QUOTIENT GROUPS AND REALIZATION OF TIGHT RIESZ GROUPS

Dedicated to the memory of Hanna Neumann

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1. Introduction

Let $(G, \preccurlyeq)$ be an l-group having a compatible tight Riesz order $\leqq$ with open-interval topology U, and H a normal subgroup. The first part of the paper concerns the question: Under what conditions on H is the structure of $(G, \preccurlyeq, \wedge, \vee, \leqq, U)$ carried over satisfactorily to $G'_H \equiv G/H$ by the canonical homomorphism; and its answer (Theorem 8°): H should be an l-ideal of $(G, \preccurlyeq)$ closed and not open in (G, U). Such a normal subgroup is here called a tangent. An essential step is to show that $\preccurlyeq'$ is the associated order of $\leqq'$.

If H is a maximal tangent then G'_H is fully ordered. The second part of the paper shows that there is a natural realization ρ of G as a subdirect product of the groups G'_H which is an order isomorphism for $\leqq$ as well as $\preccurlyeq$, if for example $(G, \preccurlyeq)$ is lattice-complete and $\leqq$ is non-androgynous, and all maximal tangents are replete. The lattice-completeness requirement can be relaxed to weak projectability. But if $\leqq$ is androgynous then ρ fails to be one-one. Extra conditions ensure that ρ is also a topological embedding of G in $A = \prod_H G'_H$ and that ρ is concordant. The topology used on A is the open-interval topology. The main results are Theorems 15° and 18°.

A realization theory for androgynous groups — those for which not every element $a > 0$ is a weak unit of $(G, \preccurlyeq)$ — remains an open question. We postpone to a later paper the use of the present realization to construct a Gelfand theory by topologizing the set of maximal tangents.

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2. Preliminaries

A poset $(X, \leqq)$ is said to have the tight Riesz (m, n) property (to be TR(m, n), for short) when

$$a_i < b_j \qquad i = 1, 2, \cdots, m, \qquad j = 1, 2, \cdots, n$$

for elements $a_1, \cdots, a_m, b_1, \cdots, b_n$ implies the existence of an element x such that $a_i < x < b_j$.

The loose Riesz (m, n) property (= LR(m, n)) is defined by replacing $<$ by $\leqq$ at all occurrences. It is easily shown that TR$(2,2) \Leftrightarrow$ TR(m, n), and LR$(2,2)$ $\Leftrightarrow$ LR(m, n) for all $m \geqq 2$, $n \geqq 2$; and TR$(1,2) \nRightarrow$ TR$(2,2) \Rightarrow$ LR$(2,2)$. Also [TR$(1,2)$ and TR$(2,1)$ and LR$(2,2)$] $\Leftrightarrow$ TR$(2,2)$. TR$(1,2)$ implies order-denseness. For some properties of these interpolation axioms on posets see Cameron and Miller [1]. A poset $(X, \leqq)$ is said to be an *antilattice* if only the trivial meets and joins exist.

A *tight Riesz* $(1,2)$ *group* (=TR$(1,2)$ group) is a directed partially ordered group $(G, \leqq)$ with the TR$(1,2)$ (equivalently, with the TR$(2,1)$) property. (In [4], [5] and [6] it was assumed that G is commutative. Here we do not insist on commutativity.) The *open-interval topology* U is the topology on G having as subbase (in fact, as base) the set of all open intervals (a, b), $a < b$. For any $x \in G$, the sets $(x - a, x + a)$, $a > 0$, form a base for U at x. For the positive cone and srtict positive cone write

$$P = \{x \in G : x \geqq 0\}, \qquad P^* = P \setminus \{0\};$$

these are normal subsets of the group. The topological boundary ∂P of P with respect to U consists of 0 together with the *pseudopositives* of $(G, \leqq)$, i.e. the elements $p \ngeqq 0$ such that $x > 0 \Rightarrow x + p > 0$, or equivalently, such that $x > 0 \Rightarrow p + x > 0$. The elements of $\partial P \cap (-\partial P) = \bar{P} \cap (-\bar{P})$ are 0 together with the pseudozeros. The set $\bar{P} = P \cup \partial P$ is the positive wedge of the *associated preordering* on G, written $\preccurlyeq$:

$$\bar{P} = \{x \in G : x \succcurlyeq 0\} = \{0\} \cup P^* \cup \{\text{pseudopositives}\}.$$

Note that

$$x > 0 \Rightarrow x \succ 0; \tag{2.1}$$

$$x < y \prec z \Rightarrow x < z; \qquad x \prec y < z \Rightarrow x < z.$$

The topological lemma Section 2, 6° of Loy and Miller [4] is valid for all not-necessarily-commutative TR$(1,2)$ groups. The associated preordering for a poset is discussed in Cameron and Miller [1].

1° Theorem. *Let* $(G, \leqq)$ *be a* TR$(1,2)$ *group. Then with the above notation*

(i) (G, U) *is a topological group, and* U *is not discrete*;

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(ii) $(G, \leqq)$ *is an order-dense antilattice*;

(iii) U *is Hausdorff* $\Leftrightarrow (G, \leqq)$ *has no pseudozeros* $\Leftrightarrow \preccurlyeq$ *is a partial order.*

(iv) *If* U *is Hausdorff, then* $(G, \preccurlyeq)$ *is a partially ordered group.*

PROOF. see Fuchs [3], p 20, and Loy and Miller [4].

We are concerned chiefly with the case where $(G, \preccurlyeq)$ is an l-group; then $\wedge$, $\vee$ denote the lattice operations with respect to $\preccurlyeq$. Given a priori a partially ordered group $(G, \preccurlyeq)$, any TR(1,2) partial order $\leqq$ on G having $\preccurlyeq$ as its associated order (and therefore having no pseudozeros) will be called a *compatible tight Riesz order for* $(G, \preccurlyeq)$ (CTRO, for short). Tight Riesz groups looked at in this light have been discussed by Wirth [11] and Reilly [7].

2°. *Let* $(G, \preccurlyeq)$ *be an l-group with compatible tight Riesz order* $\leqq$. *Then*

(i) $(G, \preccurlyeq)$ *is* $LR(2,2)$ *and* $(G, \leqq)$ *is* $TR(2,2)$;

(ii) $(G, \wedge, \vee, U)$ *is a topological lattice*;

(iii) $\preccurlyeq$ *and* $\leqq$ *are both isolated orders* (*i.e.* $nx \succ 0$ *for some positive integer* n *implies* $x \succ 0$; *and similarly for* $>$).

