LIFTING INDUCTIVE AND PROJECTIVE LIMITS

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In this paper, which is the fourth in a series of articles **(11, 12, 13)** on universal solutions in categories, a relationship between inductive limits and final structures (or projective limits and initial structures) is studied. The problems to be encountered are illustrated by the following example.

An example. For the definition of universes and the construction of concrete categories see **(11**, def. 1, p. 166 and ex. 1, p. 172**)**; for the facts concerning products and sums of groups see **(4**, th. 12, p. 16; th. 13, p. 17; th. 17, p. 39**)** and **(9**, p. 15**)**.

Let M be a non-empty universe, $\mathscr{F}$ the category of the functions of type M, $\mathscr{H}_g$ the category of the homomorphisms between groups of type M, and $\mathscr{H}_{gc}$ the category of the homomorphisms between commutative groups of type M. Furthermore, denote by F (by F') the forgetful homomorphism from $\mathscr{H}_g$ (from $\mathscr{H}_{gc}$) into $\mathscr{F}$, and by J the canonical injection from $\mathscr{H}_{gc}$ into $\mathscr{H}_g$. Clearly $F \circ J = F'$.

Let (y_ι) be a family of units of $\mathscr{H}_g$ or $\mathscr{H}_{gc}$ indexed by an element I of M. In either case the family (y_ι) admits a product in $\mathscr{H}_g$ (in $\mathscr{H}_{gc}$) which arises from the initial structure (in the classical sense of **(3**, p. 27**)** on the product in $\mathscr{F}$ of the family $(F(y_\iota))$ (the family $(F'(y_\iota))$) of the subjacent sets. If one starts out with *a family* (y_ι) *of units of* $\mathscr{H}_{gc}$, then the products of (y_ι) in $\mathscr{H}_{gc}$ and of $(J(y_\iota))$ in $\mathscr{H}_g$ coincide, since the product of $(J(y_\iota))$ in $\mathscr{H}_g$ is already commutative.

Let (x_ι) be a family of units of $\mathscr{H}_g$ or $\mathscr{H}_{gc}$ indexed by an element I of M. In either case, the family (x_ι) admits a sum in $\mathscr{H}_g$ (in $\mathscr{H}_{gc}$) *even if there does not exist* a final structure (in the classical sense of **(3**, p. 34**)**) on the sum in $\mathscr{F}$ of the family $(F(x_\iota))$ (the family $(F'(x_\iota))$) of the subjacent sets. However, a generalization of the concept of initial and final structure suggested by J. W. Duskin (McMaster University, Hamilton, Ontario) and, in a special case, implicitly used by J. R. Isbell, under the name "identification mapping" **(8**, p. 567**)**, enables one to deduce, in the above situation, the existence of a final structure (in the sense of Definition 5a) on the sum in $\mathscr{F}$ of the family $(F(x_\iota))$ (of the family $(F'(x_\iota))$) (Theorem 2a). If one starts out with *a family* (x_ι) *of units of* $\mathscr{H}_{gc}$, then the sums of (x_ι) in $\mathscr{H}_{gc}$ and of $(J(x_\iota))$ in $\mathscr{H}_g$ disagree (provided Card $(I) \geqslant 2$); however, Theorem 2a can be applied to the homomorphism J and yields the existence of a final structure (in the sense of Definition 5a) on the sum in $\mathscr{H}_g$ of the family $(J(x_\iota))$.

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1. Inductive and projective limits. In this section let Φ, $\mathscr{A}$, $\mathscr{B}$ be categories, and F a homomorphism from $\mathscr{A}$ into $\mathscr{B}$. We denote by $\mathscr{H}(\Phi, \mathscr{A})$ the discrete category of the homomorphisms from Φ into $\mathscr{A}$, and by $\mathscr{N}(\Phi, \mathscr{A})$ the category of the natural morphisms between homomorphisms from Φ into $\mathscr{A}$. Recall that each element of $\mathscr{N}(\Phi, \mathscr{A})$ is of the form (u, x, y) where u is a family of elements of $\mathscr{A}$ indexed by Φ_0, and where x, y are homomorphisms from Φ into $\mathscr{A}$ subject to the conditions $x_\phi\, u_{\beta(\phi)} = u_{\alpha(\phi)}\, y_\phi$ for all $\phi \in \Phi$. Instead of $(u, x, y) \in \mathscr{N}(\Phi, \mathscr{A})$ one frequently says that u is a natural morphism from x into y. Note that we write $\alpha(\phi)$ (we write $\beta(\phi)$) for the unique unit e of Φ such that $e\phi$ (such that ϕe) is defined; cf. (**13**, p. 15).

The mapping $x \to (x|\Phi_0, x, x)$ from $\mathscr{H}(\Phi, \mathscr{A})$ into $\mathscr{N}(\Phi, \mathscr{A})$ is injective and homomorphic. Hence it induces an isomorphism from $\mathscr{H}(\Phi, \mathscr{A})$ into the image category $\mathscr{N}(\Phi, \mathscr{A})_0$ which permits one to identify the homomorphisms from Φ into $\mathscr{A}$ with the units of $\mathscr{N}(\Phi, \mathscr{A})$.

