1 Introductory Material

1.1 Introduction

A recurring theme in the field of physics is the endeavor to unify a variety of distinct physical phenomena into a comprehensive framework that can offer both descriptions and explanations for each of them. One of the most astounding achievements in this endeavor is the unification of fundamental forces. When physicists realized that the forces of electricity and magnetism could be elegantly described using a single framework, it not only substantially enhanced our comprehension of these forces but also gave birth to the expansive domain of electromagnetism.

The remarkable success in unifying forces serves as a testament to the fact that seemingly unrelated phenomena can often be traced back to a common origin. This approach extends beyond the realm of forces and finds resonance in the burgeoning field of quantum information science. Within this field, a novel discipline has emerged, which seeks to identify shared characteristics among seemingly disparate quantum phenomena. The overarching theme of this approach lies in the recognition that various attributes of physical systems can be defined as "resources." This recognition not only alters our perspective on these phenomena but also seamlessly integrates them within a comprehensive framework known as "quantum resource theories."

For example, take the case of quantum entanglement. In the 1990s, it was transformed from a topic of philosophical debates and discussions into a valuable resource. This transformative shift revolutionized our perception of entanglement; it evolved from being an intriguing and nonintuitive phenomenon into the essential driving force behind numerous quantum information tasks. This new perspective on entanglement opened up a vast array of possibilities and applications, starting with its utilization in quantum teleportation and superdense coding. Today, entanglement stands as a fundamental resource in fields such as quantum communication, quantum cryptography, and quantum computing.

Given the success of entanglement theory, it is only natural to explore other physical phenomena that can also be recognized as valuable resources. Currently, there are several quantum phenomena that have been identified as such. These encompass areas such as quantum and classical communication, athermality (within the realm of quantum thermodynamics), asymmetry, magic (in the context of quantum computation), quantum coherence, Bell nonlocality, quantum contextuality, quantum steering, incompatibility of quantum measurements, and many more. The recognition of all

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these phenomena as resources enables us to unify them under the umbrella of quantum resource theories.

Resource theories serve as a crucial framework for addressing complex questions. They aim to unravel puzzles such as determining which sets of resources can be transformed into one another and the methods by which such conversions can occur. Additionally, they explore how to measure and detect different resources. If a direct transformation between particular resources is not feasible, resource theories examine the possibility of nondeterministic conversions and the computation of their associated probabilities. The introduction of catalysts into the equation further deepens the inquiry.

This investigative approach often yields profound insights into the underlying nature of the physical or information-theoretic phenomena under scrutiny – such as entanglement, asymmetry, athermality, and more. Furthermore, this perspective provides a structured framework for organizing theoretical findings pertaining to these phenomena. As demonstrated by the evolution of entanglement theory, the resource-theoretic perspective possesses the potential to revolutionize our understanding of familiar subjects.

In this context, chemistry exemplifies this framework, elucidating how abundant collections of chemicals can be converted into more valuable products. Similarly, thermodynamics fits this mold by addressing inquiries about the conversion of various types of nonequilibrium states – thermal, mechanical, chemical, and more – into one another, including the extraction of useful work from heat baths at differing temperatures.

Within the realm of quantum resource theories, a fundamental challenge arises in identifying equivalence classes of quantum systems that can be reversibly interconvert (or simulate each other) when considering an abundance of resource copies, and determining the rates at which these interconversions occur. The relative entropy of a resource plays a pivotal role in such reversible transformations, gauging the resourcefulness of a system by quantifying its deviation from the set of free (nonresourceful) systems. Remarkably, this function unifies essential (pseudo) metrics across seemingly disparate scientific domains. For instance, the relative entropy of a resource manifests as free energy in thermodynamics, entanglement entropy in pure state entanglement theory, and the entanglement-assisted capacity of a quantum channel in quantum communication; see Figure 1.1.

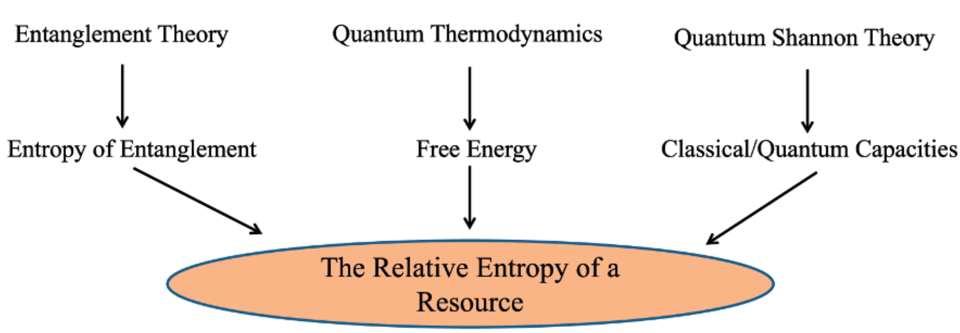

Figure 1.1 Unification of resources.

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1.2 About this Book

As mentioned in the introduction, quantum resource theories have recently emerged as a vibrant research area within the quantum information science community. Initially, the emphasis was on understanding the resources used in quantum information processing tasks. However, it has become increasingly evident that quantum resources have broad relevance, extending from quantum computing to quantum thermodynamics and the fundamental principles of quantum physics. This realization has spurred rapid developments in the field, resulting in a proliferation of publications and the development of new tools and mathematical methods that firmly underpin this area of study.

In light of the extensive literature in the field of quantum information science, one might understandably question the need for yet another book on quantum resources. Isn't this territory already covered in existing quantum information textbooks? For instance, quantum Shannon theory can be seen as a theory of interconversions among different types of resources, and Wilde [235] and Watrous [233] have produced outstanding books delving into these topics. Additionally, detailed treatments of subjects covered in this book, such as quantum divergences and Rényi entropies, can be found in Tommamichel's noteworthy work [211].

While it is accurate to say that many of the topics covered in this book are available elsewhere, what distinguishes this book is its unique approach. It explores well-trodden subjects like entropy, uncertainty, divergences, nonlocality, entanglement, and energy from a fresh perspective rooted in resource theories. Specifically, the book adopts an axiomatic approach to rigorously introduce these concepts, providing illustrative examples. Only then does it transition to operational aspects that involve the examples discussed.

Take, for instance, the topic of conditional entropy, a subject widely covered in numerous textbooks in both classical and quantum information theory. This book, however, offers a distinctive approach by presenting this concept from three distinct perspectives: axiomatic, constructive, and operational. Notably, all three perspectives converge to the same notion of conditional entropy. This approach not only provides the reader with a deeper understanding of the concept but also underscores its robust foundation.