PROOF. (i) Cameron and Miller [1], p. 10; (ii) [4], p. 236; (iii) [4], p. 236, Reilly [7], p. 11.

We are interested in carriers and l-ideals. Let $(G, \preccurlyeq)$ be an l-group. An *l-ideal* of G is a convex directed normal subgroup; it is necessarily also a sublattice. Given a subset $Q \subseteq G$, lid(Q) denotes the l-ideal generated by Q, i.e. the intersection of all l-ideals containing Q. An *o-ideal* of a partially ordered group is a convex directed normal subgroup.

Again, write

$$Q^{\perp} = \{x \in G: |x| \wedge |q| = 0 \text{ for all } q \in Q\},$$

and abbreviate $\{a\}^{\perp}$ to $a^{\perp}$. $Q^{\perp}$ is a convex subgroup and a sublattice; however, $Q^{\perp}$ is an l-ideal for every $Q \subseteq G$ if and only if all carriers of G are invariant (Fuchs [2], p. 82). Always $Q \subseteq Q^{\perp\perp}$.

The relation $a^{\perp} = b^{\perp}$ is an equivalence relation on G; the intersections of its classes with the positive cone are called the *carriers* of G; for $a \succcurlyeq 0$ the carrier of a is

$$\hat{a} = \{b \succcurlyeq 0: a \wedge x = 0 \text{ if and only if } b \wedge x = 0\}.$$

The partial order $\preccurlyeq$ induces on $\mathfrak{C}$, the set of all carriers, a partial order

$$\hat{a} \preccurlyeq \hat{b} \text{ if and only if } b \wedge x = 0 \Rightarrow a \wedge x = 0; \tag{2.2}$$

$(\mathfrak{C}, \preccurlyeq)$ is a distributive lattice.

By a *weak unit* of G we mean a weak unit of the l-group $(G, \preccurlyeq)$ i.e. an element $w \succ 0$ such that $w \wedge x = 0 \Rightarrow x = 0$. Let $\mathfrak{w}$ denote the set of weak units; it may be empty, but if not empty then it is a carrier of G, to wit, the greatest carrier. In fact $(\mathfrak{C}, \preccurlyeq)$ has a greatest element, $\mathfrak{w}$, if and only if $\mathfrak{w} \neq \emptyset$.

Now suppose that $\leqq$ is a compatible tight Riesz order of the l-group $(G, \preccurlyeq)$, and consider the following property involving $\leqq$ and $\preccurlyeq$:

(A) For all $x, y \in G$,

$$x > x \wedge y \quad \Rightarrow \quad x > y,$$

and its dual formulation

(A′) For all $x, y \in G$,

$$x < x \vee y \quad \Rightarrow \quad x < y.$$

Each implies the other. If (A) (equivalently, (A′)) fails to hold in G, we call $(G, \leqq)$ (or perhaps $\leqq$) *androgynous*, otherwise it is called non-androgynous. For an example of an androgynous group one can take $G = \mathbb{R} \times \mathbb{R}$, with $\langle x_1, x_2 \rangle > 0$ if and only if $x_1 > 0$, $x_2 \geqq 0$; here $\langle x_1, x_2 \rangle \succcurlyeq 0$ if and only if $x_1 \geqq 0$, $x_2 \geqq 0$. There are a number of equivalent formulations of property (A).

3°. *Let $(G, \preccurlyeq)$ be an l-group with compatible tight Riesz order $\leqq$. Each of the following properties is separately equivalent to (A): for all x, y:*

(i) *If $x \wedge y$ is neither x nor y, then both x, y belong to $x \wedge y + \partial P$.*

(ii) *If $x \vee y$ is neither x nor y, then $x \vee y$ belongs to both $x + \partial P$ and $y + \partial P$.*

(iii) $x > 0,\ y \succ 0 \Rightarrow x \wedge y \succ 0$.

(iv) $P^* \subseteq \mathfrak{w}$.

Proof. The pairwise equivalence of (A), (A′), (i), (ii) and (iii) is easily checked. Consider (A)$\Leftrightarrow$(iv). Assume (A) and let $a > 0$. Then $x \wedge a = 0 \Rightarrow a > x \wedge a \Rightarrow x = x \wedge a \Rightarrow x = 0$, so $a \in \mathfrak{w}$. Thus $P^* \subseteq \mathfrak{w}$. Conversely assume (iv). Let $x > x \wedge y$; the element $z = x - x \wedge y = 0 \vee (x - y)$ being in P^* is a weak unit. Let $a = y - x \wedge y$; we have

$$\begin{aligned} z \wedge a &= [0 \vee (x-y)] \wedge [0 \vee (y-x)] \\ &= 0 \vee [(x-y) \wedge (y-x)] = 0 \end{aligned}$$

so $a = 0$. Thus (A) holds.

Thus when G is non-androgynous it has weak units. If G is androgynous it may still happen that $\mathfrak{w} \neq \emptyset$, but then necessarily $P^* \nsubseteq \mathfrak{w}$. An l-group $(G, \preccurlyeq)$ can possess both androgynous and non-androgynous compatible tight Riesz orders.

4°. *Let the G in 3° be non-androgynous. A necessary and sufficient condition for $P^* = \mathfrak{w}$ is: for every $x \in \partial P$ there exists $y \in \partial P$, $y \neq 0$, with $x \wedge y = 0$.*

More generally, for $x \succcurlyeq 0$ we have

$$\text{(2.3)} \qquad x \in \bar{P} \backslash \mathfrak{w} \quad \Leftrightarrow \quad x^{\perp} \neq (0) \quad \Leftrightarrow \quad \exists\, y \in \partial P,\ y \neq 0,\ x \wedge y = 0.$$

Proof. (2.3) is easily verified, and implies the first statement.

3. Quotient groups

Throughout this section let $(G, \leqq)$ *be a nontrivial* $TR(1,2)$ *group without pseudozeros*, with open-interval topology U and associated ordering $\preccurlyeq$; *let* H *be a normal subgroup*, $H \neq (0)$, $G' = G/H$, and let $\theta: G \to G'$, $a \mapsto a' = a + H$, denote the canonical homomorphism.