Since each element of $\mathscr{N}(\Phi, \mathscr{A})$ is a triple (u, x, y) of families (or functions), it makes sense to require u, x, y to be constant. Thus one arrives at the subset $\mathscr{N}_c(\Phi, \mathscr{A})$ of $\mathscr{N}(\Phi, \mathscr{A})$ consisting of the *constant natural morphisms*. From now on, *suppose that* Φ *is not empty*. The mapping

$$s \to (\Phi_0 \times \{s\}, (\Phi \times \{\alpha(s)\}, \Phi, \mathscr{A}), (\Phi \times \{\beta(s)\}, \Phi, \mathscr{A}))$$

from $\mathscr{A}$ into $\mathscr{N}(\Phi, \mathscr{A})$ is injective and homomorphic. Hence it induces an isomorphism from $\mathscr{A}$ into the image category $\mathscr{N}_c(\Phi, \mathscr{A})$ which permits one to identify the morphisms of $\mathscr{A}$ with the constant natural morphisms of $\mathscr{N}(\Phi, \mathscr{A})$.

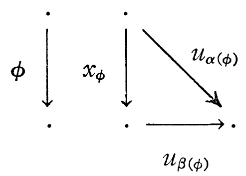
Because $\mathscr{N}_c(\Phi, \mathscr{A})$ is a subcategory of $\mathscr{N}(\Phi, \mathscr{A})$, the set ψ_r of the natural morphisms of $\mathscr{N}(\Phi, \mathscr{A})$ whose targets belong to $\mathscr{N}_c(\Phi, \mathscr{A})$ can be made an $(\mathscr{N}(\Phi, \mathscr{A}); \mathscr{N}_c(\Phi, \mathscr{A}))$-biset; cf. (**12**, def. 3, p. 203). Let x be a homomorphism from Φ into $\mathscr{A}$, considered as a unit of $\mathscr{N}(\Phi, \mathscr{A})$. The right $\mathscr{N}_c(\Phi, \mathscr{A})$-universal solutions u of x (cf. (**12**, p. 205)) generalize the classical concept of inductive limit.

Definition 1a. The conjunction of the relations

(i) u is a natural morphism from x into $a \in \mathscr{A}_0$;

(ii) whenever u' is a natural morphism from x into $a' \in \mathscr{A}_0$, then there exists one and only one $t \in \mathscr{A}$ such that $u_\iota\, t = u'_\iota$ for all $\iota \in \Phi_0$

is denoted by "*u is an inductive limit of x in $\mathscr{A}$.*"



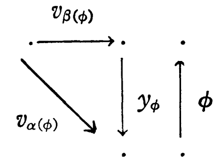
Because $\mathscr{N}_c(\Phi^0, \mathscr{A})$ is a subcategory of $\mathscr{N}(\Phi^0, \mathscr{A})$, the set ψ_l of the natural morphisms of $\mathscr{N}(\Phi^0, \mathscr{A})$ whose sources belong to $\mathscr{N}_c(\Phi^0, \mathscr{A})$ can be made an $(\mathcal{N}_c(\Phi^0, \mathcal{A}); \mathcal{N}(\Phi^0, \mathcal{A}))$-biset. Let y be a homomorphism from Φ^0 into $\mathcal{A}$, considered as a unit of $\mathcal{N}(\Phi^0, \mathcal{A})$. The left $\mathcal{N}_c(\Phi^0, \mathcal{A})$-universal solutions v of y generalize the classical concept of projective limit.

Definition 1b. The conjunction of the relations

(i) v is a natural morphism from $a \in \mathcal{A}_0$ into y;

(ii) whenever v' is a natural morphism from $a' \in \mathcal{A}_0$ into y, then there exists one and only one $s \in \mathcal{A}_0$ such that $s \cdot v_\kappa = v'_\kappa$ for all $\kappa \in \Phi_0$;

is denoted by "*v is a projective limit of y in $\mathcal{A}$.*"



Remarks. 1. As is the case with all universal solutions, inductive and projective limits of homomorphisms with values in $\mathcal{A}$ are determined uniquely up to isomorphism of $\mathcal{A}$ (**12**, prop. 3, p. 204).

2. The relations "u is an inductive limit of x in $\mathcal{A}$" and "u is a projective limit of x in $\mathcal{A}$" are in duality. More precisely, u is an inductive limit of x: $\Phi \to \mathcal{A}$ in $\mathcal{A}$ if and only if u is a projective limit of x: $\Phi^0 \to \mathcal{A}^0$ in $\mathcal{A}^0$.

3. Because in the case $\Phi \neq \emptyset$, $\mathcal{N}_c(\Phi, \mathcal{A})$ can be identified with $\mathcal{A}$, one can make ψ_r into an $(\mathcal{N}(\Phi, \mathcal{A}); \mathcal{A})$-biset. Furthermore, one can replace a natural morphism (u, x, y) belonging to ψ_r by a triple (u, x, a) where a is the common value of the constant family $(\beta(u_\iota))_{\iota \in \Phi_0}$. In this way, Definition 1a carries over to the case $\Phi = \emptyset$. The existence of an inductive limit of the empty family is equivalent to the existence of an original unit of $\mathcal{A}$. Dually, the existence of a projective limit of the empty family is equivalent to the existence of a terminal unit of $\mathcal{A}$. (For the definition of original (initial) and terminal units see (**10**, p. 2).)