The primary goal of this book is pedagogical in nature, with the hope of providing readers with a contemporary perspective on quantum resource theories. It aspires to equip readers with the necessary physical principles and advanced mathematical techniques required to comprehend recent advancements in this field. Upon completing this book, readers should have the ability to explore open problems and research directions within the field, some of which will be highlighted in the text.

In anticipation of a diverse readership, this book is designed to be inclusive, targeting both graduate students and senior undergraduate students who possess a foundational understanding of linear algebra. It aims to provide them with a comprehensive resource

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for delving into this fascinating field. Simultaneously, the book serves as a reference, offering fresh insights and innovative approaches that researchers in the early stages of their careers may find valuable. With numerous examples and exercises, it aims to serve as a textbook for courses on the subject, enhancing the learning experience for students.

While the primary audience for this textbook consists of entry-level graduate students interested in pursuing research at the master's or PhD level in quantum resource theories, encompassing quantum information science, it may also prove valuable to researchers in fields influenced by quantum information and resource theories, such as quantum thermodynamics and condensed matter physics. They may find this book to be a useful and accessible reference source.

Although we have endeavored to make the book self-contained, a basic understanding of linear algebra is essential. The goal was to create a resource accessible to graduate students from diverse backgrounds in mathematics, physics, and computer science. As a result, the book includes preliminary chapters and several appendices (online version) that fill potential knowledge gaps, given the interdisciplinary nature of the subject matter.

Quantum resource theories constitute a vast research area, with new properties of physical systems continually being recognized as resources. Consequently, the aim of this book is not to exhaustively cover all resource theories but rather to select those that illustrate the techniques used in quantum resource theories effectively. On the technical front, we have chosen to begin with the modern single-shot approach and employ it to derive asymptotic rates. Historically, asymptotic rates were studied first, but from a pedagogical standpoint, it is more intuitive to start with the single-shot regime.

To the best of our knowledge, there are currently no dedicated books specifically focused on quantum resource theories. With this book, we hope to contribute to the field by providing a comprehensive overview and integrating both new and existing results within a unified framework. While we do not claim this book to be the ultimate authority, we believe it can serve as a valuable reference that consolidates ideas scattered across various journal articles, addressing the need for a centralized resource in the field of quantum resource theories.

1.3 The Structure of the Book

In this book, we delve deep into the comprehensive framework of quantum resource theories, offering a detailed study of their general principles and equipping readers with the necessary tools and methodologies. We extensively cover three illustrative examples of resource theories – Entanglement, Asymmetry, and Thermodynamics – chosen for their pedagogical value in showcasing the diverse facets of quantum resource theories. While we do not have a dedicated chapter solely focused on quantum coherence, this concept is seamlessly woven into our broader discussions. It serves as a recurring illustrative example that enriches our understanding of various aspects of quantum resource theories throughout the book.

The initial volume of this book is structured into five main parts, with an additional sixth part containing supplementary materials.

Section 1 The opening section of this book is thoughtfully designed to cater to readers who may not possess prior knowledge of quantum mechanics or quantum information. Within this segment, we embark on a rigorous mathematical journey through quantum theory, emphasizing precise definitions and mathematical proofs of fundamental physical theorems. Key subjects covered in this section encompass quantum states, generalized quantum measurements, quantum channels, POVMs, and more. Moreover, this section extends its reach beyond the boundaries of quantum theory, delving into topics such as Ky Fan norms, the Strømer–Woronowicz theorem, the Pinching Inequality, the Reverse Hölder Inequality, certain hidden variable models, and other subjects that may not commonly cross the paths of graduate students in physics, mathematics, or computer science. Therefore, even those well-versed in these topics may find it beneficial to skim through this chapter briefly, as it has the potential to reveal previously undiscovered insights.

Section 2 The second section delves deep into the methodologies and tools employed within the realm of quantum resource theories and quantum information. While it explores numerous quantum information concepts, it distinguishes itself from conventional quantum information theory textbooks. The introductory chapter of this section provides an all-encompassing mathematical review of majorization theory, encapsulating recent groundbreaking discoveries, such as relative majorization, conditional majorization, and the intersection of probability theory with this field.

Subsequent chapters in this section adopt a distinctive approach to elucidate concepts associated with metrics, divergences, and entropies. These notions are introduced and dissected using techniques and insights drawn from the framework of quantum resource theories. For instance, entropy, conditional entropy, relative entropies, and other divergences are introduced as additive functions that adhere to monotonicity under the set of free operations, a foundational concept in quantum resource theories.

The final chapter in this section is dedicated to the asymptotic regime, focusing on the consequences of the "law of large numbers" in quantum information and quantum resource theories. This chapter introduces concepts such as weak and strong typicality, the method of types, classical and quantum hypothesis testing, and the symmetric subspace. These tools prove particularly valuable in the asymptotic domain of quantum resource theories when exploring interconversion rates among infinitely many resources.

In summary, although the contents of this second section share some commonalities with conventional quantum information theory textbooks, they diverge significantly by presenting concepts and tools in a unique manner. Rather than employing Venn diagrams to define key concepts like entropy, this part of the book aims to provide a comprehensive and rigorous approach to precisely define these concepts by employing axiomatic, constructive, and operational approaches. Leveraging the framework of quantum resource theories, this section offers a fresh and innovative perspective on these familiar topics.

Section 3 In the third section, we delve into the fundamental framework of quantum resource theories. Our journey begins with a meticulous mathematical elucidation of a quantum resource theory. We proceed to examine its foundational principles, including but not limited to the golden rule of free operations, resource nongenerating operations, physically implementable operations, convex and affine resource theories, state-based resource theories, as well as resource witnesses and their associated properties.

Next, we delve into the quantification of quantum resources. In this context, we introduce a plethora of resource measures and resource monotones, delving deep into their properties, which include additivity, sub-additivity, convexity, strong monotonicity, and asymptotic continuity. These concepts form the bedrock of quantum resource theories, and understanding them is pivotal.

Resource monotones and resource measures offer a valuable means of quantifying resources. Our emphasis is on divergence-based resource measures, such as the relative entropy of a resource, given their operational interpretations across various resource theories. We also explore techniques for computing these measures, including semidefinite programming, and delve into a practical approach for "smoothing" these measures, a technique commonly employed in single-shot quantum information science.

Concluding this section of the book, we introduce a rich array of resource interconversion scenarios. These encompass exact interconversions, stochastic (probabilistic) interconversions, approximate interconversions, and asymptotic interconversions. We delve into essential tools intricately linked to resource interconversions, such as the conversion distance within the single-shot regime, the asymptotic equipartition property, and the quantum Stein's lemma within the asymptotic domain. Additionally, we explore the uniqueness of the Umegaki relative entropy within the context of quantum resource theories. Our investigation extends to the evaluation of both the cost and distillation of resources, examining these processes within both the single-shot and asymptotic regimes. We have encapsulated the essence of this section of the book in Figure 1.2.