To ensure that a reasonable sufficiency of the structure of $(G, \leqq, \preccurlyeq, U)$ (and $\wedge$, $\vee$ when they exist) can be carried over to G', some restrictions need to be placed on H. This section is devoted to finding suitable circumstances. First, for the quotient orders $\leqq'$, $\preccurlyeq'$ to exist we need H to be $\leqq$-convex and $\preccurlyeq$-convex. Since $\preccurlyeq$-convexity implies $\leqq$-convexity, both orders exist if H is $\preccurlyeq$-convex, and then

$$a' >' 0' \Leftrightarrow a + h > 0 \text{ for some } h \in H,\ a \notin H$$

$$\Leftrightarrow a - P^* \text{ meets } H,\ a \notin H, \tag{3.1}$$

$$a' \succcurlyeq' 0' \Leftrightarrow a + h \succcurlyeq 0 \text{ for some } h \in H$$

$$\Leftrightarrow a - \bar{P} \text{ meets } H; \tag{3.2}$$

and θ is order-preserving, i.e. $a > 0 \Rightarrow a' \geqq' 0'$ and $a \succcurlyeq 0 \Rightarrow a' \succcurlyeq' 0'$. Hence $(G', \leqq')$ and $(G', \preccurlyeq')$ are partially ordered groups.

5°. *Under the assumptions of the first paragraph*:

(i) *G' is directed with respect to $\leqq'$ and $\preccurlyeq'$.*

(ii) *H is $\leqq$-directed if and only if every coset $a + H$ is $\leqq$-directed, and ikewise for $\preccurlyeq$.*

(iii) *If H is $\leqq$-directed then H meets P^*. If H is $\preccurlyeq$-directed then H meets $\bar{P}\setminus\{0\}$.*

(iv) *If H is $\leqq$-convex then H is open in (G, U) if and only if H meets P^*. So if H is $\leqq$-directed and $\leqq$-convex then H is open.*

Suppose H is $\leqq$-convex. Then:

(v) *If H is not open then $(G', \leqq')$ is a $TR(1,2)$ group.*

(vi) *If $(G, \leqq)$ is $LR(2,2)$ and H is $\preccurlyeq$-directed then $(G', \leqq')$ is $LR(2,2)$.*

(vii) *If $(G, \leqq)$ is $TR(2,2)$, and H is closed not open and $\preccurlyeq$-directed, then $(G', \leqq')$ is a $TR(2,2)$ group.*

(viii) *If $(G, \preccurlyeq)$ is $LR(2,2)$ and H is $\preccurlyeq$-directed and $\preccurlyeq$-convex (i.e. H is an o-ideal), then $(G', \preccurlyeq')$ is $LR(2,2)$.*

Proof. (i)–(iv) are straightforward. (v) Let $a' <' b_1'$, b_2' in G'. Then $a < b_{11}$, b_{21} for some $b_{11} \in b_1'$, $b_{21} \in b_2'$; since $(G, \leqq)$ is TR(1,2) there exists c such that

$$a < c < b_{11},\ b_{21}.$$

Because H does not meet P^*, $x > 0$ implies $x' >' 0'$.
Thus

$$a' <' c' <' b_1', b_2'.$$

Thus $(G', \leqq')$ is TR(1,2) and being $\leqq'$-directed is a TR(1,2) group.

(vi) The proof of Fuchs [3], p. 14 can be adapted to this case.

(vii) The conditions and (v), (vi) ensure that $(G', \leqq')$ is TR(1,2) and LR(2,2) and these together imply TR(2,2).

(viii) This is the result of Fuchs referred to in the proof of (vi).

EXAMPLE. Let $G = \mathbb{R} \times \mathbb{Z}$, with $\langle x, m \rangle > 0$ if and only if $x > 0$, $m \geqq 0$. $(G, \leqq)$ is a TR(2,2) group without pseudozeros. If $H = \mathbb{R} \times (0)$ then H is $\preccurlyeq$-convex, $\preccurlyeq$-directed, closed and open; but $G' \cong \mathbb{Z}$ is not TR(1,2).

Proposition 5° and the preceding remarks show that for a desirable theory we should require at least that H be $\neq (0)$, $\preccurlyeq$-convex, so $\leqq$-convex, closed and not open, and $\preccurlyeq$-directed. (Since H does not meet P^*, it cannot be $\leqq$-directed, and $\leqq$-convexity is satisfied vacuously.) In other words, when $(G, \preccurlyeq)$ is an l-group, H should be a closed l-ideal not meeting P^*. Since it must meet $\partial P \backslash (0)$, we call such an H a *tangent*.

Topological as well as interpolation considerations require that H be closed not open. For let $\boldsymbol{Q}$ denote the *quotient topology* on G', i.e. the strongest topology making $\theta\colon (G, \boldsymbol{U}) \to G'$ continuous; a subset $Q \subseteq G'$ belongs to $\boldsymbol{Q}$ if and only if $Q = \theta(U)$ for some $U \in \boldsymbol{U}$, and here one can take in particular $U = \theta^{-1}(Q)$; θ is an open mapping. It is known that $(G', \boldsymbol{Q})$ is a topological group, is Hausdorff if and only if H is closed in $(G, \boldsymbol{U})$, and is discrete if and only if H is open. If $\boldsymbol{U}'$ denotes the open-interval topology of $(G', \leqq')$, and this is a TR group, then $\boldsymbol{U}'$ is not discrete, so certainly $\boldsymbol{U}' \neq \boldsymbol{Q}$ if H is open.

Again, $\boldsymbol{U}' \neq \boldsymbol{Q}$ if $(G', \leqq')$ has pseudozeros while H is closed. Suppose it is known that $(G', \leqq')$ has no pseudozeros: then there is on G' the associated order $\precsim$ of $\leqq'$, as well as $\preccurlyeq'$, the quotient order of $\preccurlyeq$; and in general it is not clear that $\precsim$ and $\preccurlyeq'$ coincide, as one might hope.

We proceed to show that these troubles do not arise, and there are other benefits, when $(G, \preccurlyeq)$ is an l-group and H is a tangent. The main result is Theorem 8°, which we reach by two lemmas.