Example. Take for Φ a subcategory of the category $I \times I$ of the pairs (i, j) endowed with the internal composition law $((i, j), (k, l)) \to (i, l)$ whenever $j = k$, containing the diagonal Δ_I of $I \times I$. In other words, let Φ be the graph of a pre-order relation in I. In this case, a homomorphism from Φ into a category $\mathcal{A}$ is an inductive system of elements of $\mathcal{A}$ in the sense of (**2**, p. 89); if Φ equals Δ_I, one obtains sums. Dually, a homomorphism from Φ^0 into $\mathcal{A}$ is a projective system of elements of $\mathcal{A}$ in the sense of (**2**, p. 76); if Φ equals Δ_I, one obtains products.

2. Structure schemes and species. Let $\mathcal{A}$, $\mathcal{B}$ be categories, F a homomorphism from $\mathcal{A}$ into $\mathcal{B}$, and a, b units of $\mathcal{A}$. Since $u \in \mathcal{A}(a, b)$ (since $u \in \mathcal{A}(a, \cdot)$, $u \in \mathcal{A}_0$, $u \in \mathcal{A}^*$) implies $F(u) \in \mathcal{B}(F(a), F(b))$ (implies $F(u) \in \mathcal{B}(F(a), \cdot)$, $F(u) \in \mathcal{B}_0$, $F(u) \in \mathcal{B}^*$), F induces, by passing to subsets, a mapping from $\mathscr{A}(a, b)$ into $\mathscr{B}(F(a), F(b))$ (from $\mathscr{A}(a, \cdot)$ into $\mathscr{B}(F(a), \cdot)$, from $\mathscr{A}_0$ into $\mathscr{B}_0$, from $\mathscr{A}^*$ into $\mathscr{B}^*$) which we shall denote by F_{ab} (by $F_{a.}$, F_0, F^*).

Definition 2. The relation "for all $a \in \mathscr{A}_0$, for all $b \in \mathscr{A}_0$, F_{ab} is injective (is surjective, bijective)" is denoted by "*F* is *almost injective* (*almost surjective, bijective*)." The relation "for all $a \in \mathscr{A}_0$, $F_{a.}$ is injective (is surjective, bijective)" is denoted by "*F is locally injective* (*locally surjective, bijective*)."

Example. Let F be a homomorphism from $\mathscr{A}$ into $\mathscr{B}$, and G a homomorphism from $\mathscr{B}$ into $\mathscr{A}$. If $G \circ F$ and $F \circ G$ are naturally isomorphic to the respective identity mappings, then F is almost bijective (**1**, th., p. 1–11).

Remarks. 1. In case $\mathscr{A}$ is a subcategory of $\mathscr{B}$, and F the canonical injection from $\mathscr{A}$ into $\mathscr{B}$, the relations "F is almost surjective" and "$\mathscr{A}$ is full" are equivalent. Similarly the relations "F^* is locally surjective" and "$\mathscr{A}$ is saturated" are equivalent (**5**, def. 7, p. 5).

2. A faithful (a fully faithful) functor is a homomorphism F which is almost injective (almost bijective). A transportable functor is a homomorphism F such that F^* is locally bijective (**7**, p. I.4).

3. An injective homomorphism is almost and locally injective. Not every surjective homomorphism is almost or locally surjective.

4. Every locally injective homomorphism is almost injective. There exist homomorphisms which are locally, but not almost surjective, and there exist homomorphisms which are almost, but not locally surjective.

PROPOSITION 1. *Let F be a homomorphism from the category $\mathscr{A}$ into the category $\mathscr{B}$, F' a homomorphism from $\mathscr{B}$ into the category $\mathscr{C}$, and F'' a homomorphism from $\mathscr{A}$ into $\mathscr{C}$ such that $F'' = F' \circ F$. Then:*

(a) *if F and F' are almost injective, F'' is almost injective;*

(b) *if F and F' are almost surjective, F'' is almost surjective;*

(c) *if F'' is almost injective, F is almost injective;*

(d) *if F'' is almost surjective, and F_0 surjective, F' is almost surjective;*

(e) *if F'' is almost surjective and F' almost injective, F is almost surjective;*

(*f*) *if F'' is almost injective, F almost surjective, and F_0 surjective, F' is almost injective.*

The proposition remains true if "almost" is everywhere replaced by "locally."

The truth of the above statements follows immediately from the definitions.

