Canad. Math. Bull. Vol. **27** (3), 1984

ASYMPTOTIC ERROR EXPANSIONS FOR SPLINE INTERPOLATION

BY
H. P. DIKSHIT, A. SHARMA AND J. TZIMBALARIO

ABSTRACT. During the last decade or so there has been a revival of interest in the analysis of error-bounds $f^{(s)}-S^{(s)}$ for different classes of functions and their interpolatory splines of odd degree on a finite interval with variations on end conditions. Our object is to present a unified treatment of the asymptotic error expansion both for even and for odd degree interpolatory splines.

1. **Introduction.** We shall be interested in the class of functions $f(x)$ which are continuously differentiable with bounded derivatives up to some order on the real line and their interpolatory splines $S(x)$ on a uniform mesh. During the last decade or so, there has been a revival of interest in the analysis of error bounds $f^{(s)}-S^{(s)}$ for different classes of functions and their interpolatory splines of odd degree on a finite interval with variations on end conditions. It appears that this kind of study was first initiated by Birkhoff and de Boor [1] in 1964 for cubic splines. They showed that the error $f'-S'=O(h^3)$ for cubic splines with mesh size h. Almost all the authors consider splines of odd degree and concentrate on cubics and quintics. For detailed references, we refer to T. R. Lucas [5].

Since the problem of interpolation by even degree splines at the knots does not always have a solution, they seem to have received little attention. However, as Schoenberg [7] points out, interpolation by even degree splines at mid-intervals is uniquely solvable. For recent studies on quadratic splines interpolating f at the mid-intervals, we refer to Marsden [6] and Kammerer, Reddien and Varga [3].

Our object in this note is to present a unified treatment of the asymptotic error expansion both for even and for odd degree interpolatory splines. In §2 we give the preliminaries and a statement of the main result. §3 deals with some properties of B-splines which will be required later. The proof of the main result is presented in §4 and is based on an identity concerning the interdependence of the derivatives of a spline and the interpolatory data. Since

Received by the editors April 26, 1983 and, in revised form, December 2, 1983.
AMS Subject Classification Numbers: 41A60.

the periodic splines form a subset of cardinal splines, the results of Lucas [5] for odd-degree periodic splines are included in our study.

2. **Statement of main result.** Consider a bi-infinite sequence of points $\{kh\}_{-\infty}^{\infty}$ and denote by $\mathcal{S}_n$ the space of splines of degree n. When n is even, the knot sequence will be $\{(k+\frac{1}{2})h\}_{-\infty}^{\infty}$ and when n is odd, the knots will be taken to be $\{kh\}_{-\infty}^{\infty}$. The existence and uniqueness of a bounded spline $S(x)\in\mathcal{S}_n$ which interpolate bounded data is well known [7].

Let $W^{(2n+2)}$ denote the class of functions $f(x)$ such that $f^{(j)}(x)$ exists, is continuous and bounded on R for $j=0,1,\ldots,2n+2$. We shall prove

THEOREM 1. *Let $f(x)\in W^{(2n+2)}$ and let $S(x)\in\mathcal{S}_n$ be the spline interpolating $f(x)$ at the points $\{kh\}_{-\infty}^{\infty}$. Then*
(a) *if n is odd $(=2m-1)$ the following asymptotic formula holds for $0\le s\le 2m-2$:*

$$S_l^{(s)}=f_l^{(s)}+\sum_{k=0}^{m-1}A_{2k,m}^{(s)}h^{2m+2k-\hat{s}}f_l^{(2m+2k+s-\hat{s})}+O(h^{4m-\hat{s}}) \tag{2.1}$$

where $S_l^{(s)}=S^{(s)}(lh)$, $f_l^{(s)}=f^{(s)}(lh)$, $\hat{s}=2[s/2]$ and

(2.2)
$$A_{2k,m}^{(s)}=\frac{B_{2m+2k-\hat{s}}\,s!}{(2m+2k-s)\cdot(2m-1)!\,(2k+s-\hat{s})!}\left\{\binom{2k+s-\hat{s}}{s}-(-1)^s\binom{2m-1}{s}\right\}$$