6°. *If H is $\leqq$-convex and closed not open, then the map $\theta\colon (G, \boldsymbol{U}) \to (G', \boldsymbol{U}')$ is continuous, so $\boldsymbol{U}' \subseteq \boldsymbol{Q}$; and $\boldsymbol{U}' = \boldsymbol{Q}$ if and only if this map is also open.*

PROOF. By 5° (iv), H does not meet P^*, so

$$x > 0 \Rightarrow x' >' 0'. \tag{3.3}$$

Consider a base open set $V \equiv (a', b')$, $a' <' b'$, of $\boldsymbol{U}'$, and $x_0 \in \theta^{-1}(V)$. There exist h_1, $h_2 \in H$ such that $a + h_1 < x_0 < b + h_2$; then $U = (a + h_1, b + h_2)$ is

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open in G, contains x_0, and $U \subseteq \theta^{-1}(V)$ because of (3.3). So $\theta^{-1}(V)$ is open, θ is continuous, $\boldsymbol{U}' \subseteq \boldsymbol{Q}$.

If $\boldsymbol{U}' = \boldsymbol{Q}$ then $U \in \boldsymbol{U} \Rightarrow \theta(U) \in \boldsymbol{Q} \Rightarrow \theta(U) \in \boldsymbol{U}'$ so θ is open. Conversely if θ is open and $V \in \boldsymbol{Q}$, then $V = \theta(W)$ for some $W \in \boldsymbol{U}$, and so $V \in \boldsymbol{U}'$.

7°. *If H is $\leqq$-convex and closed not open, then the property*

$$\left[\begin{matrix} x < y;\ \textit{there exist}\ h_1, h_2 \in H \\ \textit{such that}\ x < h_1\ \textit{and}\ h_2 < y \end{matrix}\right] \Rightarrow \left[\begin{matrix} \textit{there exists}\ h_3 \in H \\ \textit{such that}\ x < h_3 < y \end{matrix}\right] \tag{3.4}$$

implies

$$\theta(a,b) = (a',b')\ \textit{for all}\ a < b, \tag{3.5}$$

and hence $\boldsymbol{U}' = \boldsymbol{Q}$.

PROOF. Let $a < b$. Since $a < x < b \Rightarrow a' <' x' <' b'$, necessarily $\theta(a,b) \subseteq (a',b')$. Let $\xi \in (a',b')$, say $\xi = x'$; there exist h_1, $h_2 \in H$ such that $a - x < h_1$ and $h_2 < b - x$, also $a - x < b - x$, so by (3.4) there exists $h_3 \in H$ for which $a < x + h_3 < b$, and here $x + h_3 \in \xi$. Therefore $\xi \in \theta(a,b)$. This proves (3.5) which in turn implies that θ is open, so $\boldsymbol{U}' = \boldsymbol{Q}$.

8°. THEOREM. *Let $(G, \preccurlyeq)$ be an l-group with a compatible tight Riesz order $\leqq$; let H be a tangent, and $G' = G/H$. Then $(G', \preccurlyeq')$ is an l-group with compatible tight Riesz order $\leqq'$;*

$$\theta: (G, \boldsymbol{U}) \to (G', \boldsymbol{U}'), \qquad x \mapsto x'$$

is a continuous open map as well as a group and lattice homomorphism; and $\boldsymbol{U}' \equiv \boldsymbol{Q}$ is Hausdorff.

PROOF. Since $(G, \preccurlyeq)$ is a lattice, it is LR(2,2). We verify (3.4) of 7°. Let $x < y$, $x < h_1$ and $h_2 < y$ with h_1, $h_2 \in H$. If $h_1 \wedge h_2 = h_1 \vee h_2$ then $h_1 = h_2$ and (3.4) is verified trivially. Assume $h_1 \wedge h_2 \prec h_1 \vee h_2$. By the tight Riesz property of $(G, \leqq)$ there exists x_1 such that $x < x_1 < y$, h_1 and then also y_1 such that $x_1, h_2 < y_1 < y$. The LR(2,2) property for $(G, \preccurlyeq)$ then implies the existence of k such tha

$$x_1,\ h_1 \wedge h_2 \preccurlyeq k \preccurlyeq h_1 \vee h_2,\ y_1.$$

Since $(H, \preccurlyeq)$ is an l-ideal, $k \in H$; and $x < k < y$. This proves (3.4).

By 7°, $\boldsymbol{U}' = \boldsymbol{Q}$; since H is closed, this topology is Hausdorff, so $(G', \leqq')$ has no pseudozeros. Since H is an l-ideal, $(G', \preccurlyeq')$ is an l-group and θ is a lattice homomorphism. By 5° (vii), $(G', \leqq')$ is a TR(2,2) group.

It remains to prove that $\preccurlyeq'$ coincides with $\precsim$, the associated order of $\leqq'$. The cones of $\leqq'$, $\preccurlyeq'$ and $\precsim$ are respectively $\theta(P)$, $\theta(\bar{P})$ and $\overline{\theta(P)}$; since θ is continuous, $\theta(\bar{P}) \subseteq \overline{\theta(P)}$, so

(3.6) $$a' >' 0' \Rightarrow a' \succcurlyeq' 0' \Rightarrow a' \succ\!\!= 0'.$$

Next, note that

(3.7) $$a' \precsim' b' <' c' \Rightarrow a' <' c',$$

and

(3.8) $$\text{if } a' >' 0' \Rightarrow a' + x' \succcurlyeq' 0', \text{ then } x' \succeq 0'.$$

Here (3.7) follows immediately from (3.6) and the fact that $\preceq$ is the associated order of $\leqq'$. Assume the premise of (3.8), and $a' >' 0'$. Since $(G', \leqq')$ is TR, there exists b' such that $a' >' b' >' 0'$; then $a' + x' >' b' + x' \succcurlyeq' 0'$ so (3.7) gives $a' + x' >' 0'$. This proves $x' \succeq 0'$.