For many purposes it is desirable to treat $(F, \mathscr{A}, \mathscr{B})$ as a new entity. In this sense we propose:

Definition 3. The conjunction of the relations (i) to (iii) (of (i) to (iv), (i) to (v)) below:

(i) $\mathscr{A}$, $\mathscr{B}$ are categories;
(ii) F is a homomorphism from $\mathscr{A}$ into $\mathscr{B}$;
(iii) F is almost injective;
(iv) F^* is locally injective;
(v) F^* is locally surjective;

is denoted by "*$F:\mathscr{A} \to \mathscr{B}$ is a structure pre-scheme (a structure scheme, structure species)*," and the relation $(\exists u)\ (u \in \mathscr{A}(a, b) \wedge F(u) = v)$ by "*v is an F-morphism from a into b.*"

Remark. Denote, for the moment, by $\mathscr{A}^{\cdot}$, $\mathscr{B}^{\cdot}$ the subjacent sets of the categories $\mathscr{A}$, $\mathscr{B}$ respectively. The graphs of the composition laws on $\mathscr{A}^{\cdot}$ and $\mathscr{B}^{\cdot}$ together with the graph of F form a structure on $(\mathscr{A}^{\cdot}, \mathscr{B}^{\cdot})$ in the sense of (**3**, p. 12) subject to the transportable axioms (i) to (iii) (to (iv), (v)). Moreover, one can introduce morphisms between structure pre-schemes (structure schemes, species) in the obvious fashion.

Example. The forgetful homomorphism $(G, U, X, U', X') \to (G, X, X')$ from the category ${}_M\mathscr{M}_{\Sigma,\sigma}$ of the σ-morphisms between Σ-structures of type M in the sense of (**3**, pp. 12, 23) into the category ${}_M\mathscr{F}$ of the functions of type M (where M is a universe) defines a structure species.

3. Supplements on $\mathscr{A}$-sets and coverings of $\mathscr{A}$. Let $\mathscr{A}$ be a category. As was shown in (**14**) and remarked in (**13**, p. 15), right $\mathscr{A}$-sets Φ and coverings of $\mathscr{A}$ by a category $\mathscr{H}$ represent equivalent structures in the sense of (**3**, p. 20). For the definition of *right $\mathscr{A}$-sets* the reader may consult (**12**, def. 2, p. 202). A morphism from the right $\mathscr{A}$-set Φ into the right $\mathscr{A}$-set Φ' is by definition a mapping f from Φ into Φ' such that $f(\phi \cdot u) = f(\phi) \cdot u$ whenever (u, ϕ) is a composable element of $\mathscr{A} \times \Phi$. By a *covering of $\mathscr{A}$ by a category $\mathscr{H}$* we mean a homomorphism h from $\mathscr{H}$ into $\mathscr{A}$ which is locally bijective (see def. 2). A morphism from the covering $h:\mathscr{H} \to \mathscr{A}$ into the covering $h':\mathscr{H}' \to \mathscr{A}$ is by definition a homomorphism g from $\mathscr{H}$ into $\mathscr{H}'$ such that $h' \circ g = h$.

Starting from a right $\mathscr{A}$-set Φ, we denote by $\mathscr{H}$ the graph of the external composition law between elements of $\mathscr{A}$ and elements of Φ. The internal composition law $((u, \phi, \phi'), (u', \bar{\phi}, \phi'')) \to (uu', \phi, \phi'')$ whenever $\phi' = \bar{\phi}$ converts $\mathscr{H}$ into a category. The mapping $(u, \phi, \phi') \to u$ from $\mathscr{H}$ into $\mathscr{A}$ is homomorphic and locally bijective, as the reader easily verifies. The mapping $\phi \to (q(\phi), \phi, \phi)$ from Φ into $\mathscr{H}$ is injective; by passing to subsets, it induces a bijection from Φ into the image $\mathscr{H}_0$ which permits one to identify quasi-morphisms of Φ with units of $\mathscr{H}$. In conclusion, we have constructed a covering $h:\mathscr{H} \to \mathscr{A}$.

Conversely, let a covering $h:\mathscr{H} \to \mathscr{A}$ be given. Denote $\mathscr{H}_0$ by Φ. Let $(u, \phi) \in \mathscr{A} \times \Phi$ be such that $h(\phi) = \alpha(u)$. By hypothesis there exists one and only one $\xi \in \mathscr{H}$ such that $\alpha(\xi) = \phi$ and $h(\xi) = u$. Denote $\beta(\xi)$ by $\phi \cdot u$.

The reader verifies easily that $(u, \phi) \to \phi \cdot u$ is an external composition law between elements of $\mathscr{A}$ and elements of Φ which makes Φ into a right $\mathscr{A}$-set. In conclusion, we have constructed a right $\mathscr{A}$-set Φ. (For details, see (**14**) or (**5**, th. 1, p. 22).)

Let Φ, Φ' be right $\mathscr{A}$-sets, $h: \mathscr{H} \to \mathscr{A}$, $h': \mathscr{H}' \to \mathscr{A}$ their associated coverings. We wish to show that to every morphism f from Φ into Φ', there corresponds a morphism g from $\mathscr{H}$ into $\mathscr{H}'$, and vice versa. If f is given, let g be the mapping $(u, \phi, \psi) \to (u, f(\phi), f(\psi))$ from $\mathscr{H}$ into $\mathscr{H}'$. Clearly $h' \circ g = h$. If g is known, let f be the mapping from $\mathscr{H}_0$ into $\mathscr{H}'_0$ deduced from g by passing to subsets. Obviously, f preserves scalar products.