Section 4 The fourth section is dedicated to the quintessential exemplar of quantum resource theories, often referred to as the "poster child" – entanglement theory. This section comprises three chapters, each focusing on distinct facets of entanglement. The first chapter delves into the realm of pure bipartite entanglement, followed by the second chapter, which explores mixed bipartite entanglement. The third chapter, in turn, delves into the intricacies of multipartite entanglement.

Within these chapters, we leverage the techniques and concepts developed in Sections 2 and 3 to delve into the theory of entanglement. This enables us to furnish a precise definition of quantum entanglement and undertake a comprehensive examination of its detection, manipulation, and quantification. Notably, the first of these three chapters serves as the cornerstone, offering an in-depth exploration of pure

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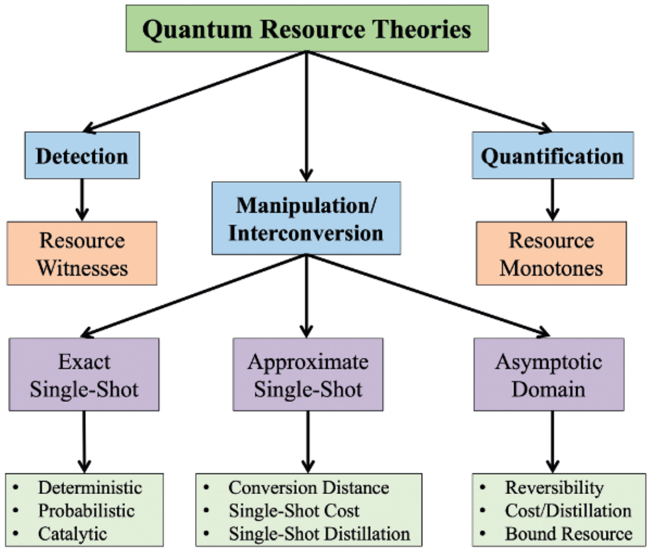

Figure 1.2 The structure of quantum resource theories

bipartite entanglement, which forms the foundational knowledge upon which the subsequent chapters on mixed and multipartite entanglement build.

Section 5 The fifth section comprises three chapters, with the first two chapters focusing on asymmetry and nonuniformity, laying the groundwork for the third chapter on quantum thermodynamics. In this section, we reveal that athermality, the resource essential for thermodynamic tasks, consists of two components: time-translation asymmetry and nonuniformity.

The first chapter explores the resource theory of asymmetry, introducing an operational framework that arises from practical constraints when multiple parties lack a common shared reference frame. This theory has found numerous applications in quantum information and beyond.

The second chapter delves into the resource theory of nonuniformity. In this theory, maximally mixed states are considered free, while all other states are regarded as valuable resources. This theory can be seen as a unique variant of thermodynamics, involving completely degenerate Hamiltonians. Indeed, we introduce this chapter to serve as a gentle introduction to the world of quantum thermodynamics.

Finally, in the third chapter of this section, we dive into quantum thermodynamics. Throughout the book, whenever we introduce a new quantum resource theory, we adhere to the structured framework outlined in Figure 1.2.

Section 6 The final section serves as a comprehensive resource aimed at ensuring the self-containment of the entire text. It exclusively includes material that directly complements the core content of the book.

In the initial three chapters, we delve into key subjects: convex analysis, operator monotonicity, and representation theory. It's important to note that each of

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these topics is vast in its own right, with numerous dedicated books solely focused on representation theory or convex analysis, for example. In this section, we have thoughtfully curated and presented the aspects of these topics that are pertinent to our book's core themes. Our approach emphasizes utilizing quantum notations and placing a strong emphasis on furnishing all the essential elements needed to ensure the book's self-contained nature.

Appendices The appendices can be downloaded from www.cambridge.org/9781009560917.

1.4 Resurrection of Quantum Entanglement: The Birth of a Fundamental Resource

In this section, we delve into the transformative protocols of quantum teleportation and superdense coding. These groundbreaking techniques marked a pivotal moment in the history of quantum physics, elevating entanglement from a purely theoretical curiosity to a precious resource with tangible applications. This paradigm shift, akin to the "resurrection" of quantum entanglement, carried profound implications for the burgeoning field of quantum information. In essence, it played a significant role in catalyzing the emergence of quantum information science. If you're new to the formalism of quantum mechanics, we recommend starting with Chapter 2 before delving into the following two sections.

Even in the early days of quantum mechanics, entanglement stood out as a distinctive and defining feature of the theory. As articulated by Schrödinger, he remarked, "I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." This statement underscores the profound departure from classical physics that entanglement embodies. Significantly, the intriguing properties of entanglement were recognized well before Bell's seminal paper on the exclusion of local hidden variable models (as discussed in Section 2.6).

To illustrate this point, consider a composite system consisting of two 1/2-spin particles, such as electrons, in the singlet state:

$$|\Psi_{-}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_{\mathbf{n}}\rangle|\downarrow_{\mathbf{n}}\rangle - |\downarrow_{\mathbf{n}}\rangle \uparrow_{\mathbf{n}}\rangle \right). \tag{1.1}$$

Here, $\{|\uparrow_{\mathbf{n}}\rangle, |\downarrow_{\mathbf{n}}\rangle\}$ forms an orthonormal basis in the complex vector space $\mathbb{C}^2$, representing the two eigenvectors of the spin observable corresponding to the "up" and "down" orientations along a direction $\mathbf{n} \in \mathbb{R}^3$. Notably, a remarkable property of this state is its independence from the specific spin direction $\mathbf{n}$ (see Chapter 2).

Now, if Alice performs a measurement in the $\mathbf{n}$ direction, it will instantaneously dictate Bob's post-measurement state to align with the opposite $\mathbf{n}$ direction. This peculiar phenomenon allows Alice to exert immediate influence on Bob's state by simply choosing whether to conduct a Stern–Gerlach measurement (as discussed in

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Section 2.1) along the $\mathbf{n}$ or $\mathbf{m}$ direction. This nonintuitive behavior of entangled composite quantum systems prompted Einstein to describe it as a "spooky action at a distance."

Beyond its profound implications from a fundamental standpoint, entanglement has gained recognition as a valuable and indispensable resource for the realization of specific quantum information processing tasks. This shift in perspective has given rise to a substantial body of research, as entanglement is no longer solely a philosophical curiosity but a powerful tool with remarkable practical applications. These applications encompass protocols like quantum teleportation, superdense coding, and numerous innovations in quantum cryptography and quantum computing.