Let $a', b' \in G'$. By (3.6), $a' \wedge b' \preccurlyeq' a', b'$ gives

$$a' \wedge b' \preceq a', b'.$$

Suppose $x' \preceq a', b'$. Then $u' >' 0'$ implies $x' <' a' + u'$, and $x' <' b' + u'$, so

$$x' \preccurlyeq' (a' + u') \wedge (b' + u') = a' \wedge b' + u'.$$

Thus $u' >' 0' \Rightarrow u' + a' \wedge b' - x' \succcurlyeq' 0$, so (3.8) gives

$$x' \preceq a' \wedge b'.$$

Therefore $a' \wedge b'$ is also the infimum of a', b' with respect to $\preceq$; and similarly for $a' \vee b'$. Take $a' \preceq b'$, to get $a' = a' \wedge b' \preccurlyeq' b'$. Recalling (3.6) we deduce that $\preccurlyeq'$ coincides with $\preceq$.

The coincidence of these two orders implies for G a separation property of some independent interest, namely

9°. *Under the conditions of* 8°, *if* $x + \bar{P}$ *does not meet* H *then there exists* $y < x$ *such that* $y + \bar{P}$ *does not meet* H.

Proof. If $x + \bar{P}$ does not meet H then not $x' \preccurlyeq' 0'$, i.e. not $x' \preceq 0'$, so there exists $a > 0$, $a' - x' \not\succcurlyeq' 0'$, i.e. $x - a + P^*$ does not meet H. Take $x - a < y < x$.

For a $\preccurlyeq$-convex normal subgroup H of G not meeting P^*, the two orders have the following descriptions:

$$x' \succcurlyeq' 0' \text{ if and only if } (\exists h \in H)\ (\forall a \in G,\ (a > 0 \Rightarrow x + h + a > 0),$$

$$x' \succeq 0' \text{ if and only if } (\forall a \in G)\ (\exists h \in H)\ (a > 0 \Rightarrow x + h + a > 0).$$

For a study of those convex l-subgroups of $(G, \preccurlyeq)$ which do meet P^*, see Reilly [7], sections 3 and 4.

4. Realization of G using maximal tangents

Theorem 8° of the previous section clears the way for the use of maximal tangents to represent G. We proceed to do this, for somewhat special circumstances; namely when $(G, \preccurlyeq)$ is weakly projectable (and so commutative), and $\leqq$ is non-androgynous. The main results are Theorem 15°, 18°. Throughout this section it is assumed that $(G, \preccurlyeq)$ *is a nontrivial l-group with compatible tight Riesz order* $\leqq$; U is the open-interval topology of $\leqq$ and $^-$ denotes closure with respect to U (except where it refers to filets). Here 'non-trivial' means that $G \neq (0)$ and $(G, \preccurlyeq)$ is not trivially or fully ordered.

10°. *If H is an l-ideal of $(G, \preccurlyeq)$, then so is $\bar{H}$.*

Proof. $\bar{H}$ is a normal subgroup of G; it is also a sublattice since (2°) the lattice operations on G are continuous.

It remains to prove that $\bar{H}$ is $\preccurlyeq$-convex. Let $u \prec x \prec v$ with $u, v \in \bar{H}$. For $a > 0$, there exist $h_1 \in (u-a, u+a) \cap H$ and $h_2 \in (v-a, v+a) \cap H$, and here

$$h_1 \wedge h_2 \in (u-a, u+a) \cap H, \qquad h_1 \vee h_2 \in (v-a, v+a) \cap H,$$

$$h_1 \wedge h_2 \preccurlyeq h_1 \vee h_2,$$

so without loss of generality we can assume $h_1 \prec h_2$.

Let $(x-b, x+b)$ be any base neighbourhood of x; take any a such that $0 < a < b$, and construct h_1, h_2 for this a as above. Write

$$k = h_1 \vee (x-a).$$

We have $k \in (x-b, x+b)$, and $h_1 \preccurlyeq k \preccurlyeq h_2$ so $k \in H$. Thus $x \in \bar{H}$.

Let H_0 be any l-ideal not meeting P^*. By Zorn's lemma there exists a maximal element in the class, ordered by $\subseteq$, of all l-ideals containing H_0 and not meeting P^*. By 10°, such a maximal element is closed. Thus

11°. *Every l-ideal maximal with respect to not meeting P^* is a maximal tangent, and conversely. Every l-ideal not meeting P^* is contained in a maximal tangent.*

Let $\mathfrak{H}$ denote the set of maximal tangents.

12°. *If H is a maximal tangent and G is commutative then for $G' = G/H$, $\leqq'$ coincides with $\preccurlyeq'$, these being order-dense full orders.*

Proof. First we show that every non-trivial l-ideal W of the l-group $(G', \preccurlyeq'$ meets $Q^* = \theta(P) \setminus \{0\}$, the strict cone of $\leqq'$. Write

$$M = \{x : x' \in W\};$$

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M is a $\preccurlyeq$-convex normal subgroup of G, and since θ is a lattice homomorphism, M is also a sublattice, and therefore an l-ideal. Moreover $H \subseteq M$. If M meets P^* in x, then $x' \in W \cap Q^*$. Suppose W does not meet Q^*; then $\bar{M}$ is a tangent, so $H = M$ and hence $W = (0)$, contradiction.

Now let $a' \succ' 0'$ in G'. Then

$$\text{lid}(a') = \{x' : 0 \preccurlyeq' |x'| \preccurlyeq' na' \text{ for some positive integer } n\}$$

being a non-trivial l-ideal, meets Q^*; i.e. there exists $x' \in G'$, $0 <' x' \preccurlyeq' na'$. By 2° (iii) and 8°, $a' >' 0'$. Thus $\leqq'$ and $\preccurlyeq'$ coincide. Since $\leqq'$ is an antilattice order (5° (vi)) and $\preccurlyeq'$ is a lattice order, the common order must be full. Since $\leqq'$ is TR, it is order-dense.

We shall need to use replete maximal tangents. In any l-group $(G, \preccurlyeq)$ write

$$\tilde{a} = \{x \in G : |x| \in |a|\hat{\ }\}$$

and call the sets $\tilde{a}$ the *filets* of G. Clearly for $a \succcurlyeq 0$, $\hat{a} = \tilde{a} \cap \bar{P}$. (For any subset A, $\bar{A}$ continues to mean the closure of A.) The filets form a lattice isomorphic with $(\mathfrak{C}, \preccurlyeq)$ when ordered by

$$\tilde{a} \preccurlyeq \tilde{b} \Leftrightarrow |a|\hat{\ } \preccurlyeq |b|\hat{\ } \Leftrightarrow a^{\perp} \supseteq b^{\perp}.$$

For some standard properties of filets in commutative l-groups see Ribenboim [8], pp 31–38. A subgroup K of G will be called *replete* if it is a union of filets, i.e. $x \in K \Rightarrow \tilde{x} \in K$.

