ENUMERATION OF INDICES OF GIVEN ALTITUDE AND POTENCY

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INDICES of the free logarithmetic $\mathfrak{L}$ correspond to bifurcating root-trees (cf. (**4**)), to Evans' non-associative numbers (**3**) and to Etherington's partitive numbers (**2**). The free commutative logarithmetic $\mathfrak{L}_c$ is the homomorph of $\mathfrak{L}$ determined by the congruence relation $P+Q \sim Q+P$. Formulæ for a_δ and p_a, i.e. the numbers of indices of $\mathfrak{L}$ of a given potency* δ and the number of indices of a given altitude a respectively, were given by Etherington (**1**), who also gave corresponding formulæ for commutative indices of $\mathfrak{L}_c$. Other enumeration formulæ are contained in (**5**).

The problem of enumeration of indices of $\mathfrak{L}$ of given potency δ $(\delta > 1)$ and given altitude a $(a+1 \leqslant \delta \leqslant 2^a$, cf. (**1**), p. 157) is essentially one of finding the number of partitions of a sequence of δ objects according to the following rules (cf. (**2**)):

(1) At the first stage the sequence of δ objects is partitioned so that the first κ objects are in the left subsequence and the remaining $\delta - \kappa$ objects in the right subsequence.

(2) At stage ν all subsequences which do not consist of single elements are again partitioned into a left subsequence and a right subsequence.

(3) There are a stages. After stage a all subsequences consist of single elements.

The corresponding problem for indices of $\mathfrak{L}_c$ is equivalent to the enumeration of partitions of an unordered set of δ identical objects according to similar rules.

As there is an index of potency 1 and altitude 0 we may say that a set of a single element can be partitioned at stage 0.

Let $p(a, \delta)$ denote the number of indices of altitude a and potency δ. Obviously $p(0, 1) = 1$. If $a \geqslant 1$, any index X of altitude a and potency δ is the sum of its left sub-index X' and its right sub-index X'', i.e. $X = X' + X''$. We can obtain all required indices by :

(1) Letting sub-index X' run through all indices of altitude $a-1$ and X'' through all indices of altitude less than $a-1$ and potency $\delta - \delta_{X'}$ (where $\delta_{X'}$ denotes the potency of X'). There are

$$\sum_{d=a}^{\delta-1} \{p(a-1, d) \sum_{\alpha=0}^{a-2} p(\alpha, \delta-d)\}$$

such indices ;

(2) as in (1) but interchanging the roles of X' and X'' ; and

(3) if $\delta - a \geqslant a$, letting X' run through all indices of altitude $a-1$ and

* *Potency*, representing the number of free knots in a tree, was called *degree* by Etherington and *length* by Evans.

potency d ($d=a, a+1, \ldots, \delta-a$), and X'' through all indices of altitude $a-1$ and potency $\delta-d$. There are

$$\sum_{d=a}^{\delta-a} p(a-1, d)p(a-1, \delta-d)$$

of these.
Hence

$$p(a, \delta)=\sum_{d=a}^{\delta-1}\left\{p(a-1, d)\left(\sum_{a=0}^{a-2} 2p(a, \delta-d)+p(a-1, \delta-d)\right)\right\}$$

where $p(x, y)=0$ whenever $x+1>y$ or $y>2^x$.

Denote the number of commutative indices of $\mathfrak{L}_c$ of altitude a and potency δ by $q(a, \delta)$. Then $q(0, 1)=1$. If $a\geqslant 1$ and $X=X'+X''$ is an index of altitude a and potency δ, we obtain all such non-congruent indices by :

(1) letting X' run through all indices of $\mathfrak{L}_c$ of altitude $a-1$ and X'' through all indices of altitude less than $a-1$ and of potency $\delta-\delta_{X'}$. There are

$$\sum_{d=a}^{\delta-1}\left\{q(a-1, d)\sum_{a=0}^{a-2} q(a, \delta-d)\right\}$$

such indices ; and

(2) (*a*) if δ is odd and $\frac{1}{2}(\delta-1)\geqslant a$, letting X' run through all indices of $\mathfrak{L}_c$ of altitude $a-1$ and potency d ($d=a, a+1, \ldots, \frac{1}{2}(\delta-1)$) and X'' through all indices of altitude $a-1$ and potency $\delta-d$. There are

$$\sum_{d=a}^{\frac{1}{2}(\delta-1)} q(a-1, d)q(a-1, \delta-d)$$

of these.

(*b*) if δ is even and $\frac{1}{2}\delta-1\geqslant a$

(i) letting X' run through all indices of $\mathfrak{L}_c$ of altitude $a-1$ and potency d ($d=a, a+1, \ldots, \frac{1}{2}\delta-1$) and X'' through all indices of altitude $a-1$ and potency $\delta-d$. There are

$$\sum_{d=a}^{\frac{1}{2}\delta-1} q(a-1, d)q(a-1, \delta-d)$$

of these ; and

(ii) letting both X' and X'' run through all indices of $\mathfrak{L}_c$ of altitude $a-1$ and potency $\frac{1}{2}\delta$ but taking only one index from each thus obtained pair of congruent indices except when $X'\sim X''$. There are

$$\tfrac{1}{2}q(a-1, \tfrac{1}{2}\delta)\{q(a-1, \tfrac{1}{2}\delta)+1\}$$

of these.
Thus

$$q(a, \delta)=\sum_{d=a}^{\delta-1}\left\{q(a-1, d)\sum_{a=0}^{a-2} q(a, \delta-d)\right\}+Q(a, \delta)$$

where

$$Q(a, \delta)=\begin{cases}\displaystyle\sum_{d=a}^{\frac{1}{2}(\delta-1)} q(a-1, d)q(a-1, \delta-d), \text{ if } \delta \text{ is odd,}\\ \displaystyle\sum_{d=a}^{\frac{1}{2}\delta-1} q(a-1, d)q(a-1, \delta-d)\\ \qquad +\tfrac{1}{2}q(a-1, \tfrac{1}{2}\delta)\{q(a-1, \tfrac{1}{2}\delta)+1\}, \text{ if } \delta \text{ is even.}\end{cases}$$

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We calculate

a	0	1	2	2	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4
δ	1	2	3	4	4	5	6	7	8	5	6	7	8	9	10	11	12	13	14	15	16
$p(a, \delta)$	1	1	2	1	4	6	6	4	1	8	20	40	68	94	114	116	94	60	28	8	1
$q(a, \delta)$	1	1	1	1	1	2	2	1	1	1	3	5	7	8	9	7	7	4	3	1	1

REFERENCES

(1) I. M. H. Etherington, On non-associative combinations, *Proc. Roy. Soc. Edin.*, **59** (1939), 153-162.

(2) I. M. H. Etherington, Non-associative arithmetics, *Proc. Roy. Soc. Edin.*, A, **62** (1949), 442-453.

(3) Trevor Evans, Non-associative number theory, *Amer. Math. Monthly*, **64** (1957), 299-309.

(4) H. Minc, Index polynomials and bifurcating root-trees, *Proc. Roy Soc. Edin.*, A, **64** (1957), 319-341.

(5) H. Minc, The free commutative entropic logarithmetic, *Proc. Roy. Soc. Edin.*, A, **65** (1959), 177-192.

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