Lemma 1. *Let f be a morphism from the right $\mathscr{A}$-set Φ into the right $\mathscr{A}$-set Φ', and let g be the corresponding morphism from the covering $\mathscr{H}$ of $\mathscr{A}$ into the covering $\mathscr{H}'$ of $\mathscr{A}$ where $\mathscr{H}$, $\mathscr{H}'$ are the graphs of the external composition laws on Φ, Φ' respectively. Denote the canonical injection from Φ (from Φ') into $\mathscr{H}$ (into $\mathscr{H}'$) by j (by j'). Then $j' \circ f = g \circ j$; further f is an isomorphism if and only if g is an isomorphism.*

Note first that the (true) relation $f(\phi \cdot q(\phi)) = f(\phi)\, q(\phi)$ implies

$$q(f(\phi)) = q(\phi).$$

It follows that

$$j'(f(\phi)) = (q(f(\phi)), f(\phi), f(\phi)) = (q(\phi), f(\phi), f(\phi)) = g(q(\phi), \phi, \phi) = g(j(\phi))$$

for all $\phi \in \Phi$. If g is bijective, it has a reciprocal, and the same is true of f. Conversely, if f is bijective, it has a reciprocal f^{-1} to which corresponds a mapping g' from $\mathscr{H}'$ into $\mathscr{H}$. Then

$$g'(g(u, \phi, \psi)) = g'(u, f(\phi), f(\psi)) = (u, \phi, \psi),$$
$$g(g'(u, \phi', \psi')) = g(u, f^{-1}(\phi'), f^{-1}(\psi')) = (u, \phi', \psi')$$

and g is bijective.

Recall that a unit ω of a category $\mathscr{H}$ is said to be *original* (to be *terminal*) if, for all $\omega' \in \mathscr{H}_0$, the set $\mathscr{H}(\omega, \omega')$ (the set $\mathscr{H}(\omega', \omega)$) is reduced to one element (**10**, p. 2).

Lemma 2. *Let g be an isomorphism from the category $\mathscr{H}$ into the category $\mathscr{H}'$, and let $\omega \in \mathscr{H}_0$. In order that ω be original (be terminal) in $\mathscr{H}$ it is necessary and sufficient that $g(\omega)$ be original (be terminal) in $\mathscr{H}'$.*

Note that, for all $\omega \in \mathscr{H}_0$, $\omega' \in \mathscr{H}_0$, g induces a bijection from $\mathscr{H}(\omega, \omega')$ into $\mathscr{H}'(g(\omega), g(\omega'))$.

For the remainder of this section, let Φ be an $(\mathscr{A}; \mathscr{B})$-biset, Φ' an $(\mathscr{A}'; \mathscr{B})$-biset, $a \in \mathscr{A}_0$, $a' \in \mathscr{A}'_0$. Then $\Phi(a, \cdot)$ and $\Phi'(a', \cdot)$ are right $\mathscr{B}$-sets. Denote by $h: \mathscr{H} \to \mathscr{B}$, $h': \mathscr{H}' \to \mathscr{B}$ their respective coverings, and by j, j' the respective canonical injections from $\Phi(a, \cdot)$, $\Phi'(a', \cdot)$ into $\mathscr{H}$, $\mathscr{H}'$.

Lemma 3. *Let $\phi \in \Phi(a, \cdot)$. In order that ϕ be a right universal solution of a in Φ it is necessary and sufficient that $j(\phi)$ be an original unit in $\mathscr{H}$.*

Indeed, the relation

$$(\forall v)(\forall v')((v \in \mathscr{B} \wedge v' \in \mathscr{B} \wedge \phi v = \phi v' = \psi) \Rightarrow (v = v'))$$

signifies that $\mathscr{H}(j(\phi), j(\psi))$ has at most one element, and the relation $(\exists v)(v \in \mathscr{B} \wedge \phi v = \psi)$ signifies that $\mathscr{H}(j(\phi), j(\psi))$ has at least one element.

LEMMA 4. *Let f be an isomorphism from the right $\mathscr{B}$-set $\Phi(a, \cdot)$ into the right $\mathscr{B}$-set $\Phi'(a', \cdot)$, and let $\phi \in \Phi(a, \cdot)$. In order that ϕ be a right universal solution of a in Φ it is necessary and sufficient that $f(\phi)$ be a right universal solution of a' in Φ'.*

Note that f indices an isomorphism g from $\mathscr{H}$ into $\mathscr{H}'$ (Lemma 1), and apply Lemma 2 once, Lemma 3 twice.

4. Final and initial structures. In what follows let $F: \mathscr{A} \to \mathscr{B}$ be a structure pre-scheme. The elements (a, u) of $\mathscr{A}_0 \times \mathscr{B}$ such that $F(a) = \alpha(u)$ form a subset $\mathscr{L}$ of $\mathscr{A}_0 \times \mathscr{B}$. Each element (a, u) of $\mathscr{L}$ is said to be *a left structure on* $\beta(u)$. The category $\mathscr{A}$ operates on $\mathscr{L}$ from the left by means of the external composition law

$$(s, (a, u)) \to (\alpha(s), F(s)u) \qquad (\beta(s) = a);$$

the category $\mathscr{B}$ operates on $\mathscr{L}$ from the right by means of the external composition law

$$(v, (a, u)) \to (a, uv) \qquad (\beta(u) = \alpha(v)).$$

Thus $\mathscr{L}$ becomes an $(\mathscr{A}; \mathscr{B})$-biset (**12**, def. 3, p. 203). Write $s \perp (a, u)$ instead of $(\alpha(s), F(s)u)$ for convenience.

