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A MEAN VALUE RELATED TO D. H. LEHMER'S PROBLEM AND THE RAMANUJAN'S SUM*

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Abstract. Let $q > 1$ be an odd integer and c be a fixed integer with $(c, q) = 1$. For each integer a with $1 \leq a \leq q - 1$, it is clear that there exists one and only one b with $0 \leq b \leq q - 1$ such that $ab \equiv c \pmod q$. Let $N(c, q)$ denotes the number of all solutions of the congruence equation $ab \equiv c \pmod q$ for $1 \leq a, b \leq q - 1$ in which a and $\overline{b}$ are of opposite parity, where $\overline{b}$ is defined by the congruence equation $b\overline{b} \equiv 1 (\bmod q)$. The main purpose of this paper is using the mean value theorem of Dirichlet L-functions to study the mean value properties of a summation involving $(N(c, q) - \frac{1}{2}\phi(q))$ and Ramanujan's sum, and give two exact computational formulae.

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1. Introduction. Let p be an odd prime and c be a fixed integer with $(c, p) = 1$. For each integer a with $1 \leq a \leq p - 1$, it is clear that there exists one and only one b with $0 \leq b \leq p - 1$ such that $ab \equiv c \pmod p$. Let $M(c, p)$ denotes the number of cases in which a and b are of opposite parity. In [**6**], Professor D. H. Lehmer asked to study $M(1, p)$ or at least to say something non-trivial about it. It is known that $M(1, p) \equiv 2$ or $0 \pmod 4$ when $p \equiv \pm 1 \pmod 4$. For general odd number, $q \geq 3$, Wenpeng [**7**] and [**8**] studied the asymptotic properties of $M(1, q)$, and obtained a sharp asymptotic formula:

$$M(1, q) = \frac{1}{2}\phi(q) + O\left(q^{\frac{1}{2}} d^2(q) \ln^2 q\right),$$

where $\phi(q)$ denotes the Euler function, and $d(q)$ is the number of divisors of q.

Wenpeng [**11**] also studied the asymptotic properties of the mean square value of the error term $M(a, p) - \frac{p-1}{2}$ and gave the asymptotic formula

$$\sum_{a=1}^{p-1} \left(M(a, p) - \frac{p-1}{2} \right)^2 = \frac{3}{4}p^2 + O\left(p \exp\left(\frac{3 \ln p}{\ln \ln p} \right) \right).$$

Now, we let q be an odd integer, c be any integer with $(c, q) = 1$, $N(c, q)$ denotes the number of pairs of integers a, b with $ab \equiv c \pmod q$ for $1 \leq a, b \leq q - 1$ in which a and $\overline{b}$ are of opposite parity, and

$$E(c, q) = N(c, q) - \frac{1}{2}\phi(q).$$

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The main purpose of this paper is to use the analytic method and the properties of the Dirichlet L-functions to study the mean value computational problem of the function $R_q(c+1)E(c,q)$, where $R_q(c)$ is the Ramanujan's sum, defined as follows (see Theorem 8.6 in [**1**]):

$$R_q(c) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{\frac{2\pi ikc}{q}} = \sum_{d|(c,q)} d\mu(q/d),$$

where $\mu(n)$ is the famous Möbius function.

About the mean value of $R_q(c+1)E(c,q)$, it seems that none has studied it yet, at least we have not seen any related result till now. In this paper we shall prove the following two conclusions.

THEOREM 1. *Let $q \geq 3$ is an odd square-full number (that is, for any prime p, $p|q$ if and only if $p^2|q$), we have the identity*

$$\sum_{\substack{c=1 \\ (c,q)=1}}^{q} R_q(c+1)E(c,q) = \frac{1}{2}\phi^2(q)\prod_{p|q}\left(1+\frac{1}{p}\right),$$

where $\phi(q)$ is Euler function, $\prod_{p|q}$ denotes the product over all prime divisor of q.

THEOREM 2. *For any prime $p \geq 3$, we have the identity*

$$\sum_{c=1}^{p-1} R_p(c+1)E(c,p) = \frac{1}{2}p(p-1).$$

For general odd number $q \geq 3$, whether there exists a computational formula for $\sum_{c=1\,(c,q)=1}^{q} R_q(c+1)E(c,q)$ is an open problem.

We believe that it is true. But now, we can only give an asymptotic formula.

2. Several lemmas. In this section we shall give several Lemmas, which are necessary for the proof of our theorems. First we have the following.