In this section, we embark on a journey through some of these protocols, known as unit protocols, as they exclusively rely on unit noiseless resources. These protocols serve as a testament to the versatility of entanglement and employ three distinct resource types: a noiseless quantum communication channel, a noiseless classical communication channel, and the entangled bit, abbreviated as ebit.

1.4.1 Quantum Teleportation

Quantum teleportation is a groundbreaking protocol enabling Alice to transmit an unknown quantum state $|\psi\rangle$ to Bob, all without the need for a dedicated quantum communication channel. Instead, it relies on the clever utilization of entanglement and a classical communication channel to achieve this remarkable feat, as illustrated in Figure 1.3. To elucidate, consider the scenario where Alice and Bob share a composite system comprising two electrons initially prepared in the singlet state:

$$|\Psi_-^{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{1.2}$$

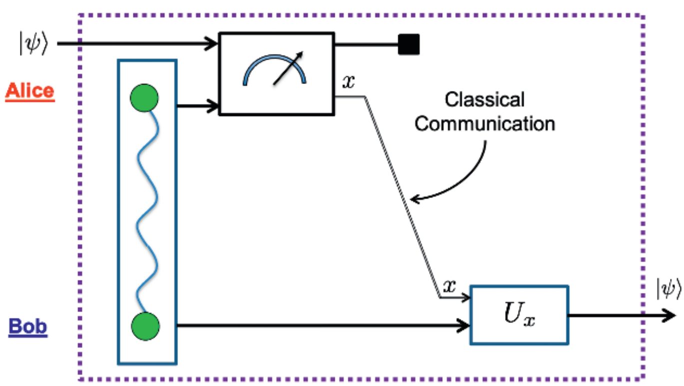

Figure 1.3 Quantum teleportation. Single-line arrows correspond to quantum systems. Double line arrows correspond to classical systems.

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Furthermore, let's consider the scenario where Alice possesses an additional electron in her system, characterized by a quantum state $|\psi^{\tilde{A}}\rangle = a|0\rangle + b|1\rangle$. Importantly, both Alice and Bob lack knowledge regarding the spin state of this electron, which means they are unaware of the specific values of a and b. According to the principles of quantum mechanics, the collective quantum state of these three electrons – two under Alice's control and one under Bob's – is described by the tensor product:

$$\begin{aligned} |\psi^{\tilde{A}}\rangle \otimes |\Psi_-^{AB}\rangle &= \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle) \otimes (|01\rangle - |10\rangle) \\ \underset{\text{Parentheses}}{\textbf{Openning}} \rightarrow &= \frac{1}{\sqrt{2}}\big(a|001\rangle + b|101\rangle - a|010\rangle - b|110\rangle\big). \end{aligned} \tag{1.3}$$

It's noteworthy that in our description in (1.3), we represented the state $|\psi^{\tilde{A}}\rangle \otimes |\Psi_-^{AB}\rangle$ using the computational basis of the vector space $\tilde{A}AB$. However, we can achieve a more insightful representation by substituting the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ of system $\tilde{A}A$ with the Bell basis consisting of $|\Phi_\pm^{\tilde{A}A}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi_\pm^{\tilde{A}A}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. This substitution allows us to express the state as follows:

$$\begin{aligned} |\psi\rangle^{\tilde{A}} \otimes |\Psi_-^{AB}\rangle &= \frac{1}{2}\Big[a\big(|\Phi_+^{\tilde{A}A}\rangle + |\Phi_-^{\tilde{A}A}\rangle\big)|1\rangle + b\big(|\Psi_+^{\tilde{A}A}\rangle - |\Psi_-^{\tilde{A}A}\rangle\big)|1\rangle \\ &\quad - a\big(|\Psi_+^{\tilde{A}A}\rangle + |\Psi_-^{\tilde{A}A}\rangle\big)|0\rangle - b\big(|\Phi_+^{\tilde{A}A}\rangle - |\Phi_-^{\tilde{A}A}\rangle\big)|0\rangle\Big] \\ \textbf{Collecting terms} \rightarrow &= \frac{1}{2}\Big[|\Phi_+^{\tilde{A}A}\rangle(a|1\rangle - b|0\rangle) + |\Phi_-^{\tilde{A}A}\rangle(a|1\rangle + b|0\rangle) \\ &\quad + |\Psi_+^{\tilde{A}A}\rangle(b|1\rangle - a|0\rangle) - |\Psi_-^{\tilde{A}A}\rangle(a|0\rangle + b|1\rangle)\Big]. \end{aligned} \tag{1.4}$$

Therefore, if Alice performs the Bell measurement on her two qubits $\tilde{A}A$, that is the basis (projective) measurement

$$\Big\{P_0 = |\Psi_-^{\tilde{A}A}\rangle\langle\Psi_-^{\tilde{A}A}|,\ P_1 = |\Phi_-^{\tilde{A}A}\rangle\langle\Phi_-^{\tilde{A}A}|,\ P_2 = |\Phi_+^{\tilde{A}A}\rangle\langle\Phi_+^{\tilde{A}A}|,\ P_3 = |\Psi_+^{\tilde{A}A}\rangle\langle\Psi_+^{\tilde{A}A}|\Big\}, \tag{1.5}$$

she will get with equal probability four possible outcomes (denoted $x = 0, 1, 2, 3$, and global phase is ignored):

Outcome	Post-Measurement State	Simplification (Up to a global phase)
$x = 0$	$\lvert\Psi_-^{\tilde{A}A}\rangle \otimes (a\lvert 0\rangle + b\lvert 1\rangle)$	$\lvert\Psi_-^{\tilde{A}A}\rangle \otimes \lvert\psi\rangle$
$x = 1$	$\lvert\Phi_-^{\tilde{A}A}\rangle \otimes (a\lvert 1\rangle + b\lvert 0\rangle)$	$\lvert\Phi_-^{\tilde{A}A}\rangle \otimes \sigma_1\lvert\psi\rangle$
$x = 2$	$\lvert\Phi_+^{\tilde{A}A}\rangle \otimes (a\lvert 1\rangle - b\lvert 0\rangle)$	$\lvert\Phi_+^{\tilde{A}A}\rangle \otimes \sigma_2\lvert\psi\rangle$
$x = 3$	$\lvert\Psi_+^{\tilde{A}A}\rangle \otimes (b\lvert 1\rangle - a\lvert 0\rangle)$	$\lvert\Psi_+^{\tilde{A}A}\rangle \otimes \sigma_3\lvert\psi\rangle$

where we denote by $\{\sigma_x\}_{x=0,1,2,3}$ the identity matrix $\sigma_0 = I_2$, and the three Pauli matrices σ_1, σ_2, and σ_3. Hence, up to a global phase, Bob's state after outcome x occurred is $\sigma_x|\psi\rangle$. After Alice sends (via a classical communication channel) the measurement outcome x to Bob, Bob can then perform the unitary operation $U_x = \sigma_x$ to obtain the state

$$\sigma_x\left(\sigma_x|\psi\rangle\right) = \sigma_x^2|\psi\rangle = |\psi\rangle. \tag{1.6}$$

Therefore, by using shared entanglement, and after transmitting two classical bits (cbits), Alice was able to transfer her unknown qubit state $|\psi\rangle$ to Bob's side.