The elements (a, v) of $\mathscr{A}_0 \times \mathscr{B}$ such that $F(a) = \beta(v)$ form a subset $\mathscr{R}$ of $\mathscr{A}_0 \times \mathscr{B}$. Each element (a, v) of $\mathscr{R}$ is said to be *a right structure on* $\alpha(v)$. The category $\mathscr{A}$ operates on $\mathscr{R}$ from the right by means of the external composition

$$(t, (a, v)) \to (\beta(t), vF(t)) \qquad (\alpha(t) = a);$$

the category $\mathscr{B}$ operates on $\mathscr{R}$ from the left by means of the external composition law

$$(u, (a, v)) \to (a, uv) \qquad (\beta(u) = \alpha(v)).$$

Thus $\mathscr{R}$ becomes a $(\mathscr{B}; \mathscr{A})$-biset. Write $(a, v) \perp t$ instead of $(\beta(t), vF(f))$ for convenience.

Definition 4. Let $b \in \mathscr{B}_0$. We say "*the families* $(x_\iota, u_\iota)_{\iota \in I}$ *of left structures on* b *and* $(y_\kappa, v_\kappa)_{\kappa \in K}$ *of right structures on* b *are compatible*" instead of "for all $\iota \in I$, for all $\kappa \in K$, $u_\iota v_\kappa$ is an F-morphism (def. 3)."

The above definition can, in particular, be applied to the case where I or K is reduced to one element.

The right structures (y, v) on b which are compatible with a given family $(x_\iota, u_\iota)_{\iota\in I}$ of left structures on b form a stable subset $\mathscr{R}_1$ of the right $\mathscr{A}$-set $\mathscr{R}$. We are interested in the right universal solutions of b in $\mathscr{R}_1$. More precisely:

Definition 5a. The conjunction of the relations:
(i) (y, v) is a right structure on b which is compatible with the family (x_ι, u_ι);
(ii) whenever (y', v') is a right structure on b which is compatible with the family (x_ι, u_ι), then there exists one and only one morphism t of $\mathscr{A}$ such that $(y, v) \perp t$ is defined and equal to (y', v')

is denoted by "(y, v) *is a final structure on* b *with respect to the family* (x_ι, u_ι) *of left structures on* b."

The left structures (x, u) on b which are compatible with a given family $(y_\kappa, v_\kappa)_{\kappa\in K}$ of right structures on b form a stable subset $\mathscr{L}_1$ of the left $\mathscr{A}$-set $\mathscr{L}$. By considering left universal solutions of b in $\mathscr{L}_1$, one is led to:

Definition 5b. The conjunction of the relations:
(i) (x, u) is a left structure on b which is compatible with the family (y_κ, v_κ);
(ii) whenever (x', u') is a left structure on b which is compatible with the family (y_κ, v_κ), then there exists one and only one morphism s of $\mathscr{A}$ such that $s \perp (x, u)$ is defined and equal to (x', u')

is denoted by "(x, u) *is an initial structure on* b *with respect to the family* (y_κ, v_κ) *of right structures on* b."

As is the case with all universal solutions, final (initial) structures on b are determined uniquely up to isomorphisms of $\mathscr{A}$ operating from the right (from the left) (**12**, prop. 3, p. 204).

PROPOSITION 2. *Let* $F: \mathscr{A} \to \mathscr{B}$ *be a structure species. Furthermore, let* $b \in \mathscr{B}_0$ *and let* $(x_\iota, u_\iota)_{\iota\in I}$ *be a family of left structures on* b. *In order that* b *admit a final structure with respect to the family* (x_ι, u_ι) *of the form* (y, b) *it is necessary and sufficient that* b *admit a final structure* (y', v) *with respect to the family* (x_ι, u_ι) *where* $v \in \mathscr{B}^*$.

Since $\mathscr{B}_0 \subset \mathscr{B}^*$ the necessity of the condition is clear. Let (y', v) be a final structure on b with respect to the family (x_ι, u_ι) where $v \in \mathscr{B}^*$. By hypothesis, F^* is locally bijective. Hence there exists one (and only one) $\bar{v} \in \mathscr{A}^*$ such that $\beta(\bar{v}) = y'$ and $F(\bar{v}) = v$. Denote $\alpha(\bar{v})$ by y'. Because $(y, b) \perp \bar{v} = (y', v)$ and since $\bar{v} \in \mathscr{A}^*$, (y, b) is also a final structure on b with respect to the family (x_ι, u_ι) (**12**, prop. 3, p. 204).