(b) *If n is even $(=2m)$, we have for $0\le s\le 2m$*

$$S_l^{(s)}=f_l^{(s)}+\sum_{k=0}^{m}\tilde{A}_{2k,m}^{(s)}h^{2m+2k-\hat{s}}f_l^{(2m+2k+s-\hat{s})}+O(h^{4m+2-\hat{s}}) \tag{2.3}$$

where

$$\tilde{A}_{2k,m}^{(s)}=\frac{(2^{2k+2m-1-\hat{s}}-1)B_{2k+2m-\hat{s}}\,s!\left\{\binom{2k-1+s-\hat{s}}{s}-(-1)^s\binom{2m}{s}\right\}}{(2k+2m-\hat{s})2^{2k+2m-1-\hat{s}}(2m)!\,(2k-1+s-\hat{s})!} \tag{2.4}$$

and B_k in (2.2) *and* (2.4) *are Bernoulli numbers.*

REMARKS. When $n=2$, $\mathcal{S}_2$ denotes the class of quadratic splines with knots $\{(k+\frac{1}{2})h\}_{-\infty}^{\infty}$ and the nodes of interpolation $\{kh\}_{-\infty}^{\infty}$. In this case it follows from (2.3) and (2.4) that

$$S_l^{(1)}=f_l^{(1)}+\frac{h^2}{24}f_l^{(3)}-\frac{7h^4}{960}f_l^{(5)}+O(h^6) \tag{2.5}$$

$$S_l^{(2)}=f_l^{(2)}-\frac{h^3}{24}f_l^{(5)}+O(h^5). \tag{2.6}$$

If $x=(l+t)h$, with $|t|\le\frac{1}{2}$, we have from (2.5) and (2.6) on expanding in Taylor

https://doi.org/10.4153/CMB-1984-051-7 Published online by Cambridge University Press

series

$$S(x)-f(x)=\left(\frac{t}{24}-\frac{t^3}{6}\right)h^3f_l^{(3)}-\frac{t^4h^4}{120}f_l^{(4)}+O(h^5)$$

and

$$S'(x)-f'(x)=\left(\frac{1}{24}-\frac{t^2}{2}\right)h^2f_l^{(3)}-\frac{t^3h^3}{30}f_l^{(4)}+O(h^6).$$

This shows that when $t=\frac{1}{2}$, $S(x)-f(x)=O(h^4)$ and when $t=1/2\sqrt{3}$, $S'(x)-f'(x)O(h^3)$.

3. **Auxiliary results.** We recall that the forward B-splines for cardinal splines of degree n are given by

$$Q_{n+1}(x)=\frac{1}{n!}\sum_{\nu=0}^{n+1}(-1)^\nu\binom{n+1}{\nu}(x-\nu)_+^n \tag{3.1}$$

and the central B-spline is given by

$$M_{n+1}(x)=Q_{n+1}\left(x+\frac{n+1}{2}\right). \tag{3.2}$$

Let $\psi_{n+1}(u)=(2\sin\frac{1}{2}u/u)^{n+1}$. It is known [6] that

$$M_{n+1}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_{n+1}(u)e^{ixu}\,du.$$

Then for any integer ν and for any integer s for which $M_{n+1}^{(s)}(\nu)$ has a meaning, we have

$$M_{n+1}^{(s)}(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(iu)^s\psi_{n+1}(u)e^{i\nu u}\,du=\frac{1}{2\pi}\int_0^{2\pi}\phi_{n+1,s}(u)e^{i\nu u}\,du \tag{3.3}$$

where

$$\phi_{n+1,s}(u)=(-1)^{s/2}\sum_{-\infty}^{\infty}(u+2\pi j)^s\psi_{n+1}(u+2\pi j) \tag{3.4}$$

By inversion, from (3.3) we get

$$\phi_{n+1,s}(u)=\sum_{-\infty}^{\infty}M_{n+1}^{(s)}(\nu)e^{-i\nu u}. \tag{3.5}$$

From (3.4) we can rewrite $\phi_{n+1,s}(u)$ in a form which we shall need later. Indeed we have

$$\begin{aligned}\phi_{n+1,s}(u)&=(-1)^{s/2}\sum_{-\infty}^{\infty}(u+2\pi j)^s\left(\frac{2\sin\frac{u+2\pi j}{2}}{u+2\pi j}\right)^{n+1}\\&=(1)^{s/2}\left(2\sin\frac{u}{2}\right)^{n+1}\sum_{-\infty}^{\infty}\frac{(-1)^{(n+1)j}}{(u+2\pi j)^{n+1-s}}.\end{aligned} \tag{3.6}$$

https://doi.org/10.4153/CMB-1984-051-7 Published online by Cambridge University Press