If Bob did not receive the classical message from Alice, then his state is one of the four states $\{\sigma_x|\psi\rangle\}_{x=0,1,2,3}$. Since he does not know x, from his perspective his state is (see Exercise 1.1)

$$\rho = \frac{1}{4}\sum_{x=0}^{3}\sigma_x|\psi\rangle\langle\psi|\sigma_x = \frac{1}{2}I. \tag{1.7}$$

That is, without the knowledge of x, Bob's resulting state is the maximally mixed state, and contains no information about $|\psi\rangle$.

Exercise 1.1. *Show that for any density matrix $\rho \in \mathfrak{D}(\mathbb{C}^2)$,*

$$\frac{1}{4}\sum_{x=0}^{3}\sigma_x\rho\sigma_x = \frac{1}{2}I. \tag{1.8}$$

Hint: Prove first that the left-hand side of (1.8) is invariant under a conjugation by σ_x.

Exercise 1.2. *Show that if instead of the singlet state $|\Psi_-^{AB}\rangle$, Alice and Bob share another maximally entangled state $|\Phi^{AB}\rangle$ (i.e. the reduced density matrix of $|\Phi^{AB}\rangle$ is the maximally mixed state), then, by modifying slightly the protocol, they can still teleport an unknown quantum state from Alice to Bob.*

This protocol can be generalized in several different ways. First, in Exercise 1.3 you will generalize it to d-dimensions. Moreover, in general, if Alice and Bob do not share the singlet state, but instead their particles are prepared in some other nonseperable state (i.e. entangled state, but not maximally entangled state) $\rho^{AB} \in \mathfrak{D}(AB)$, then typically perfect/faithful teleportation will not be possible. Still in this case one can design a protocol achieving quantum teleportation with probability that is less than 1 (see Exercise 1.4), and/or in the end of the protocol the state in Bob's lab is not exactly equal to Alice's original state $|\psi\rangle^{\tilde{A}}$ but only close to it up to some treshold. Thus, the protocol described here is called *faithful* teleportation, since the protocol teleport perfectly $|\psi\rangle$ form Alice to Bob with 100% success rate.

Exercise 1.3. *Let $|\Phi^{AB}\rangle := \frac{1}{\sqrt{d}}\sum_{z\in[d]}|zz\rangle$ be a 2-qudit (normalized) maximally entangled state in $AB \cong \mathbb{C}^d \otimes \mathbb{C}^d$. Consider a family of d^2 states in AB defined by*

$$|\psi_{xy}^{AB}\rangle = T^x \otimes S^y|\Phi^{AB}\rangle, \qquad x, y \in [d], \tag{1.9}$$

where T and S are the phase and shift operators defined by $T|z\rangle = e^{i\frac{2\pi z}{d}}|z\rangle$ and $S|z\rangle = |z \oplus 1\rangle$, where $\oplus$ is the plus modulo d, and $z \in [d]$.

1. *Show that* $\{|\psi_{xy}^{AB}\rangle\}_{x,y\in[d]}$ *is an orthonormal basis of* AB.
2. *Show that the reduced density matrix of* $|\psi_{xy}^{AB}\rangle$ *is the maximally mixed state for all* $x, y \in [d]$.
3. *Find a protocol for faithful teleportation of a qudit from Alice's lab to Bob's lab. Assume that the joint measurement that Alice performs on her two qudits is a basis measurement in the basis* $\{|\psi_{xy}^{AB}\rangle\}_{x,y\in[d]}$. *What are the unitary operators performed by Bob? How many classical bits (cbits) Alice transmits to Bob?*

Exercise 1.4. *Suppose Alice and Bob share the state* $|\psi^{AB}\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$. *Show that there exists a 2-outcome (basis) measurement that Alice can perform, such that with some probability greater than zero, the state of Alice and Bob after the measurement becomes the maximally entangled state* $|\Phi_+^{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

So far we assumed that the teleported state is a pure state. However, the exact same protocol works even if the unknown state $|\psi\rangle$ is replaced with a mixed state ρ. This is because we can view any mixed state as some ensemble of pure states $\{p_x, |\psi_x\rangle\}$ in which the parameter x is unknown. Irrespective of the value of x, the protocol in this section will teleport $|\psi_x\rangle$ from Alice to Bob, and thereby, given that the value of x is unknown, Alice effectively teleported to Bob the mixed state $\rho := \sum_x p_x|\psi_x\rangle\langle\psi_x|$. Alternatively, note that the quantum teleportation protocol in Figure 1.3 can be described as a realization of the identity quantum channel $\mathsf{id} \in \mathrm{CPTP}(A \to B)$ (with $|A| = |B| := d$) given by

$$\mathsf{id}^{A\to B}(\rho^A) = \sum_{x\in[d^2]} \mathrm{Tr}_{A\tilde{A}}\left[\left(P_x^{\tilde{A}A} \otimes U_x^B\right)\left(\rho^A \otimes \Phi^{\tilde{A}B}\right)\left(P_x^{\tilde{A}A} \otimes U_x^B\right)^*\right], \tag{1.10}$$

where $\{P_x^{\tilde{A}A}\}_{x\in[d^2]}$ corresponds to the measurement on systems $\tilde{A}$ and A in the maximally entangled basis, U_x is the unitary performed by Bob after he received the value x from Alice, and Φ^{AB} is the maximally entangled state on system AB. The quantum teleportation protocol states that there exists $\{P_x^{\tilde{A}A}\}$ and $\{U_x\}$ such that the quantum channel $\mathsf{id}^{A\to B}$ above is indeed the identity channel. Although, in the protocol above we proved it only for pure input states $|\psi\rangle\langle\psi|$, from the linearity of the quantum channel $\mathsf{id}^{A\to B}$, it follows that $\mathsf{id}^{A\to B}$ is the identity quantum channel on all mixed states.

Exercise 1.5 (Entanglement Swapping). *Consider four qubit systems A, B, C, and D, in the double-singlet state* $|\Psi_-^{AB}\rangle \otimes |\Psi_-^{CD}\rangle$.

1. *Show that a joint Bell measurement on system* BC *generates a maximally entangled state on system* AD *(along with another maximally entangled state on system* BC*) for all four possible outcomes of the measurement (see Figure 1.4).*
2. *Show that the singlet state in* AD *can be generated by quantum teleportation between system* BC *and system* D.
3. *Generalize the entanglement swapping protocol to four qudit systems each of dimension* d.

