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A DIFFERENTIAL-GEOMETRIC PERSPECTIVE ON MAGNETO-HYDRODYNAMIC EQUILIBRIA

DAVID PERRELLA

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This thesis considers magneto-hydrodynamic (MHD) equilibria under the lens of differential geometry and topology. An MHD equilibrium is a reduced model describing the magnetic field which guides a steady and globally neutral plasma. In this thesis, we explore the mathematical foundations of their study.

One fundamental tool used in the study of MHD equilibria is the ability to write the magnetic field in simplified ways using suitable coordinates. These coordinates are known as flux or magnetic coordinates in plasma physics. In the mathematical literature, these coordinates are established as part of Arnold's structure theorem for certain MHD equilibria. Here, we extend this theory to include more general magnetic fields, such as Beltrami fields and vacuum fields which are not covered by the structure theorem. Interestingly, it is found that first integrals of Beltrami fields and vacuum fields with regular toroidal level sets force the magnetic field to (1) have no zeros in the vicinity of the toroidal surface and (2) admit a nontrivial continuous symmetry. We show how items (1) and (2) together enable one to establish the existence of flux coordinates. Integrals are constructed which enable direct calculation of the coefficients appearing in the definition of flux coordinates which capture the long-term behaviour of the field lines, corresponding to what is known as the rotational transform in plasma physics. This study prompts further investigation of the connection between flux coordinates and symmetries. We are led to an efficient proof of the existence of flux coordinates for any nonvanishing divergence-free vector fields admitting first integrals with regular toroidal level-sets. An example magnetic field is found where one may naively expect the existence of flux coordinates where there is in fact none.

We then turn to the famous Grad conjecture which loosely states that well-behaved MHD equilibria on solid toroidal domains in $\mathbb{R}^3$ must admit geometric symmetries.

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We show that if one considers MHD equilibria in arbitrary Riemannian 3-manifolds with boundary, there are examples of well-behaved MHD equilibria which do not admit any continuous symmetries. Moreover, these Riemannian 3-manifolds with boundary can be made arbitrarily close in geometry to a domain in $\mathbb{R}^3$. This work leverages the formalism of Eulerisable vector fields introduced by Peralta and Salas, Rechtman, and Torres de Lizaur, which splits the MHD equilibrium equations into an exterior calculus part and a metric compatibility part.

After considering general MHD equilibria, we return to the most special case of MHD equilibria, vacuum fields or Neumann harmonic vector fields, as they are called in the study of boundary value problems. We prove that if a vacuum field on a solid toroidal domain has a sufficiently nice first integral (nested toroidal flux surfaces), then the field necessarily has no zeros. This is a global version of the result about vacuum and Beltrami fields previously mentioned. Lastly, we develop an integrability theory of (vector-) distributions (into foliations) as a generalisation of the Stefan–Sussman theorem and make some comments about how this relates to the study of MHD equilibria.

Some of this research has appeared in [1–4].

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DAVID PERRELLA, Department of Mathematics and Statistics,
University of Western Australia, Perth, Western Australia 6009, Australia
e-mail: david.perrella@outlook.com

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