LEMMA 1. *Let χ be a primitive character modulo m with $\chi(-1) = -1$. Then we have*

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi}\tau(\chi)L(1,\overline{\chi}),$$

where $\tau(\chi)$ is the Gaussian sum associated with χ, $e(y) = e^{2\pi iy}$, and $L(1,\chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. This can be easily deduced by Theorems 12.11 and 12.20 in [**1**]. □

LEMMA 2. *Suppose χ is an odd character* mod q*, then we have the identity*

$$(1-2\chi(2))\sum_{a=1}^{q} a\chi(a) = \chi(2)q\sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

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Proof. See [**5**]. □

LEMMA 3. *Let $q > 1$ be an odd number, then we have the identity*

$$\sum_{\substack{c=1\\(c,q)=1}}^{q} R_q(c+1)E(c,q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |\tau(\chi)|^2 \cdot |1-2\chi(2)|^2 \cdot \left| \frac{1}{q} \sum_{a=1}^{q} a\chi(a) \right|^2 ,$$

where $\sum_{\chi \bmod q \, \chi(-1)=-1}$ denotes the summation over all Dirichlet characters χ mod q such that $\chi(-1) = -1$.

Proof. From the orthogonality relation for character sums mod q and the definition of $N(c,q)$, we have

$$\begin{aligned}
N(c,q) &= \frac{1}{2} \sum_{a=1}^{q} \sum_{\substack{b=1\\ ab\equiv c(q)}}^{q} (1-(-1)^{a+\bar b}) = \frac{1}{2}\phi(q) - \frac{1}{2} \sum_{a=1}^{q} \sum_{\substack{b=1\\ ab\equiv c(q)}}^{q} (-1)^{a+\bar b} \\
&= \frac{1}{2}\phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \left(\sum_{a=1}^{q} (-1)^a \chi(a) \right) \left(\sum_{b=1}^{q} (-1)^{\bar b} \chi(b) \right) \\
&= \frac{1}{2}\phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \left(\sum_{a=1}^{q} (-1)^a \chi(a) \right) \left(\sum_{b=1}^{q} (-1)^{b} \chi(\bar b) \right) \\
&= \frac{1}{2}\phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \left(\sum_{a=1}^{q} (-1)^a \chi(a) \right) \overline{\left(\sum_{b=1}^{q} (-1)^{b} \chi(b) \right)} \\
&= \frac{1}{2}\phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \left| \sum_{a=1}^{q} (-1)^a \chi(a) \right|^2 . \qquad (1)
\end{aligned}$$

If $\chi(-1) = 1$, then

$$\sum_{a=1}^{q} (-1)^a \chi(a) = 0. \qquad (2)$$

If $\chi(-1) = -1$, then

$$\sum_{a=1}^{q} (-1)^a \chi(a) = 2\chi(2) \sum_{a=1}^{\frac{q-1}{2}} \chi(a). \qquad (3)$$

Note that the identity

$$\sum_{\substack{c=1\\(c,q)=1}}^{q} \overline{\chi}(c) R_q(c+1) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{a}{q}\right) \sum_{\substack{c=1\\(c,q)=1}}^{q} \overline{\chi}(c) e\left(\frac{ac}{q}\right) = \overline{\chi}(-1)\, |\tau(\chi)|^2 ,$$

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combining (1), (2), (3) and Lemma 2, we may immediately deduce

$$\sum_{\substack{c=1\\(c,q)=1}}^{q} R_q(c+1)E(c,q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |\tau(\chi)|^2 \cdot |1-2\chi(2)|^2 \cdot \left| \frac{1}{q} \sum_{a=1}^{q} a\chi(a) \right|^2 .$$

This proves Lemma 3.

To introduce Lemma 4, we need to give the definition of the Dedekind sums. For a positive integer q and an arbitrary integer h, the classical Dedekind sums $S(h,q)$ is defined by

$$S(h,q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer} \end{cases}.$$

The various properties of $S(h,k)$ have been studied by many authors (see [**2**–**4**, **9**, **10**]). For this sum, there is also another kind of expression, which is as follows. □

LEMMA 4. *Let $q > 2$ be an integer, then for any integer a with $(a,q)=1$, we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2.$$

Proof. See Lemma 2 in [**9**]. □

LEMMA 5. *Let $q > 2$ be an odd square-full number (that is, for any prime p, $p|q$ if and only if $p^2|q$), then we have the identities*

$$\text{(I)} \qquad \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} |L(1,\chi)|^2 = \frac{\pi^2}{12} \frac{\phi^3(q)}{q^2} \prod_{p|q} \left(1+\frac{1}{p}\right);$$

$$\text{(II)} \qquad \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24} \frac{\phi^3(q)}{q^2} \prod_{p|q} \left(1+\frac{1}{p}\right),$$

where $\sum^{*}_{\chi \bmod q \, \chi(-1)=-1}$ denotes the summation over all odd primitive characters mod q.