We recall that

$$\cot\frac{u}{2}=\sum_{-\infty}^{\infty}\frac{1}{\frac{u}{2}+\pi j},\quad \operatorname{cosec}\frac{u}{2}=\sum_{-\infty}^{\infty}\frac{(-1)^j}{\frac{u}{2}+\pi j}$$

so that from (3.6), it follows that when n is odd $(=2m-1)$,

$$\phi_{2m,s}(u)=\frac{(-1)^{-(s/2)-1}2^{2m-1}}{(2m-s-1)!}\left(\sin\frac{u}{2}\right)^{2m}\left(\cot\frac{u}{2}\right)^{(2m-s-1)} \tag{3.7}$$

and when n is even $(=2m)$,

$$\phi_{2m+1,s}(u)=\frac{(-1)^{3s/2}2^{2m}}{(2m-s)!}\left(\sin\frac{u}{2}\right)^{2m+1}\left(\operatorname{cosec}\frac{u}{2}\right)^{(2m-s)}, \tag{3.8}$$

where $(\cot u/2)^{(k)}$ denotes the kth derivative of $\cot u/2$. Set

$$\left(\frac{2\sin\frac{u}{2}}{u}\right)^n=\sum_0^{\infty}\alpha_{n,k}u^{2k} \tag{3.9}$$

and let

$$\beta_{n+1,k}^{(s)}=\begin{cases}\dfrac{(-1)^{k+s+1}B_{2k}}{2k(2k-2m+s)!\,(2m-1-s)!}, & n=2m-1\\[2ex] \dfrac{(-1)^{k+s+1}(2^{2k-1}-1)B_{2k}}{2k2^{2k-1}(2k-2m+s-1)!\,(2m-s)!}, & n=2m\end{cases} \tag{3.10}$$

where B_{2k} denote Bernoulli numbers. We shall prove

Lemma 1. *The function $\phi_{n+1,s}(u)$ has the following power series expansion:*
(a) *If $n=2m-1$, we have*

$$\phi_{2m,s}(u)=(-1)^{3s/2}\sum_0^{\infty}\alpha_{2m,k}^{(s)}u^{2k+s} \tag{3.11}$$

where

$$\alpha_{2m,k}^{(s)}=\begin{cases}\alpha_{2m,k}, & k=0,1,\ldots,m-\left[\frac{s}{2}\right]-1\\[2ex] \alpha_{2m,k}+\sum\limits_{j=m-[s/2]}^{k}\beta_{2m,j}^{(s)}\alpha_{2m,k-j}, & k\geq m-\left[\frac{s}{2}\right].\end{cases} \tag{3.12}$$

(b) *If $n=2m$, we have*

$$\phi_{2m+1,s}(u)=(-1)^{s/2}\sum_0^{\infty}\alpha_{2m+1,k}^{(s)}u^{2k+s} \tag{3.13}$$

where

$$(3.14)\qquad \alpha^{(s)}_{2m+1,k}=\begin{cases}\alpha_{2m+1,k}, & k=0,1,\ldots,m-\left[\frac{s}{2}\right]-1\\ \alpha_{2m+1,k}+\sum\limits_{j=m-[s/2]}^{k}\beta^{(s)}_{2m+1,j}\alpha_{2m+1,k-j}, & k\geq m-\left[\frac{s}{2}\right].\end{cases}$$

REMARK. We must observe that when $s=0$, the numbers $\alpha^{(0)}_{2m,k}$ are not always the same as $\alpha_{2m,k}$ without the superscript 0.

Proof. The proof of this Lemma is a simple consequence of the expansion (3.9) and the following known ([2], pp. 334–335) Laurent expansions of cot $u/2$ and cosec u/s:

$$(3.15)\qquad \cot\frac{u}{2}=\frac{2}{u}+2\sum_{k=1}^{\infty}\frac{(-1)^kB_{2k}}{(2k)!}u^{2k-1}$$

$$(3.16)\qquad \operatorname{cosec}\frac{u}{2}=\frac{2}{u}+2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2^{2k-1}-1)B_{2k}}{(2k)!\,2^{2k-1}}u^{2k-1}.$$

Differentiating (3.15) $2m-s-1$ times and using (3.7) and (3.9), we get (3.11). Similarly from (3.16), (3.8) and (3.9), we get (3.13).