For the groups G at present under consideration, the maximal tangents may not be replete. For example, if $G = C[0,1]$ (the continuous functions on $[0,1]$) with $f > 0$ if and only if $f(t) > 0$ for $0 \leqq t \leqq 1$, then $f \succcurlyeq 0$ if and only if $f(t) \geqq 0$ for $0 \leqq t \leqq 1$ (i.e. $\leqq$ and $\preccurlyeq$ are the tight and loose pointwise orderings respectively), and $(G, \preccurlyeq)$ is an l-group with non-androgynous CTRO $\leqq$, and U is the metric topology of the sup norm. The filets $\tilde{f}$ can be identified with the closed supports $\text{supp}(f)$ (Ribenboim [8], pp 42, 43), and the maximal tangents are the maximal l-ideals

$$H_{t_0} = \{f : f(t_0) = 0\}, \qquad 0 \leqq t_0 \leqq 1.$$

It is easily seen that they are not replete. In fact, if $g(t_0) = 0$ and $g(t) > 0$ for $t \neq t_0$, there exists no replete tangent containing g.

On the other hand $B[0,1]$ (the bounded functions on $[0,1]$) with the same orders is again an l-group with non-androgynous CTRO, but now the filets $\tilde{f}$ can be identified with the zero sets

$$Z(f) = \{t : f(t) = 0\},$$

while the maximal tangents are precisely the subsets of the form $\{f: Z(f) \in J\}$ where J is a maximal filter of subsets of $[0,1]$; so the maximal tangents are all replete.

13°. *If* $(G, \preccurlyeq)$ *is a non-trivial commutative l-group with non-androgynous compatible tight Riesz order* $\leqq$, *then* $\mathfrak{H}$ *is not empty.*

PROOF. The equation $\mathfrak{w} = \bar{P} \setminus \{0\}$ would imply that $x \succ 0$, $y \succ 0 \Rightarrow x \wedge y \succ 0$, hence that $(G, \preccurlyeq)$ is an antilattice as well as a lattice, and so fully ordered, contrary to assumption. Hence there exists $c \succ 0$, $c \notin \mathfrak{w}$. Write

$$K = \bigcup \{\bar{a}: \bar{a} \preccurlyeq \bar{c}\} = c^{\perp\perp}.$$

K is an l-ideal; and K cannot meet P^* since distinct carriers are disjoint subsets, $\hat{c} \prec \mathfrak{w}$ and $P^* \subseteq \mathfrak{w}$ (by 3°). By 11°, K is contained in some maximal tangent.

In the extreme case $P^* = \mathfrak{w}$ of 13°, the maximal tangents are all replete. For let $H \in \mathfrak{H}$ and write

$$L \equiv L_H = \bigcup \{\bar{a}: \bar{a} \preccurlyeq \bar{h} \text{ for some } h \in H\}.$$

L is an l-ideal. If L meets P^*, in p say, we have $\bar{p} \preccurlyeq \bar{h}$ for some $h \succ 0$, $h \in H$, so $P^* = \mathfrak{w} = \hat{p} = \hat{h}$, whence $h \in H \cap P^*$, contradiction. Since $H \subseteq L$ and H is maximal we have $H = L$, so H is replete. $B[0,1]$ is a case where $P^* = \mathfrak{w}$.

Generally, note that a maximal tangent H is replete if and only if $L_H = H$.

We proceed to discuss the realization of G, assumed commutative. For an arbitrary choice of $H \in \mathfrak{H}$ write (when emphasis or clarity requires it) $G'_H = G/H$, x'_H, $U'_H, \cdots$ for the corresponding entities, and form the full direct product

$$A = \prod_{H \in \mathfrak{H}} G'_H.$$

Let p_H denote the projection $\xi \mapsto \xi_H$, $A \to G'_H$ onto the Hth factor. We make A partially ordered group in two ways, writing

$$\xi > 0 \text{ if and only if } \xi_H >' 0' \text{ for all } H \in \mathfrak{H},$$

$$\xi \succcurlyeq 0 \text{ if and only if } \xi_H \geqq' 0' \text{ for all } H \in \mathfrak{H}$$

(recall 12°). It is easily verified that

14°. $(A, \preccurlyeq)$ *is an l-group with* $(\xi \wedge \eta)_H = \xi_H \wedge \eta_H$, $(\xi \vee \eta)_H = \xi_H \vee \eta_H$, *so that each* p_H *is a lattice homomorphism*; $\leqq$ *is a compatible tight Riesz order for* $\preccurlyeq$

Let N denote the open-interval topology of $(A, \leqq)$, and write

$$Q^* = \{\xi \in A: \xi > 0\},\ Q = Q^* \cup \{0\},$$

so that $\bar{Q} = \overline{Q^*} = \{\xi \in A: \xi \succcurlyeq 0\}$. For a subset $F \subseteq A$ let N_F denote N relativized to F.

Write ρ for the natural group homomorphism of G into A, and for $x \in G$ write $\tilde{x} = (x_H)_{H \in \mathfrak{H}}$ for the element $\xi \in A$ for which $p_H(\xi) = x'_H$, so that $\rho: x \mapsto \tilde{x}$. Clearly ρ preserves order: $x > 0 \Rightarrow \rho(x) > 0$, and $x \succcurlyeq 0 \Rightarrow \rho(x) \succcurlyeq 0$. Also $p_H[\rho(G)] = G'_H$. We look for conditions on G making ρ one-one and also making ρ^{-1} order-preserving. (If ρ^{-1} exists and ρ and ρ^{-1} are both order-preserving we call ρ *isotone*, for the order in question.) If ρ has these properties, it is a *realization* of G, in the sense of Ribenboim [8], p. 61.

