\frac{1}{2}} \in A$$

(with $H(z, s) = z(1+s\,z/2)^{-1}$ and $G(z, s) = z(1-s\,z^2/4)^{-\frac{1}{2}}$).

On the other hand it was shown [2], [8] that all the meromorphic functions of Ω, except for the Möbius linear functions, belong to B. It is the purpose of the present paper to show that the function:

(5) $$F(z) = \frac{z}{\sqrt{1+z}}$$

(which does not fall in the above category, nor does its inverse) belongs to the class B.

The interesting fact about this function is that it is the composition of two functions of the class A. Indeed, $G(z)$ and $H(z)$ being the functions defined in (4), we have $F(z) = G[H(z)]$, which proves that A is not a subgroup of Ω.

The method used is Lewin's geometrical solution of the functional equation for the infinitesimal transformation ([6], [7]).

Suppose that $f(z) \in A$. Define

(6) $$L(z) = \left(\frac{\partial f(z, s)}{\partial s}\right)_{s=0}.$$

It can be shown ([3], [5]) that $L(z)$ satisfies the functional equation

(7) $$L[f(z)] = f'(z)L(z).$$

By Lewin's method we shall show, that the functional equation (7) in the case of the function $F(z) = z(1+z)^{-\frac{1}{2}}$ has no other solution, regular at $z = 0$, than $L(z) \equiv 0$. As, in (6), $L(z) \equiv 0$ only for $f(z) = z$, this proves that the function $F(z) = z(1+z)^{-\frac{1}{2}}$ is non-embeddable.

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We shall, first, note the following properties of the function $F(z) = z(1+z)^{-\frac{1}{2}}$.

(a) $F(z)$ is analytic and single-valued in the complex plane cut along the ray $-\infty < z \leqq -1$. (We consider the branch that maps the positive ray $0 < z < +\infty$ on itself, so that $-(\pi/2) < \arg\sqrt{1+z} < (\pi/2)$.)

(b) $F(z)$ maps the real segment $-1 < z < 0$ on the negative ray $-\infty < z < 0$.

(c) $F(z)$ maps the circle $|z+1| = 1$ on the imaginary segment $\{z = iy \mid |y| \leqq 2\}$.

(d) If $|z_0+1| < 1$ and $\operatorname{Im}\{z_0\} \neq 0$, then there is a point z_1, such that $|z_1+1| < 1$, $\operatorname{Im}\{z_1\} \neq 0$ and $F(z_1) = z_0$.

Proof. We proceed to construct the point z_1. Consider the equation

(8) $$\omega^2 - z_0\omega - 1 = 0.$$

Let ω_1 and ω_2 be its roots and choose $-\pi \leqq \arg \omega_1 \leqq \pi$, and $|\omega_1| \leqq |\omega_2|$. Since $\omega_1\omega_2 = -1$ we have $|\omega_1| \leqq 1 \leqq |\omega_2|$; and we may put $\omega_1 = |\omega_1|e^{i\phi}$ and $\omega_2 = |\omega_2|e^{i(\pi-\phi)}$. As from $|z_0+1| < 1$ follows $\mathrm{Re}\{z_0\} < 0$, we get from (8)

$$\mathrm{Re}\{\omega_1+\omega_2\} = |\omega_1| \cos \phi + |\omega_2| \cos (\pi - \phi) < 0$$

or

$$(|\omega_1| - |\omega_2|) \cos \phi < 0.$$

Hence $|\omega_1| \neq |\omega_2|$, so that $0 < |\omega_1| < 1 < |\omega_2|$. We have, also $\cos \phi > 0$ and $-\pi/2 < \phi < \pi/2$. Furthermore

$$\mathrm{Im}\{\omega_1+\omega_2\} = (|\omega_1|+|\omega_2|) \sin \phi = \mathrm{Im}\, z_0 \neq 0,$$

so that $\phi \neq 0$.

Now put $z_1 = \omega_1^2 - 1$. We have

$$\mathrm{Im}\{z_1\} = |\omega_1|^2 \sin 2\phi \neq 0 \quad \text{and} \quad |1+z_1| = |\omega_1|^2 < 1.$$

Moreover, by (8)

$$z_0 = \frac{\omega_1^2 - 1}{\omega_1} = \frac{z_1}{\sqrt{1+z_1}}.$$

Noting that $-\pi/2 < \arg \omega_1 = \arg \sqrt{1+z_1} < \pi/2$ we see that $z_0 = F(z_1)$ and z_1 is the required point.

(e) In the mapping $z \to F(z)$ we have $|z| > |F(z)|$, $|z| = |F(z)|$ or $|z| < |F(z)|$ according to whether z is outside, on, or inside the circle $|z+1| = 1$.

(f) $F(z)$ maps the right half plane $\mathrm{Re}\{z\} \geqq 0$ into itself.