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Proof. From the definition of the Dedekind sums, Lemma 4 and the Möbius inversion formula (see Theorem 2.9 in [**1**]) we have

$$\begin{aligned}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2 &= \frac{\phi(q)}{q^2}\pi^2 \sum_{d|q}\mu(d)\frac{q}{d}S\left(a,\frac{q}{d}\right) \\ &= \pi^2\frac{\phi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d}S\left(a,\frac{q}{d}\right). \qquad (4)\end{aligned}$$

If $a = 1$, then it is easy to compute

$$S(1,q) = \sum_{k=1}^{q-1}\left(\frac{k}{q}-\frac{1}{2}\right)^2 = \frac{1}{12}\left(q-3+\frac{2}{q}\right).$$

So from this formula and (4) we have

$$\begin{aligned}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 &= \frac{\pi^2}{12}\frac{\phi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d}\left(\frac{q}{d}-3+\frac{2d}{q}\right) \\ &= \frac{\pi^2}{12}\phi(q)\sum_{d|q}\frac{\mu(d)}{d^2} - \frac{\pi^2}{4}\frac{\phi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d} + \frac{\pi^2}{6}\frac{\phi(q)}{q^2}\sum_{d|q}\mu(d) \\ &= \frac{\pi^2}{12}\frac{\phi^2(q)}{q}\left[\prod_{p|q}\left(1+\frac{1}{p}\right)-\frac{3}{q}\right]. \qquad (5)\end{aligned}$$

Note that q is a square-full number, $\mu(q)$ and $\phi(q)$ are two multiplicative functions,

$$\sum_{d|q}\mu(d)\frac{\phi^2(q/d)}{q^2/d^2} = 0 \quad \text{and} \quad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \sum_{d|q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod \frac{q}{d} \\ \chi(-1)=-1}} |L(1,\chi\chi_0)|^2,$$

from the Möbius inversion formula and (5) we may immediately get

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 &= \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod \frac{q}{d} \\ \chi(-1)=-1}} |L(1,\chi\chi_0)|^2 \\ &= \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod \frac{q}{d} \\ \chi(-1)=-1}} |L(1,\chi)|^2 \\ &= \sum_{d|q}\mu(d)\left\{\frac{\pi^2}{12}\frac{\phi^2(q/d)}{q/d}\left[\prod_{p|\frac{q}{d}}\left(1+\frac{1}{p}\right)-\frac{3}{q/d}\right]\right\} \\ &= \frac{\pi^2}{12}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right),\end{aligned}$$

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where we have used the fact that $\sum_{d|q}\mu(d)\sum_{\chi \bmod \frac{q}{d}\ \chi(-1)=-1}|L(1,\chi\chi_0)|^2 = \sum_{d|q}\mu(d)\sum_{\chi \bmod \frac{q}{d}\ \chi(-1)=-1}|L(1,\chi)|^2$ if q be a square-full number, and χ_0 denotes the principal character mod q. This proves the Formula (I) of Lemma 5.

If $a = 2$, then note that q is an odd number, so from the definition of the Dedekind sums we have

$$S(2,q) = \sum_{k=1}^{\frac{(q-1)}{2}}\left(\frac{k}{q}-\frac{1}{2}\right)\left(\frac{2k}{q}-\frac{1}{2}\right) + \sum_{k=\frac{q+1}{2}}^{q-1}\left(\frac{k}{q}-\frac{1}{2}\right)\left(\frac{2k}{q}-\frac{3}{2}\right)$$
$$= \frac{1}{24}\left(q-6+\frac{5}{q}\right).$$

Then from this identity and (4) we have

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24}\frac{\phi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d}\left(\frac{q}{d}-6+\frac{5d}{q}\right)$$
$$= \frac{\pi^2}{24}\phi(q)\sum_{d|q}\frac{\mu(d)}{d^2} - \frac{\pi^2}{4}\frac{\phi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d} + \frac{5\pi^2}{24}\frac{\phi(q)}{q^2}\sum_{d|q}\mu(d)$$
$$= \frac{\pi^2}{24}\frac{\phi^2(q)}{q}\left[\prod_{p|q}\left(1+\frac{1}{p}\right)-\frac{6}{q}\right]. \tag{6}$$