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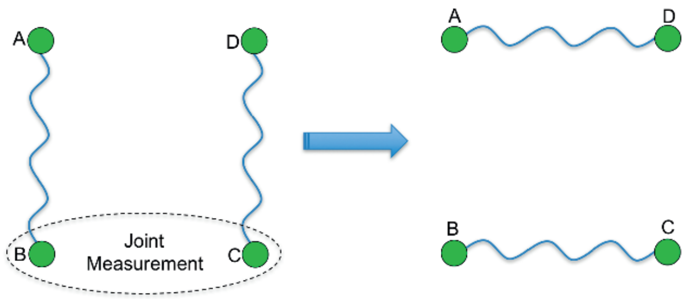

Figure 1.4 Entanglement swapping.

1.4.2 Superdense Coding

How much classical information can be transmitted with a single qubit? Suppose Alice wants to transmit a classical message to Bob, but all she has at her disposal is a single qubit (e.g. spin of an electron) and a perfect noiseless quantum communication channel that she can use to transmit the electron to Bob. She can prepare the single electron in the spin that she wants and send the electron to Bob over the noiseless quantum channel. Alice and Bob agree at the beginning that a message 0 corresponds to spin up in the z direction, and a message 1 corresponds to spin down in the z direction. In this way, Alice can transmit one cbit with one use of a perfectly noiseless quantum channel. Can they do better? We will see later on that it is not possible to encode more than one classical bit into a single electron, as long as Alice's electron is not entangled with another electron in Bob's system.

Suppose now that Alice's electron is maximally entangled with another electron in Bob's lab, so that Alice and Bob share the singlet state $|\Psi_-^{AB}\rangle$. In the first step of the protocol, Alice encodes a message x into a her qubit. She is doing it by performing one out of several (unitary) rotations $\{U_x^A\}_{x=0}^{m-1}$ on her qubit (electron). If Alice chose to do the x rotation, the state of the system after the rotation is

$$|\psi_x^{AB}\rangle := \left(U_x^A \otimes I^B\right)|\Psi_-\rangle^{AB}. \tag{1.11}$$

Taking $U_x = \sigma_x$ to be the four Pauli matrices (with $\sigma_0 = I_2$) we get that the four states $\{|\psi_x^{AB}\rangle\}_{x=0}^3$ are orthonormal and form a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$. In fact, this is the Bell basis we encountered in the previous subsection. In the next step of the protocol, Alice sends her electron (over a noiseless quantum communication channel) to Bob. Upon receiving Alice's electron, Bob has in his lab two electrons in the state $|\psi_x^{AB}\rangle$. Given that the set of states $\{|\psi_x^{AB}\rangle\}_{x=0}^3$ form an orthonormal basis, in the last step of the protocol, Bob performs a joint basis measurement on his two electrons, in the basis $\{|\psi_x^{AB}\rangle\}_{x=0}^3$, and thereby learns the outcome x. The outcome x is the message that Alice intended to send Bob.

Exercise 1.6. *Show that the set of states $\{|\psi_x^{AB}\rangle\}_{x=0}^3$ is an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.*

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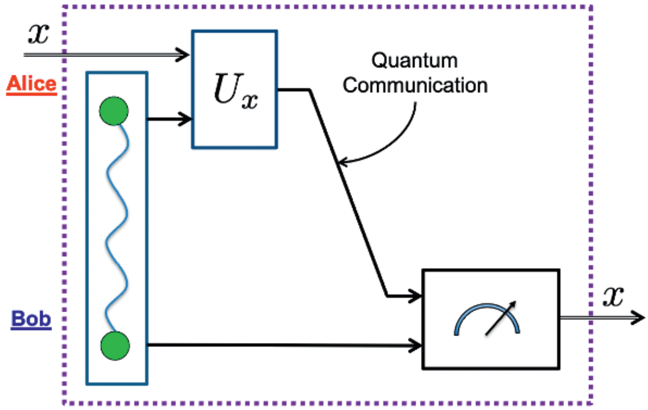

Figure 1.5 Superdense coding. Double lines correspond to classical systems, and single lines to quantum systems.

Exercise 1.7. *Let* $|\Phi^{AB}\rangle := \frac{1}{\sqrt{d}} \sum_{z\in[d]} |zz\rangle$ *be a maximally entangled state in* $\mathbb{C}^d \otimes \mathbb{C}^d$*. Show that Alice can use it to transmit to Bob* $2\log_2(d)$ *cbits.*

1.5 Resource Analysis and Reversibility

The previous two protocols demonstrate that entanglement is a valuable resource with which certain tasks will not be possible without it. We have seen in the protocols above that entanglement can be converted to other types of resources such as quantum or classical communication channels. We will use the following notations to denote these resources:

1. $[qq]$ denotes one ebit; that is, a unit of a static noiseless resource comprising of two qubits in a maximally entangled state.
2. $[q \to q]$ denotes one use of an ideal (noiseless) qubit channel.
3. $[c \to c]$ denotes one use of a classical bit channel capable of transmitting perfectly one classical bit.
4. $[cc]$ denotes one bit of shared randomness.

Observe that these resource units can be classified into being classical or quantum, and static (such as $[qq]$ and $[cc]$) or dynamic (such as $[q \to q]$ or $[c \to c]$).

With these notations, the teleportation can be viewed as a process in which one ebit plus two uses of a classical bit channel are consumed to simulate a qubit channel. In resource symbols this can be characterized as the following resource inequality:

$$[qq] + 2[c \to c] \geqslant [q \to q]. \tag{1.12}$$

Note that the use of the inequality here is justified by the fact that a single use of a quantum channel cannot generate both an entangled state *and* a double use of a classical channel.

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For superdense coding, an ebit plus one use of a quantum channel is used to simulate two uses of a classical channel. This can be expressed as the resource inequality

$$[qq] + [q \to q] \geqslant 2[c \to c]. \tag{1.13}$$

Note that if entanglement is not considered as a resource, that is, the parties are supplied with unlimited singlet states, then we can remove the ebit cost $[qq]$ in (1.12) and (1.13) and get that for teleportation $2[c \to c] \geqslant [q \to q]$ and for superdense coding $[q \to q] \geqslant 2[c \to c]$. This makes teleportation and superdense coding the dual protocols of each other, and in this case we can say that $[q \to q] = 2[c \to c]$.