Remarks. A simple analysis shows that the final structures of the form (y, b) are precisely the final structures in the sense of N. Bourbaki (**3**, p. 34) in categorical language. Consequently, b admits a final structure in the sense of N. Bourbaki if and only if it admits a final structure (y', v) in the sense of J. W. Duskin where $v \in \mathscr{B}^*$ (everything with respect to the family (x_ι, u_ι)). Similarly for initial structures. For another approach, compare (**6**).

PROPOSITION 3a. *Let (x, u) be a left structure on $b \in \mathscr{B}_0$, and let (y, v) be a final structure on b with respect to (x, u). If u is surjective, then the unique $\bar{u} \in \mathscr{A}(x, y)$ such that $F(\bar{u}) = uv$ is surjective.*

PROPOSITION 3b. *Let (y, v) be a right structure on $b \in \mathscr{B}_0$, and let (x, u) be an initial structure on b with respect to (y, v). If v is injective, then the unique $\bar{v} \in \mathscr{A}(x, y)$ such that $F(\bar{v}) = uv$ is injective.*

We give a proof of the surjective case. Let s, t be elements of $\mathscr{A}$ such that $\bar{u}s = \bar{u}t$. Then s, t are morphisms from y into some $y' \in \mathscr{A}_0$. Moreover, the equations

$$uvF(s) = F(\bar{u})F(s) = F(\bar{u}s) = F(\bar{u}t) = F(\bar{u})F(t) = uvF(t)$$

hold. Since u is surjective by hypothesis, $vF(s) = vF(t)$. The last equation reads $(y, v) \perp s = (y, v) \perp t$, and implies $s = t$, because (y, v) is a final structure.

In the situation of Proposition 3a, y is said to be *a quotient-structure of x on b*. In the situation of Proposition 3b, x is said to be *a substructure of y on b*. As mentioned earlier, the quotient-structures of x (the substructures of y) on b are determined uniquely up to isomorphisms of $\mathscr{A}$.

Example. Let M be a non-empty universe, $\mathscr{F}$ the category of the functions of type M, and $\mathscr{H}$ the category of the homomorphisms between categories of type M; cf. (**11**, def. 1, p. 166 and ex. 1, p. 172). Furthermore, denote by F the forgetful homomorphism from $\mathscr{H}$ into $\mathscr{F}$. In view of the example at the end of No. 2, F defines a structure species.

Let (x, u) be a left structure on $b \in \mathscr{F}_0$. Suppose u is a canonical surjection from $F(x)$ into a quotient-set $F(x)/R$ where R is compatible with the internal composition law of x (i.e.

$$R\{\xi, \xi'\} \Rightarrow R\{\alpha(\xi), \alpha(\xi')\} \wedge R\{\beta(\xi), \beta(\xi')\};$$
$$R\{\xi, \xi'\} \wedge R\{\eta, \eta'\} \wedge \xi\eta = \zeta \wedge \xi'\eta' = \zeta' \Rightarrow R\{\zeta, \zeta'\};$$

$\xi, \xi', \eta, \eta', \zeta, \zeta'$ elements of $F(x)$). J. R. Isbell has shown (**8**, prop. 1.6, p. 568) that b admits a quotient-structure in our sense even if it does not admit a quotient structure in the sense of N. Bourbaki.

5. Lifting theorems. In what follows, let $F: \mathscr{A} \to \mathscr{B}$ be a structure pre-scheme and Φ a category. For the time being, let $(x_\iota, u_\iota)_{\iota \in I}$ be a family of left structures on the unit b of $\mathscr{B}$. Furthermore, let (y, v) be a right structure on b which is compatible with the family (x_ι, u_ι). Since for each $\iota \in I$ the set of the $\mathscr{A}$-morphisms s from x_ι into y such that $F(s) = u_\iota v$ is not empty, one can find a family $\bar{u}$ of $\mathscr{A}$-morphisms indexed by I such that $\iota \in I$ implies $\bar{u}_\iota: x_\iota \to y$ and $F(\bar{u}_\iota) = u_\iota v$. Moreover, these conditions guarantee the uniqueness of each $\bar{u}_\iota$, and hence of the family $\bar{u}$; for F is almost injective. Now let x be a homomorphism from Φ into $\mathscr{A}$, and u a natural morphism from $F \circ x$ into b. We know that $(x_\iota, u_\iota)_{\iota \in \Phi_0}$ is a family of left structures on

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the common value b of the constant family $(\beta(u_\iota))_{\iota\in\Phi_0}$. Let $\bar{u}$ be chosen as above. We claim that $\bar{u}$ is a natural morphism from x into y. Indeed, for each $\phi \in \Phi$, $x_\phi\, \bar{u}_{\beta(\phi)}$ and $\bar{u}_{\alpha(\phi)}$ are morphisms from $x_{\alpha(\phi)}$ into y, and one has

$$F(x_\phi\, \bar{u}_{\beta(\phi)}) = F(x_\phi)F(\bar{u}_{\beta(\phi)}) = F(x_\phi)u_{\beta(\phi)}\, v = u_{\alpha(\phi)}\, v = F(\bar{u}_{\alpha(\phi)}).$$

We conclude that the equations $x_\phi\, \bar{u}_{\beta(\phi)} = \bar{u}_{\alpha(\phi)}$ hold for all $\phi \in \Phi$ since F is almost injective.