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Equation (7) for $F(z) = z(1+z)^{-\frac{1}{2}}$ becomes

(9) $$L[z(1+z)^{-\frac{1}{2}}] = \tfrac{1}{2}(2+z)(1+z)^{-\frac{3}{2}}L(z).$$

We take the function $z(1+z)^{-\frac{1}{2}}$ defined on the plane cut along $-\infty < z \leqq -1$.

Let $L(z)$ be a solution of (9), regular at $z = 0$. First we shall show that $L(z)$ is an entire function.

We note that if z is not on the ray $-\infty < z \leqq -1$ then z is a regular point of the functions $\frac{1}{2}(2+z)(1+z)^{-\frac{3}{2}}$ and $z(1+z)^{-\frac{1}{2}}$ and hence it follows from (9) that the points z and $z(1+z)^{-\frac{1}{2}}$ are either both regular points of the function $L(z)$ or are both singular points of the function $L(z)$. Suppose now that $L(z)$ in (9) is not an entire function. Then $L(z)$ has a singularity z_0, with minimal distance from the origin. We distinguish four cases:

I) Suppose that z_0 is on the negative axis. Then, from (b) we deduce the existence of a point z_1 such that $-1 < z_1 < 0$, $|z_0| > |z_1|$, $F(z_1) = z_0$ (thus z_1 is not on the cut and $F(z_1)$ is defined). $L(z)$ is regular for $z = z_1$ and hence by (9) $L(z)$ is regular for $z = z_0$, which is a contradiction.

II) Suppose that z_0 is outside the circle $|z+1| = 1$, but not on the negative axis. From (e) follows, that $z_1 = F(z_0)$ satisfies $|z_1| < |z_0|$ and is therefore a regular point of $L(z)$ and hence so is z_0, which is a contradiction.

III) Suppose that z_0 lies on the circle $|z+1| = 1$, but $z_0 \neq -2$. Then $z_1 = F(z_0)$, by (c) and (e) satisfies $|z_1| = |z_0|$, $\operatorname{Re}\{z_1\} = 0$; so that, by II, z_1 cannot be a singular point of $L(z)$, and z_0 is also a regular point of $L(z)$, which is a contradiction.

IV) Suppose that z_0 is inside the circle $|z+1| = 1$ not on the negative axis. By (d) we can find a point z_1 inside the circle such that $F(z_1) = z_0$. From (e) follows that $|z_1| < |z_0|$, and z_1 is, therefore, a regular point of $L(z)$, and so is z_0, again a contradiction.

Therefore $L(z)$ is an entire function. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence with $\operatorname{Im}\{z_n\} > 0$ for all n, and $\lim_{n\to\infty} z_n = -2$. Put $w_n = F(z_n)$. We have then $\lim_{n\to\infty} w_n = 2i$ and $\lim_{n\to\infty} F'(z_n) = 0$. By (7), as $L(z)$ is continuous, we have

$$L(2i) = L(\lim_{n\to\infty} w_n) = \lim_{n\to\infty} L(w_n) = \lim_{n\to\infty} F'(z_n)L(z_n) = 0. \tag{10}$$

Consider the sequence $\{u_n\}_{n=1}^{\infty}$ given by $u_1 = 2i$, $u_{n+1} = F(u_n)$. By (10) we have $L(u_1) = 0$ and from (7) follows that $L(u_n) = 0$ implies $L(u_{n+1}) = 0$; hence $L(u_n) = 0$ for all n. By (*f*) we have $\operatorname{Re}\{u_n\} \geqq 0$ for all n; therefore all the points $\{u_n\}$ are outside the circle $|1+z| = 1$, and from (e) follows $|u_{n+1}| < |u_n|$ for all n. The points u_n form thus an infinite and bounded set, which has at least one point of accumulation. The zeros of the entire function $L(z)$ have thus a point of accumulation in the complex plane, hence $L(z) \equiv 0$.

We have thus shown that $L(z) \equiv 0$ is the only solution of (9) regular at the origin, and therefore $F(z) \in B$.

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We may note that from $f(z) \in B$ follows $g^{-1}\{f[g(z)]\} \in B$ for every $g \in \Omega$; taking $f(z) = G[H(z)]$, $g(z) = G(z)$ we see that $G[H(z)] \in B$ implies $H[G(z)] \in B$. If $G(z)$, $H(z)$ are defined as in (4), we get that

$$H[G(z)] = \frac{2z}{z+\sqrt{(4-z^2)}} \in B$$

as well.

https://doi.org/10.1017/S144678870000464X Published online by Cambridge University Press

Acknowledgements

The author is indebted to Professor E. Jabotinsky for his guidance which led to this paper, and to Dr. M. Lewin who checked it.

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Faculty of Mathematics, Technion
Israel Institute of Technology, Haifa