LEMMA 2. *For any $S(x)\in\mathcal{S}_n$ and for any non-negative integer s for which $D^sS_l=D^sS|_{s=lh}$, $(D\equiv d/dx)$ has a meaning, we have the following identity:*

$$(3.17)\qquad \phi_{n+1}(-ihD)D^sS_l=h^{-s}\phi_{n+1,s}(-ihD)S_l$$

for any integer l.

Proof. Set

$$\tilde{Q}_{n+1}(x)=\frac{1}{n!}\sum_{\nu=0}^{n+1}(-1)^{\nu}\binom{n+1}{\nu}(x-\nu h)^n_+.$$

Then $\tilde{Q}_{n+1}(x)=h^nQ_{n+1}(x/h)$. Similarly we have

$$(3.18)\qquad \tilde{M}_{n+1}(x)\equiv\tilde{Q}_{n+1}\left(x+\frac{n+1}{2}h\right)=h^nM_{n+1}(x).$$

Since every spline $S(x)\in\mathcal{S}_n$ has a unique representation in terms of B-splines, viz.,

$$S(x)=\sum_{-\infty}^{\infty}c_{\nu}\tilde{M}_{n+1}(x-\nu h),$$

and since obviously we have

$$\sum_{-\infty}^{\infty}\tilde{M}_{n+1}(x+jh)\tilde{M}^{(s)}_{n+1}(x-jh+lh)=\sum_{-\infty}^{\infty}\tilde{M}^{(s)}_{n+1}(x+jh)\tilde{M}_{n+1}(x-jh+lh)$$

https://doi.org/10.4153/CMB-1984-051-7 Published online by Cambridge University Press

for any integer l, it follows that

$$(3.19)\quad \sum_{-\infty}^{\infty} \tilde{M}_{n+1}(x+jh)S^{(s)}(x-jh+lh)=\sum_{-\infty}^{\infty} \tilde{M}_{n+1}^{(s)}(x+jh)S(x-jh+lh).$$

Putting $x=0$ in (3.19) and using (3.18), we get

$$(3.20)\quad \sum_{-\infty}^{\infty} M_{n+1}(j)D^sS_{l-j}=h^{-s}\sum_{-\infty}^{\infty} M_{n+1}^{(s)}(j)S_{l-j}.$$

If we use the shift operator $E=e^{hD}$, we have

$$D^sS_{l-j}=E^{-j}D^sS_l=D^se^{-jhD}S_l$$

so that (3.20) becomes

$$(3.21)\quad \left\{\sum_{j=-\infty}^{\infty} M_{n+1}(j)e^{-jhD}\right\}D^sS_l=h^{-s}\left\{\sum_{-\infty}^{\infty} M_{n+1}^{(s)}(j)e^{-jhD}\right\}S_l$$

Using (3.5) and (3.21), we get (3.17) where $\phi_{n+1,0}(u)=\phi_{n+1}(u)$.

4. **Proof of Theorem 1.** Since the proofs for the cases n odd and n even are very close and the same is true for s odd and s even, we shall sketch an outline of the proof only when n is even $(=2m)$ and s is even.

If in the identity (3.17), we replace D^sS_l by

$$(4.1)\quad F_l^{(s)}\equiv f_l^{(s)}+\sum_{k=0}^{m} \tilde{A}_{2k,m}^{(2)}h^{2m+2k-\hat{s}}f_l^{(2m+2k+s-\hat{s})}$$

and S_l by f_l, then the error $E_{n,s,l}$ is given by

$$(4.2)\quad E_{n,s,l}=\phi_{n+1}(-ihD)F_l^{(s)}-h^{-s}\phi_{n+1,s}(-ihD)f_l$$

where $\bar{A}_{2k,m}^{(s)}$ are given by (2.4). We shall use (3.10), (3.13) and (3.14) in (4.2) This gives

This gives

$$E_{n,s,l}=\left(\sum_0^{\infty} \alpha_{2m+1,k}^{(0)}h^{2k}D^{2k}\right)\left(D^s+\sum_{k=0}^{m} \tilde{A}_{2k,m}^{(s)}h^{2m+2k-s}D^{2m+2k}\right)f_l$$
$$-h^{-s}\left(\sum_0^{\infty} \alpha_{2m+1,k}^{(s)}h^{2k+s}D^{2k+s}\right)f_l$$
$$=\left(\sum_{k=0}^{\infty} \gamma_{2m+1,k}^{(s)}h^{2k}D^{2k+s}\right)f_l.$$