We can note straight away that, without further conditions on $(G, \preccurlyeq)$, if ρ is one-one then in fact it is isotone for both orderings. For let ρ be one-one, and suppose $x \in G$, $\rho(x) \succcurlyeq 0$. Then for every $H \in \mathfrak{H}$ we have $x'_H \succcurlyeq' 0'$, and since the canonical homomorphism θ_H is a lattice homomorphism, also $0 \preccurlyeq' (x \wedge 0)'_H \preccurlyeq' 0$. Therefore $\rho(x \wedge 0) = 0$, $x \succcurlyeq x \wedge 0 = 0$. Thus ρ is $\preccurlyeq$-isotone. Now suppose $y \in G$, $\rho(y) > 0$. By what has just been proved, $y \succ 0$. If $y \not> 0$ then $\mathrm{lid}(y)$ is an l-ideal not meeting P^* (by 2° (iii)), so there exists $K \in \mathfrak{H}$, $y \in \mathrm{lid}(y) \subseteq K$; whence $y'_K = 0'$ whereas in fact $y'_H >' 0'$ for all $H \in \mathfrak{H}$. Thus $y > 0$; ρ is $\leqq$-isotone.

An l-group $(G, \preccurlyeq)$ is called *Stone* if for every $a \in G$

$$G = a^{\perp} \oplus a^{\perp\perp}. \tag{4.2}$$

Every lattice-complete l-group is a Stone l-group, and commutative (cf. Fuchs [2] Theorems 16, 18, p. 91).

Strzelecki has introduced the concept of weak projectability: a commutative l-group $(G, \preccurlyeq)$ is called *weakly projectable* if for every a, $b \in G$ there exists $z \in a^{\perp}$ such that $b \in (|a| + |z|)^{\perp\perp}$. Every commutative Stone l-group is weakly projectable. On the other hand, $C[0, 1]$ with the loose pointwise ordering is weakly projectable but not Stone (Spirason and Strzelecki [10], Speed and Strzelecki [9]).

15°. THEOREM. *If $(G, \preccurlyeq)$ is a weakly projectable l-group with a non-androgynous compatible tight Riesz order $\leqq$, whose maximal tangents are all replete, then ρ is isotone for $\leqq$ and $\preccurlyeq$, and is thus a realization of $(G, \leqq, \preccurlyeq)$ as a subdirect product of fully ordered groups.*

PROOF. By 13°, $\mathfrak{H}$ is nonempty (we assume that $\preccurlyeq$ is not a full order). By previous remarks it suffices to show that

$$\ker(\rho) = \bigcap_{H \in \mathfrak{H}} H = (0). \tag{4.3}$$

We prove that if $b \succ 0$ in G then there exists $H_b \in \mathfrak{H}$ with $b \notin H_b$. Then given any $c \neq 0$ we have $|c| \succ 0$ and so $c \notin \cap H$.

So let $b \succ 0$. Now $b^{\perp}$ is an l-ideal; since $(G, \leqq)$ is non-androgynous $b^{\perp}$ does not meet P^*, for otherwise there would exist $x > 0$, $x \wedge b = 0$, contradicting 3°(iii). By 11° there exists $H_b \in \mathfrak{H}$, $H_b \supseteq b^{\perp}$. Suppose that $b \in H_b$: then we prove $H_b = G$ as follows. Let $a \in G$; by weak projectability there exists $z \in b^{\perp}$ such that $a \in (b + |z|)^{\perp\perp}$. In any commutative l-group, $y^{\perp\perp}$ is the smallest replete l-ideal containing y. Since $b + |z| \in H_b$ and H_b is replete, $a \in (b + |z|)^{\perp\perp} \subseteq H_b$. Thus $H_b = G$, contradiction; so $b \notin H_\eta$, as required.

If $(G, \leqq)$ is androgynous then ρ cannot be one-one. For suppose $(G, \leqq)$ is androgynous; by 3°(iii) there exist $x > 0$, $y \succ 0$ with $x \wedge y = 0$. If $H \in \mathfrak{H}$ then H is a prime l-ideal by 12°, so either $x \in H$ or $y \in H$, i.e. $y \in H$ since $H \cap P^* = \emptyset$. Therefore $0 \neq y \in \bigcap_{H \in \mathfrak{H}} H$, so ρ is not one-one. (This argument is due to Davis.)

Topologically the situation for ρ is somewhat less satisfactory. As well as the open-interval topology $\boldsymbol{N}$ on A, there is the product topology $\boldsymbol{T}$, which in this context is less relevant that $\boldsymbol{N}$. We shall also need to consider the penetration of A by the image under ρ: the following possibilities suggest themselves.

(α) $\rho(P^*)$ is dense in $(Q^*, \boldsymbol{N})$: given $0 \leqq \eta < \xi$ in A there exists $a > 0$ in G such that $\eta < \rho(a) < \xi$.

(β) $0 \in \overline{\rho(P^*)}$: given $\xi > 0$ in A there exists $a > 0$ in G such that $0 < \rho(a) < \xi$.

(γ) $\rho(G)$ is dense in $(A, \boldsymbol{N})$: given $\eta < \xi$ in A there exists $x \in G$ such that $\eta < \rho(x) < \xi$.

It is clear that $(\alpha) \Rightarrow (\beta)$; it is less obvious, but true, that $(\alpha) \Rightarrow (\gamma)$. Sherman has shown, by using a result of Reilly's, that $(\gamma) \Rightarrow (\alpha)$ (personal communication). However, (β) appears to be a weaker condition. For example, let G be $C[0,1]$, ordered as before; $\mathfrak{H}$ can be identified with $[0,1]$, as we saw, and each f under ρ with itself. It follows easily that (β) does not hold.

16°. *Let $(G, \preccurlyeq)$ be an l-group with a compatible tight Riesz order $\leqq$. Then*

(i) *$\boldsymbol{T} \subseteq \boldsymbol{N}$, so that $\boldsymbol{T}_{\rho(G)} \subseteq \boldsymbol{N}_{\rho(G)}$.*
If $\mathfrak{H}$ is infinite then $\boldsymbol{T}_{\rho(G)} \neq \boldsymbol{N}_{\rho(G)}$.

(ii) *If ρ is one-one, then for all $a < b$ in G,*

$$\rho(a, b) = \rho(G) \cap \prod_{H \in \mathfrak{H}} (a'_H, b'_H) = \rho(G) \cap (\tilde{a}, \tilde{b}),$$

so $\rho: (G, \boldsymbol{U}) \to (\rho(G), \boldsymbol{N}_{\rho(G)})$ is an open map.

