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METRICAL PROBLEMS IN DIOPHANTINE APPROXIMATION

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This thesis is concerned with metric approximation problems. The classical paragon is that of Dirichlet's theorem: the approximation of irrationals by rationals. Many questions can be answered with Dirichlet's theorem, but also many remain, such as how can one distinguish between two sets each of which has 'asymptotic size zero'.

The well-known results of Khinchin and Jarník from the 1920s are fundamental in the theory of asymptotic Diophantine approximation, which is concerned with strengthening a corollary of Dirichlet's theorem. By contrast, the theory of uniform Diophantine approximation investigates improving Dirichlet's theorem itself. In this thesis, looking at both classical results as well as relatively recent work by Kleinbock and Wadleigh [4], we present several new results concerning metrical description of the sets of Dirichlet nonimprovable numbers.

Chapter 1 gives a short overview of classical results in Diophantine approximation and its metrical theory and ends with the definition of the set of Ψ-Dirichlet improvable numbers, whose detailed study is one of the motivations of this thesis.

Chapter 2 is devoted to some auxiliary results from metrical theory along with elementary results on continued fractions.

Chapter 3 includes the Lebesgue measure result for the set of Dirichlet nonimprovable numbers established by Kleinbock and Wadleigh [4], who were the first to characterise the set of Dirichlet nonimprovable numbers in terms of the growth of their continued fraction entries.

Chapters 4–6 contain the main new results and solve three problems in uniform Diophantine approximation.

Chapter 4 provides the first main result of this thesis. To state the result, we introduce some notation. Let $\Psi : [1, \infty) \to \mathbb{R}^+$ be a nondecreasing function, $a_n(x)$ the nth partial quotient and $q_n(x)$ the denominator of the nth convergent of the continued

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fraction of x. The set

$$G(\Psi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) > \Psi(q_n(x)) \text{ for infinitely many } n \in \mathbb{N}\}$$

of Dirichlet nonimprovable numbers, represented in terms of entries of continued fractions, is related to the classical set of $1/(q\Psi(q))$-well approximable numbers $\mathcal{K}(\Psi)$ in the sense that $\mathcal{K}(3\Psi) \subset G(\Psi)$.

It is shown that $G(\Psi) \setminus \mathcal{K}(3\Psi)$ is uncountable by calculating the Hausdorff dimension of this set. Not surprisingly it turns out to be equal to the Hausdorff dimension of $\mathcal{K}(\Psi)$ and $G(\Psi)$. The content of this chapter is published in [3].

Chapter 5 presents the second main result of the thesis. It is very natural to ask what happens if in the sets $\mathcal{K}(\Psi)$ and $G(\Psi)$ defined above, the approximation function Ψ is evaluated at the index n and not at q_n. This question is addressed in this chapter. It is proved that the Hausdorff dimension of the difference set, that is,

$$\mathcal{F}(\Phi) := \left\{x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array}\right\},$$

where $\Phi : \mathbb{N} \to (1, \infty)$ is any function with $\lim_{n\to\infty} \Phi(n) = \infty$, is positive and hence the set is nontrivial. This in turn contributes to the metrical theory of continued fractions. The content of this chapter is published in [2].

Chapter 6 consists of the third main result of this thesis and generalises the Jarník–Besicovitch set in the setting of uniform Diophantine approximation. It takes into account the set of points $x \in [0,1)$ for which the product of an arbitrary block of consecutive partial quotients in their continued fraction expansions is growing. Motivation for considering this problem arose from the observation that the Jarník–Besicovitch set is concerned with the growth of one partial quotient, whereas the set of Dirichlet nonimprovable numbers is concerned with the growth of the product of consecutive partial quotients. For any $r \in \mathbb{N}$, we investigate the Hausdorff dimension of the set

$$\mathcal{R}_r(\tau; h) := \left\{x \in [0,1) : \prod_{d=1}^{r} a_{n+d}(x) \geq e^{\tau(x)(S_n h(x))} \text{ for infinitely many } n \in \mathbb{N}\right\},$$

where h and τ are positive continuous functions, $S_n h(x) := h(x) + \cdots + h(T^{n-1}(x))$ is the ergodic sum and T represents the Gauss map. The content of this chapter is published in [1].

Chapter 7 summarises some recent advances in the theory of uniform Diophantine approximation concerned with the metrical theory of the sets of Dirichlet nonimprovable numbers.

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