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A SIMPLE TEST FOR NUCLEARITY OF INTEGRAL OPERATORS ON $L_2(R^n)$

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Abstract

This note shows how the rate of decay of the singular values can be easily estimated for a wide class of integral operators on $L_2(R^n)$.

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If A is a compact operator on a Hilbert space, then the spectrum of A^*A is a sequence of positive eigenvalues converging to zero,

$$s_1(A)^2 \geqslant s_2(A)^2 \geqslant \cdots$$

The number $s_k(A)$ is called the kth singular value of A. The collection of operators,

$$S_p = \left\{ A \mid \sum_{k=1}^{\infty} s_k(A)^p < \infty \right\},$$

is an ideal in the algebra of bounded operators, and is complete in the topology induced by the norm

$$|A|_p = \left[\sum_{k=1}^{\infty} s_k(A)^p \right]^{1/p}.$$

In particular, S_1 and S_2 consist of nuclear and Hilbert-Schmidt operators, respectively. The ideals are nested, with

$$S_p \subset S_q, \qquad p < q,$$

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and

(1) $$S_p S_q \subset S_r,$$

where

$$r^{-1} = p^{-1} + q^{-1}.$$

From (1) we see that whenever an operator A can be factorized,

(2) $$A = BC,$$

with $B \in S_p$ and $C \in S_q$, then it will follow that $A \in S_r$. These assertions are well known and can be found in the text on non-selfadjoint operators by Gohberg and Krein (1969).

In this note we consider an integral operator A on $L_2(R^n)$ with kernel $A(\cdot, \cdot)$,

$$(Af)(y) = \int_{R^n} A(y, x) f(x)\, d^n x,$$

and represent it in the form (2) with

$$B = AD \quad \text{and} \quad C = D^{-1},$$

where D is a specially chosen differential operator. After integration by parts, the factor B is an integral operator. If its kernel is a Hilbert-Schmidt kernel, a condition that is easy to check, then we obtain an estimate for the rate of decay of the singular values of A. The operator C, which is the resolvent of the differential operator, need not be constructed explicitly.

LEMMA. *Suppose that A is a bounded integral operator whose kernel, $A(y, x)$, is twice continuously differentiable with respect to x. Construct the kernel*

$$B(y, x) = \left(-\nabla^2 + |x|^a\right) A(y, x), \qquad a > 0.$$

If

$$\int_{R^n} \int_{R^n} |B(y, x)|^2\, d^n y\, d^n x < \infty,$$

so that $B \in S_2$, then $A \in S_r$ for all $r > t$, where

$$t = \frac{2n(a+2)}{4a + n(a+2)}, \quad \textit{and} \quad s_k(A) = o(k^{-1/t}) \quad \textit{as } k \to \infty.$$

This trick of extracting the resolvent of a differential operator as a factor in an integral operator is due to Gohbert and Krein (1969), page 120, who used it to prove the nuclearity of an integral operator on $L_2(a, b)$, $-\infty < a < b < \infty$. The factorization trick has also been extensively studied by Birman and Solomyak (1977) in a series of papers on the asymptotic rate of decay of the singular values

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of integral operators, and their 1977 paper provides an excellent guide to the literature. The merit of the factorization in this note is its simplicity and its direct application to kernels on the whole of R^n, rather than a compact domain.

PROOF. Consider the differential operator

$$D = -\nabla^2 + |x|^a$$

on $L_2(R^n)$ with domain the set of test functions with rapid decrease at infinity. Titchmarsh (1958) shows that this symmetric operator is essentially self-adjoint, and that the spectrum of its closure consists of a sequence of discrete eigenvalues, $\lambda_1 \leq \lambda_2 \leq \cdots$, whose corresponding eigenvectors are complete. The eigenvalues are strictly positive, because $-\nabla^2$ is positive and $|x|^a$ is strictly positive. Titchmarsh (1958) also shows that the number $N(\lambda)$ of eigenvalues in the interval $[0, \lambda)$ is asymptotically given by

$$N(\lambda) \sim \left[2^n \pi^{n/2} \Gamma(\tfrac{1}{2}n + 1)\right]^{-1} \int_{|x|^a < \lambda} \left[\lambda - |x|^a\right]^{n/2} d^n x = c\lambda^p$$

where $p = n(2^{-1} + a^{-1})$ and

$$c = \frac{1}{2^{n-1}a} \frac{\Gamma(n/a)}{\Gamma(n/2)\Gamma(n/a + n/2 + 1)}$$

Thus, $k = N(\lambda_k) \sim c\lambda_k^p$ and so

$$\sum_{k=1}^{\infty} \lambda_k^{-q} < \infty \quad \text{for all } q > p.$$

Let C denote the inverse of the differential operator. It is clearly compact and its singular values are $s_k(C) = \lambda_k^{-1}$, so $C \in S_q$. Now, for any $f \in L_2(R^n)$, let

$$g(x) = \int_{R^n} C(x, z) f(z)\, d^n z,$$

where $C(x, z)$ is the kernel of C (Green's function for D), so that

$$(-\nabla^2 + |x|^a) g(x) = f(x).$$

Then, after integrating by parts, we obtain

$$\int_{R^n} A(y, x) f(x)\, d^n x = \int_{R^n} A(y, x)(-\nabla^2 + |x|^a) g(x)\, d^n x$$

$$= \int_{R^n} B(y, x) \left(\int_{R^n} C(x, z) f(z)\, d^n z \right) d^n x.$$

Thus, we have factorized A with $B \in S_2$ and $C \in S_q$ so we conclude that $A \in S_r$ for all

$$r > (2^{-1} + p^{-1})^{-1} = t.$$

Using the relation $s_{k+l-1}(A) \leqslant s_l(B)s_k(C)$, valid for any $k, l \geqslant 1$, with $l = k$ and again with $l = k + 1$, we find that the sequences $\{k^{1/p}s_{2k-1}(A)\}$ and $\{k^{1/p}s_{2k}(A)\}$ are square summable, so

$$\sum_{k=1}^{\infty} s_k(A)^2 k^{2/p} < \infty.$$

Hence $s_k(A)^2 k^{1+2/p} \to 0$ and

$$s_k(A) = o(k^{-1/t}). \tag{3}$$

(The author thanks the referee for help in sharpening estimate (3).)

Finally, we note that potentials other than $|x|^a$ could be used in the differential operator to obtain possibly sharper estimates.

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