From the above considerations, one obtains the existence of a mapping L ("lifting") *from the set of the right structures on* b *which are compatible with the family* $(x_\iota, u_\iota)_{\iota\in\Phi_0}$ *into the set of the natural morphisms from* x *into some unit of* $\mathscr{A}$. L is characterized by the following property:

(*) *Let* (y, v) *be a right structure on* b *which is compatible with the family* (x_ι, u_ι); *let* $\bar{u}$ *be a natural morphism from* x *into some* $y' \in \mathscr{A}_0$. *In order that the pair* $((y, v), \bar{u})$ *belong to the graph of* L *it is necessary and sufficient that* $y = y'$ *and that* $F(\bar{u}_\iota) = u_\iota v$ *for all* $\iota \in \Phi_0$.

It will be convenient to write condition (ii) of Definition 1a as the conjunction of the following relations:

(a) (Uniqueness) If v, v' are morphisms of $\mathscr{B}$ such that, for all $\iota \in \Phi_0$, $u_\iota v = v_\iota v'$, then $v = v'$.

(b) (Existence) Let u' be a natural morphism from $F \circ x$ into some unit of $\mathscr{B}$. There exists a morphism v of $\mathscr{B}$ such that, for all $\iota \in \Phi_0$, $u_\iota v = u'_\iota$.

THEOREM 1. *Let* x *be a homomorphism from* Φ *into* $\mathscr{A}$, *and let* u *be a natural morphism from* $F \circ x$ *into* $b \in \mathscr{A}_0$. *Then*

(0) *L is a homomorphism qua mapping between right $\mathscr{A}$-sets* (No. 3).

(1) *If u verifies condition* (a), *then L is injective.*

(2) *If u verifies condition* (b), *then L is surjective.*

(3) *If u is an inductive limit of $F \circ x$, then L is bijective.*

Assume $(y, v) \perp t$ is defined; write $\bar{u}$ for $L(u)$ and y' for $\beta(t)$. Then $\beta(\bar{u}_\iota) = y = \alpha(t)$ for all $\iota \in \Phi_0$. Consequently $\bar{u}t$ is defined and equal to $(\bar{u}_\iota\, t)_{\iota\in\Phi_0}$ where t is considered as an element of $\mathscr{N}_c(\Phi, \mathscr{A})$. But $\bar{u}_\iota\, t$: $x_\iota \to y'$ and $F(\bar{u}_\iota\, t) = F(\bar{u}_\iota)F(t) = u_\iota vF(t)$. Since $(y, v) \perp t = (y', vF(t))$, one infers that $L((y, v) \perp t) = L(y, v)t$.

Assume u verifies condition (a), and let $L(y, v) = L(y', v') = \bar{u}$ where $\bar{u}$ is a natural morphism from x into some $a \in \mathscr{A}_0$. Then $y = a = y'$, and $u_\iota v = F(\bar{u}_\iota) = u_\iota v'$ for all $\iota \in \Phi_0$ by (*). The last statement implies that $v = v'$. Hence $(y, v) = (y', v')$, and L is injective.

Suppose that u verifies condition (b), and let $\bar{u}$ be a natural morphism from x into some $y \in \mathscr{A}_0$. As is well known (**1**, pp. 1–9), the family $(F(\bar{u}_\iota))_{\iota\in\Phi_0}$ is a natural morphism from $F \circ x$ into $F(y) \in \mathscr{B}_0$. By hypothesis, there exists $v \in \mathscr{B}$ such that, for all $\iota \in \Phi_0$, $u_\iota v = F(\bar{u}_\iota)$. This signifies $F(y, v) = \bar{u}$, due to (*).

The last statement follows by conjunction of cases.

THEOREM 2a. *Let x be a homomorphism from Φ into $\mathscr{A}$. Suppose u is an inductive limit of the composite homomorphism $F \circ x$ from Φ into $\mathscr{B}$; denote by b the common value of the constant family $(\beta(u_\iota))_{\iota\in\Phi_0}$. In order that x admit an inductive limit it is necessary and sufficient that there exist, on b, a final structure with respect to the family $(x_\iota, u_\iota)_{\iota\in\Phi_0}$ of left structures.*

THEOREM 2b. *Let y be a homomorphism from Φ^0 into $\mathscr{A}$. Suppose v is a projective limit of the composite homomorphism $F \circ y$ from Φ^0 into $\mathscr{B}$; denote by b the common value of the constant family $(\alpha(v_\kappa))_{\kappa\in\Phi_0}$. In order that y admit a projective limit it is necessary and sufficient that there exist, on b, an initial structure with respect to the family $(y_\kappa, v_\kappa)_{\kappa\in\Phi_0}$ of right structures.*

We give proof in the inductive case. According to Theorem 1, L is an isomorphism of right $\mathscr{A}$-sets. By definition, final structures and inductive limits are right universal solutions. Therefore $L(y, v)$ is an inductive limit of x in $\mathscr{A}$ if and only if (y, v) is a final structure on b with respect to the family $(x_\iota, u_\iota)_{\iota\in\Phi_0}$, by virtue of Lemma 4.

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