However, in almost all practical scenarios, entanglement is an expensive resource that can be difficult to generate over long distances and that is also highly sensitive to decoherence and noise. Therefore, specifically pure maximally entanglement is scarce, and must be treated as a resource. The question then becomes if it is possible to change slightly the protocols of teleportation and superdense coding, making them more symmetric, in the sense that the two resource inequalities in (1.12) and (1.13) merge into a single resource equality. This is indeed possible if we replace $2[c \to c]$ in the right-hand side of (1.13) with two uses of an isometry channel known as the coherent bit channel.

1.5.1 The Coherent Bit Channel

We introduce here another unit resource that is called the *coherent bit channel*, or in short the *cobit* channel, and is denoted by $[q \to qq]$. As the symbol suggests, this unit resource represents one use of a channel. The channel is defined by the isometry, $V: A \to A \otimes B$, with $|A| = |B| = 2$, according to the following action on the basis of A:

$$V|x\rangle^A = |x\rangle^A|x\rangle^B \quad \forall\, x \in \{0,1\} \quad \text{or equivalently} \quad V = \sum_{x=0}^{1} |x\rangle\langle x|^A \otimes |x\rangle^B. \tag{1.14}$$

We will denote by

$$\mathcal{V}_Z(\rho) := V\rho V^*, \qquad \forall\, \rho \in \mathfrak{L}(A), \tag{1.15}$$

where the subscript Z indicates that the basis $\{|0\rangle, |1\rangle\}$ is an eigenbasis of the third Pauli operator (i.e. eigenvectors of the spin observable in the z-direction). One can define V with respect to other bases. For example, we will denote by $\mathcal{V}_X(\cdot) = U(\cdot)U^*$ the coherent bit channel with respect to the basis $\{|+\rangle, |-\rangle\}$, where U is the isometry defined by $U|\pm\rangle^A = |\pm\rangle^A|\pm\rangle^B$.

How is this resource related to other resources? First note that with such a resource Alice can transmit a classical bit to Bob. Indeed, Alice can encode a cbit $x \in \{0,1\}$ in the state $|x\rangle^A$ and send it over the channel $\mathcal{V}_Z$. Then, Bob receives $|x\rangle^B$ on his system and performs a basis measurement to learn x. We therefore have

$$[q \to qq] \geqslant [c \to c]. \tag{1.16}$$

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Exercise 1.8 shows that we also have $[q \to qq] \geqslant [qq]$. Among other things, this also implies that $[c \to c] \ngeqslant [q \to qq]$ or in other words, $[c \to c]$ is strictly less resourceful than $[q \to qq]$.

Exercise 1.8. *Show that* $\mathcal{V}_Z\left(|+\rangle\langle+|^A\right) = |\Phi_+^{AB}\rangle\langle\Phi_+^{AB}|$.

1.5.2 Coherent Superdense Coding

For superdense coding, we saw that an ebit, $[qq]$, plus one use of a quantum channel, $[q \to q]$, can be used to simulate two uses of a classical channel, $2[c \to c]$. We now show that the same resources can also be used to simulate two uses of the coherent map $\mathcal{V}$. That is, we will show that

$$[qq] + [q \to q] \geqslant 2[q \to qq]. \tag{1.17}$$

Note that due to (1.16), the above equation also implies the resource inequality (1.13). The quantum protocol that achieves this resource conversion is called *coherent* superdense coding.

Coherent superdense coding protocol (see Figure 1.6) consists of several steps. Initially, Alice and Bob share the maximally entangled state $|\Phi_+^{AB}\rangle$. Alice then prepares an input state $|x\rangle^{A_1}|y\rangle^{A_2}$ so that Alice and Bob's initial state (time t_0 is the figure) is

$$|x\rangle^{A_1}|y\rangle^{A_2}|\Phi_+^{AB}\rangle. \tag{1.18}$$

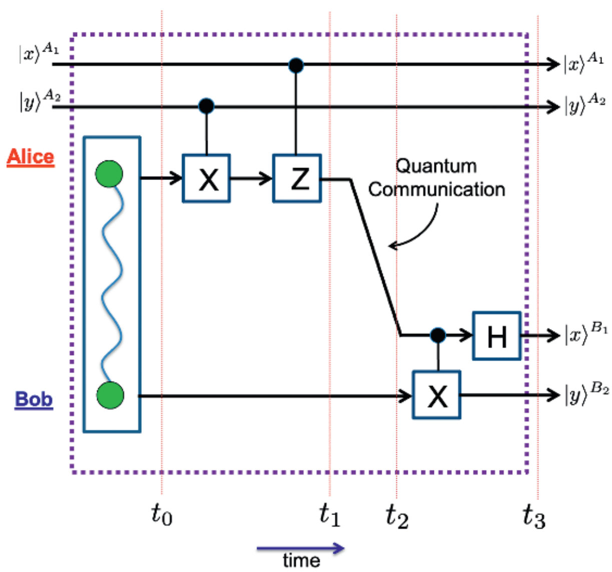

Figure 1.6 Coherent superdense coding. One ebit plus one use of a noiseless qubit channel are implemented to realize two uses of the cobit channel.

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Alice then performs a sequence of two controlled unitary gates, controlled X on system A_2 and A, followed by controlled Y gate on system A_1 and A. The resulting state at time t_1 is

$$|x\rangle^{A_1}|y\rangle^{A_2}|\phi_{xy}^{AB}\rangle, \quad \text{where} \quad |\phi_{xy}^{AB}\rangle := \left(Z^x X^y \otimes I^B\right)|\Phi_+^{AB}\rangle \quad \text{and} \quad x,y \in \{0,1\}, \tag{1.19}$$

where Z^x equals the identity matrix for $x = 0$, and the third Pauli matrix for $x = 1$ (X^y is defined similarly). A key observation is that $\{|\phi_{xy}^{AB}\rangle\}_{x,y\in\{0,1\}}$ is precisely the Bell basis, and therefore forms an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$. Note also that this encoding $(x,y) \to |\phi_{xy}^{AB}\rangle$ is done by Alice alone, and therefore essentially identical to the superdense coding protocol we encountered earlier.