In this sum, $\gamma_{2m+1,k}^{(s)}\equiv 0$ for $k=0,1,\ldots,m-(s/2)-1$ and for $m-(s/2)\le k\le 2m-(s/2)$, $\gamma_{2m+1,k}^{(s)}$ again vanishes because of (3.14) and (2.4). This proves that

$$(4.3)\quad E_{n,s,l}=O(h^{4m+2-s}).$$

Set

$$\delta_l^{(s)} = S_l^{(s)} - F_l^{(s)}. \tag{4.4}$$

Subtracting (4.2) from (3.17), we get from (4.3)

$$\phi_{n+1}(-ihD)\delta_l^{(s)} = O(h^{4m+2-s}). \tag{4.5}$$

Observing that (3.17) and (3.20) are equivalent, we replace (4.5) by

$$\sum_{-\infty}^{\infty} M_{2m+1}(i-j)\delta_j^{(s)} = O(h^{4m+2-s}). \tag{4.6}$$

The matrix $A = (M_{2m+1}(i-j))$ is a banded symmetric Toeplitz matrix with $2m+1$ successive non-zero elements in each row. The associated polynomial in this case is $\rho_{2m}(x)$ where

$$\rho_{2m}(x) = 2^{2m}(2m)!\, x^m \sum_{j=-m}^{m} M_{2m+1}(j)x^j.$$

Schoenberg [7] has shown that this polynomial has simple, negative zeros

$$\mu_{2m} < \mu_{2m-1} < \cdots < \mu_{m+1} < -1 < \mu_m < \cdots < \mu_1 < 0$$

such that

$$\mu_j \mu_{2m-j} = 1, \quad j = 0, 1, \ldots, m-1.$$

Hence

$$\rho_{2m}(x) = K_m x^m \prod_{j=1}^{m} (x + c_j + x^{-1})$$

where $c_j = \mu_j + \mu_{2m-j} > 2$. Following Kershaw [4], we see that for the matrix A defined above, we have

$$A = K'_m \prod_{j=1}^{m} A_j$$

where A_j is a banded three diagonal circulant matrix with a row of the form $(0 \cdots 1\ c_j\ 1 \cdots)$ with $c_j > 2$. Hence

$$\|A^{-1}\| \le \frac{1}{K'_m} \prod_{j=1}^{m} \|A_j^{-1}\| < \infty.$$

Since $\|A^{-1}\|$ is uniformly bounded it follows from (4.6) that

$$\|\delta_j^{(s)}\|_\infty = O(h^{4m+2-s}),$$

which concludes the proof of the theorem.

5. **Conclusion.** The methods used above can be adapted to find higher order terms in the expansions (2.1) and (2.3) when the function has a higher degree

https://doi.org/10.4153/CMB-1984-051-7 Published online by Cambridge University Press

of differentiability. Also for odd-degree splines of degree n similar expansions can be obtained for $S_{l^+}^{(n)}$ and $S_{l^-}^{(n)}$ by using the above results. It would be interesting to know what kind of results hold for other kinds of interpolatory conditions such as Hermite and lacunary conditions.

References

1. G. Birkhoff and C. De Boor, *Error bounds for spline interpolation*, J. Math. Mech., **13** (1964), 827–836.

2. E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, 1975.

3. W. J. Kamerer, G. W. Reddien and R. S. Varga, *Quadratic interpolatory spline*, Numer. Math., **22** (1974), 241–259.

4. D. Kershaw, *A bound on the inverse of a band matrix which occurs in interpolation by periodic odd order splines*, J. Inst. Maths. Applics, **20** (1977), 227–228.

5. T. R. Lucas, *Asymptotic expansions for interpolating periodic splines*, SIAM J. Numer. Anal., **19** (1982), 1051–1066.

6. M. Marsden, *Quadratic spline interpolation*, Bull. Amer. Math. Soc., **80** (1974), 903–906.

7. I. J. Schoenberg, *Cardinal spline interpolation*, SIAM Regional Conf. Series, 12 (1973).

University of Alberta
Edmonton, Alberta T6G2H1.
Canada

https://doi.org/10.4153/CMB-1984-051-7 Published online by Cambridge University Press