(iii) *The map $\rho: (G, \boldsymbol{U}) \to (A, \boldsymbol{T})$ is continuous. If ρ is one-one and (β holds then*

$$\rho: (G, \boldsymbol{U}) \to (A, \boldsymbol{N})$$

is continuous.

The proof of 16° is straightforward.

17° Corollary. *Let $(G, \preccurlyeq)$ be a commutative l-group with compatible tight Riesz order $\leqq$, and let ρ be one-one. If (β)holds, then ρ is a topological embedding of $(G, \boldsymbol{U})$ in $(A, \boldsymbol{N})$. If (γ) holds, then ρ is a concordant realization, i.e. $\rho:(G, \preccurlyeq) \to (A, \preccurlyeq)$ is a lattice isomorphism into.*

PROOF. By 16° (ii) and (iii), $\rho: (G, U) \to (\rho(G), N_{\rho(G)})$ is a homeomorphism, so ρ embeds G.

To prove ρ concordant, note that $\rho(x \wedge y) \preccurlyeq \rho(x) \wedge \rho(y)$ since ρ preserves $\preccurlyeq$, and let $\xi \preccurlyeq \rho(x)$, $\rho(y)$ in A. Let $\eta > 0$ in A; then $\xi - \eta < \rho(x)$, $\rho(y)$, and by assumption (γ) there exists $z \in G$ with $\xi - \eta < \rho(z) < \rho(x)$, $\rho(y)$, whence $z < x \wedge y$, $\xi < \rho(z) + \eta < \rho(x \wedge y) + \eta$, so $\xi \preccurlyeq \rho(x \wedge y)$. Thus $\rho(x \wedge y) = \rho(x) \wedge \rho(y)$, and similarly for $\vee$.

Putting 15° and 17° together we obtain

18° THEOREM. *Let $(G, \preccurlyeq)$ be a commutative l-group with a non-androgynous compatible tight Riesz order $\leqq$, whose maximal ideals are all replete. If $(G, \preccurlyeq)$ is weakly projectable and (β) holds then ρ is a topological embedding of (G, U) in (A, N) as well as a realization of $(G, \leqq, \preccurlyeq)$ in A. If also (γ) holds, then ρ is concordant for $\preccurlyeq$.*

Theorem 15° applies only for non-androgynous CTRO's; for these we know that $P^* \subseteq \mathfrak{w}$, so that $\mathfrak{w}$ is the maximal non-androgynous CTRO, provided it is a CTRO. The final lemma deals with this point. Let $(G, \precsim)$ be any l-group whose set of weak units $\mathfrak{w}$ is not empty, and write Q for the positive cone of $\precsim$. Since $\mathfrak{w}$ is a subsemigroup and $\mathfrak{w} \subseteq Q^*$, $\mathfrak{w}$ is the strict cone of a partial ordering on G, call it $\leqq$, making $(G, \leqq)$ a partially ordered group. Let $\preccurlyeq$ denote the associated preorder of $\leqq$. It is easily proved that $x \succ 0 \Rightarrow x \succ 0$. Let comparison of orders refer to comparison (with respect to $\subseteq$) of their positive cones. We have

19°. *Let $(G, \precsim)$ be an l-group with $\mathfrak{w} \neq \emptyset$. If $\leqq$, the order having $\mathfrak{w}$ as strict positive cone, is $TR(1,2)$ without pseudozeros, then it is the largest non-androgynous compatible tight Riesz order for $\precsim$.*

PROOF. We have only to show that $\leqq$ is a CTRO for $\precsim$. Now

$$a > 0 \quad \Rightarrow \quad \alpha \succ 0 \quad \Rightarrow \quad a \succ 0,$$

and here $\leqq$ is order-dense, $\precsim$ is a lattice order, and $\preccurlyeq$ is the associated order of $\leqq$. This is the basis on which the argument following (3.6) depends; that argument shows here that $\precsim$ and $\preccurlyeq$ coincide. Therefore $\leqq$ is compatible for $\precsim$.

In general $\mathfrak{w}$ does not give the largest CTRO: Wirth [11] has shown that for an abelian divisible l-group $(G, \preccurlyeq)$, there is a largest CTRO if and only if $(G, \preccurlyeq)$ is fully ordered.

At the other extreme, let

$\sigma = \{s$: for each $x \succ 0$, there exists a positive integer n such that $x \prec ns\}$

denote the set of strong units of $(G, \preccurlyeq)$. Wirth has shown that for an abelian divisible para-archimedean l-group $(G, \preccurlyeq)$, there is a smallest CTRO if and only if $\sigma \neq \emptyset$, and then σ is the strict positive cone of that CTRO.

In the circumstances of Theorem 15°, the carrier lattice $\mathfrak{C}$ has greatest element $\mathfrak{w}$. So if $\hat{c}$ is covered in $\mathfrak{C}$ by $\mathfrak{w}$, then it determines by

$$H_{\hat{c}} = \bigcup\{\bar{a}\colon \bar{a} \preccurlyeq \bar{c}\} = c^{\perp\perp}$$

an element of $\mathfrak{H}$. If G has only finitely many carriers, so that $\mathfrak{C}$ is a Boolean attice (Fuchs [2], p. 82), every (replete) maximal tangent is of this form, and $\mathfrak{H}$ is in one-one correspondence with the set of atoms of $\mathfrak{C}$.

A final incidental remark: the maximal tangents H need not be lattice-closed, in the sense

$$\left.\begin{array}{l} X \subseteq H \\ \bigvee X \text{ exists in } (G, \preccurlyeq) \end{array}\right\} \Rightarrow \bigvee X \in H.$$

For a counter-example take $G = B(0,1)$ with $\leqq$ and $\preccurlyeq$ the tight and loose pointwise ordering respectively; $(G, \preccurlyeq)$ is lattice-complete and $\leqq$ is a non-androgynous CTRO. Any maximal tangent containing

$$\operatorname{gr}\{f \succcurlyeq 0 : f(x) = 0 \text{ for } 0 < x < \alpha_f, \text{ for some } \alpha_f \in (0,1)\}$$

is not lattice-closed.

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