In the next step Alice uses a noiseless qubit channel to transmit system A to Bob. Therefore, at time t_2 the state of the system is $|x\rangle^{A_1}|y\rangle^{A_2}|\phi_{xy}^{B_1B_2}\rangle$, where as before,

$$|\phi_{xy}^{B_1B_2}\rangle := \left(Z^x X^y \otimes I^{B_2}\right)|\Phi_+^{B_1B_2}\rangle. \tag{1.20}$$

Since both $\{|\phi_{xy}^{B_1B_2}\rangle\}_{x,y\in\{0,1\}}$ and $\{|xy\rangle^{B_1B_2}\}_{x,y\in\{0,1\}}$ are orthonormal bases of Bob's two qubit space B_1B_2, we conclude that there exists a unitary matrix $U^{B_1B_2}$ such that

$$|xy\rangle^{B_1B_2} = U^{B_1B_2}|\phi_{xy}^{B_1B_2}\rangle \quad \forall\, x,y \in \{0,1\}. \tag{1.21}$$

It turns out that the unitary U^{AB} as already defined can be expressed as a CNOT gate followed by a Hadamard gate on system B_1 (see Bob's side in Figure 1.6 between time steps t_2 and t_3). Explicitly,

$$\begin{aligned} U^{B_1B_2} &= \left(H|0\rangle\langle 0|^{B_1}\right) \otimes I^{B_2} + \left(H|1\rangle\langle 1|^{B_1}\right) \otimes X^{B_2} \\ &= |+\rangle\langle 0|^{B_1} \otimes I^{B_2} + |-\rangle\langle 1|^{B_1} \otimes X^{B_2}. \end{aligned} \tag{1.22}$$

Hence, after the application of the unitary $U^{B_1B_2}$ on Bob's system, Alice and Bob state is $|x\rangle^{A_1}|y\rangle^{A_2}|x\rangle^{B_1}|y\rangle^{B_2}$. That is, the quantum circuit in Figure 1.6 simulates the linear transformation

$$|x\rangle^{A_1}|y\rangle^{A_2} \to |x\rangle^{A_1}|x\rangle^{B_1} \otimes |y\rangle^{A_2}|y\rangle^{B_2}, \tag{1.23}$$

which is equivalent to two coherent channels. The resources we used to simulate these two coherent channels are precisely the same ones used in superdense coding to simulate two noiseless classical channels.

Exercise 1.9. *Show that the unitary matrix $U^{B_1B_2}$ above satisfies (1.21).*

Exercise 1.10. *Suppose the initial ebit shared between Alice and Bob was given in the singlet state $|\Psi_-^{AB}\rangle$ instead of $|\Phi_+^{AB}\rangle$, and consider the exact same protocol as in Figure 1.6, until time step t_2. Revise the unitary matrix $U^{B_1B_2}$ after time step t_2 so that the protocol still simulates two coherent channels.*

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1.5.3 Coherent Teleportation

The coherent teleportation protocol is the resource reversal of the coherent superdense coding protocol. Particularly, it reveals that two uses of a cobit channel are sufficient to generate one ebit and at the same time simulate one use of a qubit channel. That is, the coherent teleportation protocol demonstrates that

$$2[q \to qq] \geqslant [qq] + [q \to q]. \tag{1.24}$$

The protocol achieving this resource inequality is depicted in Figure 1.7.

In the first step of the protocol Alice sends a qubit $|\psi^A\rangle = a|0\rangle^A + b|1\rangle^A$ into the first cobit channel. The cobit channel $\mathcal{V}_Z$ transforms this state into the state

$$a|0\rangle^A|0\rangle^{B_1} + b|1\rangle^A|1\rangle^{B_1} = \frac{1}{\sqrt{2}}\left[|+\rangle^A\left(a|0\rangle^{B_1} + b|1\rangle^{B_1}\right) + |-\rangle^A\left(a|0\rangle^{B_1} - b|1\rangle^{B_1}\right)\right]. \tag{1.25}$$

In the next step, system A goes through the second cobit channel, $\mathcal{V}_X$, yielding the state

$$\frac{1}{\sqrt{2}}\left[|+\rangle^A|+\rangle^{B_2}\left(a|0\rangle^{B_1} + b|1\rangle^{B_1}\right) + |-\rangle^A|-\rangle^{B_2}\left(a|0\rangle^{B_1} - b|1\rangle^{B_1}\right)\right]. \tag{1.26}$$

Finally, in the last step, Bob sends his systems through a CNOT gate. Note that $X|+\rangle = |+\rangle$ so that the CNOT gate only changes $|-\rangle^{B_2}|1\rangle^{B_1}$ to $-|-\rangle^{B_2}|1\rangle^{B_1}$ while keeping all the other terms intact. Hence, after Bob's CNOT gate, Alice and Bob share the state

$$\frac{1}{\sqrt{2}}\left[|+\rangle^A|+\rangle^{B_2}\left(a|0\rangle^{B_1} + b|1\rangle^{B_1}\right) + |-\rangle^A|-\rangle^{B_2}\left(a|0\rangle^{B_1} + b|1\rangle^{B_1}\right)\right] = |\Phi_+^{AB_2}\rangle|\psi^{B_1}\rangle. \tag{1.27}$$

That is, at the end of the protocol Alice teleported her quantum state $|\psi\rangle$ to Bob's system B_1, and also share with Bob's system B_2 the maximally entangled state $\Phi_+^{AB_2}$.

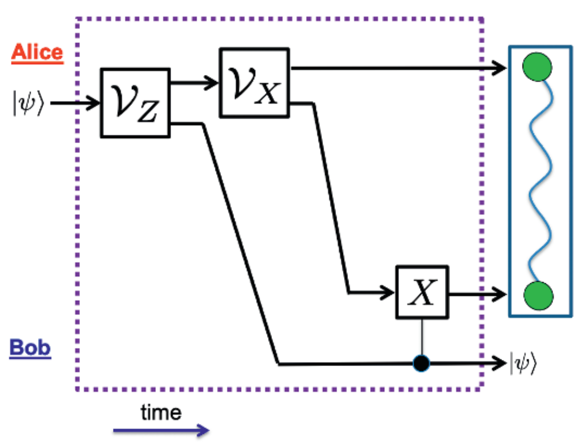

Figure 1.7 Coherent quantum teleportation. Two cobit channels produce one ebit plus one use of a noiseless qubit channel.

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Coherent quantum teleportation and coherent superdense coding demonstrate that two cobit channels have the same resource value as one ebit and one use of a qubit channel:

$$[qq] + [q \to q] = 2[q \to qq]. \tag{1.28}$$

This means that coherent teleportation is the reversal process of coherent superdense coding and vice versa.

1.6 Notes and References

Quantum teleportation was discovered in Ref. [20], and quantum superdense coding in Ref. [23]. These seminal papers paved the way for the development of quantum Shannon theory and entanglement theory, as they demonstrated that entanglement, besides being interesting from a fundamental point of view, is a resource that can be consumed to achieve certain exotic tasks such as quantum teleportation. Moreover, these protocols are considered as the *unit* protocols (see, for example, Ref. [235]), since they form the building blocks with which one studies the capabilities of noisy quantum channels to transmit information in asymptotic settings involving many uses of the channels.

The resource analysis using notations such as $[q \to q]$ was first introduced in Ref. [62] where the rules of this "resource calculus" developed. The coherent bit, coherence teleportation, and coherent superdense coding are due to Ref. [112]. More details on coherent communication can be found in the book of Wilde [235].