Applying the Möbius inversion formula and (6) we can also deduce that

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(2)|L(1,\chi)|^2 = \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod \frac{q}{d}\\ \chi(-1)=-1}}\chi(2)\chi_0(2)|L(1,\chi\chi_0)|^2$$
$$= \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod \frac{q}{d}\\ \chi(-1)=-1}}\chi(2)|L(1,\chi)|^2$$
$$= \sum_{d|q}\mu(d)\left\{\frac{\pi^2}{24}\frac{\phi^2(q/d)}{q/d}\left[\prod_{p|\frac{q}{d}}\left(1+\frac{1}{p}\right)-\frac{6}{q/d}\right]\right\}$$
$$= \frac{\pi^2}{24}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right).$$

This completes the proof of Lemma 5. □

3. Proof of the theorems. In this section, we will use lemmas from Section 2 to prove our theorems. First we prove Theorem 1. For any odd square-full number $q \geq 3$ and $\chi \bmod q$, note that the Gauss sum $\tau(\chi) = 0$, if χ is not a primitive character mod q; If χ is a primitive character mod q, then we have $|\tau(\chi)|^2 = q$. So from Lemmas 1, 3

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and 5 we have

$$\sum_{\substack{c=1\\(c,q)=1}}^{q} R_q(c+1)E(c,q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |\tau(\chi)|^2 \cdot |1-2\chi(2)|^2 \cdot \left|\frac{1}{q}\sum_{a=1}^{q} a\chi(a)\right|^2$$

$$= \frac{2}{\pi^2}\frac{q^2}{\phi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} (5-2\chi(2)-2\overline{\chi}(2))\,|L(1,\chi)|^2$$

$$= \frac{2}{\pi^2}\frac{q^2}{\phi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} (5-4\chi(2))\,|L(1,\chi)|^2$$

$$= \frac{2}{\pi^2}\frac{q^2}{\phi(q)}\left[\frac{5\pi^2}{12}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right) - \frac{4\pi^2}{24}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right)\right]$$

$$= \frac{1}{2}\phi^2(q)\prod_{p|q}\left(1+\frac{1}{p}\right).$$

This proves Theorem 1.

Applying Lemma 4 we may get identities

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2}$$

and

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24}\frac{(p-1)^2(p-5)}{p^2}.$$

From these two formulae, Lemmas 1 and 3 we can deduce that

$$\sum_{c=1}^{p-1} R_p(c+1)E(c,p)$$

$$= \frac{2}{\pi^2}\frac{p^2}{p-1}\left(5\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 - 4\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2\right)$$

$$= \frac{2}{\pi^2}\frac{p^2}{p-1}\left[\frac{5\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2} - \frac{4\pi^2}{24}\frac{(p-1)^2(p-5)}{p^2}\right]$$

$$= \frac{1}{2}p(p-1).$$

This completes the proof of Theorem 2.

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REFERENCES

1. T. M. Apostol, *Introduction to analytic number theory* (Springer-Verlag, New York, 1976).

2. L. Carlitz, The reciprocity theorem of Dedekind sums, *Pacific J. Math.* **3** (1953), 513–522.

3. J. Chaohua, On the mean value of Dedekind sums, *J. Number Theory* **87** (2001), 173–188.

4. J. B. Conrey, E. Fransen, R. Klein and C. Scott, Mean values of Dedekind sums, *J. Number Theory* **56** (1996), 214–226.

5. T. Funakura, On Kronecker's limit formula for Dirichlet series with periodic coefficients, *Acta Arith*. **55** (1990), 59–73.

6. R. K. Guy, *Unsolved problems in number theory*, 2nd edn. (Springer-Verlag, New York, 1994).

7. Z. Wenpeng, On a problem of D. H. Lehmer and its generalization, *Composito Math.* **86** (1993), 307–316.

8. Z. Wenpeng, A problem of D. H. Lehmer and its generalization (II), *Compositio Math.* **91** (1994), 47–56.

9. Z. Wenpeng, A problem of D. H. Lehmer and its mean square value formlua, *Jap. J. Math.* **29** (2003), 109–116.

10. W. Zhang, On the mean values of Dedekind sums, *J. Theor. Nombres*. **8** (1996), 429–442.

11. W. Zhang, A note on the mean square value of the Dedekind sums, *Acta Math. Hung*. **86** (2000), 275–289.