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A Characterization of $PSU_{11}(q)$

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Abstract. Order components of a finite simple group were introduced in [4]. It was proved that some non-abelian simple groups are uniquely determined by their order components. As the main result of this paper, we show that groups $PSU_{11}(q)$ are also uniquely determined by their order components. As corollaries of this result, the validity of a conjecture of J. G. Thompson and a conjecture of W. Shi and J. Bi both on $PSU_{11}(q)$ are obtained.

1 Introduction

For an integer n, let $\pi(n)$ be the set of prime divisors of n. If G is a finite group then $\pi(G)$ is defined to be $\pi(|G|)$. The prime graph $\Gamma(G)$ of a group G is a graph whose vertex set is $\pi(G)$, and two distinct primes p and q are linked by an edge if and only if G contains an element of order pq. Let π_i, $i = 1, 2, \dots, t(\Gamma(G))$ be the connected components of $\Gamma(G)$. For $|G|$ even, π_1 will be the connected component containing 2. Then $|G|$ can be expressed as a product of some positive integers m_i, $i = 1, 2, \dots, t(\Gamma(G))$, with $\pi(m_i) = \pi_i$. The integers m_i are called the order components of G. The set of order components of G will be denoted by $OC(G)$. If the order of G is even, it is assumed that m_1 is the even order component and $m_2, \dots, m_{t(\Gamma(G))}$ are the odd order components of G. The order components of non-abelian simple groups having at least three prime graph components are obtained by G. Y. Chen [8, Tables 1–3]. The order components of non-abelian simple groups with two order components can be obtained according to [19, 25; see also 12, 13]. The following groups are uniquely determined by their order components : $G_2(q)$ where $q \equiv 0 \pmod 3$ [2], sporadic simple groups [3], Suzuki-Ree groups [6], $E_8(q)$ [7], $PSL_2(q)$ [8], A_p where p and $p-2$ are primes [10], $PSL(3,q)$ [12, 13], $PSL(5,q)$ [11], $F_4(q)$ [14,17], $C_2(q)$ where $q > 5$ [15], $PSU(3,q)$ for $q > 5$ [18] and $PSU_5(q)$ [16].

In this paper, we prove that the groups $PSU_{11}(q)$, for any prime power q, are also uniquely determined by their order components, that is we have:

The Main Theorem *Let G be a finite group, $M = PSU_{11}(q)$ with $OC(G) = OC(M)$. Then $G \cong M$.*

2 Preliminary Results

In order to prove the main theorem, we present some lemmas.

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Definition 2.1 ([9]) A finite group G is called a 2-*Frobenius* group if it has a normal series $G > K > H > 1$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.2 ([25, Theorem A]) *If G is a finite group with prime graph of more than one component, then G is one of the following groups:*

(a) *a Frobenius or 2-Frobenius group;*
(b) *a simple group;*
(c) *an extension of a π_1-group by a simple group ;*
(d) *an extension of a simple group by a π_1-solvable group;*
(e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.3 ([25, Lemma 3]) *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group.*

The next lemma follows from [1, Theorem 2]:

Lemma 2.4 *Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(\Gamma(G)) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$, and G has one of the following structures:*

(a) $2 \in \pi(K)$ *and all Sylow subgroups of H are cyclic.*
(b) $2 \in \pi(H)$*, K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the* 2*-Sylow subgroups of H are cyclic or generalized quaternion groups.*
(c) $2 \in \pi(H)$*, K is an abelian group and there exists $H_0 \leq H$ such that $|H : H_0| \leq 2$, $H_0 = Z \times SL(2,5)$, $(|Z|, 2.3.5) = 1$ and the Sylow subgroups of Z are cyclic.*

The next lemma follows from [1, Theorem 2] and Lemma 2.3

Lemma 2.5 *Let G be a* 2*-Frobenius group of even order. Then $t(\Gamma(G)) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that*

(a) $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi(K/H) = \pi_2$;
(b) *G/K and K/H are cyclic, $|G/K|$ divides $|\operatorname{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$ and $|G/K| < |K/H|$;*
(c) *H is nilpotent and G is a solvable group.*

Lemma 2.6 ([5, Lemma 8]) *Let G be a finite group with $t(\Gamma(G)) \geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not a π_i-number, then $m_1 m_2 \ldots m_r$ is a divisor of $|N| - 1$.*

Lemma 2.7 ([4, Lemma 1.4]) *Suppose G and M are two finite groups satisfying $t(\Gamma(M)) \geq 2$, $N(G) = N(M)$, where $N(G) = \{n \mid G$ has a conjugacy class of size $n\}$, and $Z(G) = 1$. Then $|G| = |M|$.*

The next lemma follows from [4, Lemma 1.5].

Lemma 2.8 *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(\Gamma(G_1)) = t(\Gamma(G_2))$ and $OC(G_1) = OC(G_2)$.*

Lemma 2.9 *Let G be a finite group and let M be a non-abelian simple group with $t(\Gamma(M)) = 2$ satisfying $OC(G) = OC(M)$. Let $|M| = m_1m_2$, $OC(M) = \{m_1, m_2\}$, and $\pi(m_i) = \pi_i$ for $i = 1$ or 2. Then $|G| = m_1m_2$ and one of the following holds:*

(a) *G is a Frobenius or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover, $OC(K/H) = \{m'_1, m'_2, \dots, m'_s, m_2\}$, $|K/H| = m'_1m'_2\cdots m'_sm_2$ and $m'_1m'_2\cdots m'_s \mid m_1$, where $\pi(m'_j) = \pi_j(K/H)$, $1 \le j \le s$. Also, $|G/K| \mid |\operatorname{Out}(K/H)|$.*

Proof The first part of the lemma follows from the above lemmas. Since $t(\Gamma(G)) \ge 2$, we have $t(\Gamma(G/H)) \ge 2$. Otherwise, $t(\Gamma(G/H)) = 1$, hence $t(\Gamma(G)) = 1$, which is a contradiction, since H is a π_1-group. Moreover, we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, xH induces an automorphism of K/H and this automorphism is trivial if and only if $xH \in Z(G/H)$. Therefore, $G/K \le \operatorname{Out}(K/H)$ and since $Z(G/H) = 1$, it follows that $|G/K| \mid |\operatorname{Out}(K/H)|$.

Lemma 2.10 *Let $M = PSU_{11}(q)$. Suppose $D(q) = \frac{q^{11}+1}{k(q+1)}$, where $k = (11, q+1)$.*

(a) *If $p \in \pi(M)$, then $|S_p| \le q^{55}$ where $S_p \in Syl_p(M)$;*
(b) *If $p \in \pi_1(M)$ and $p^\alpha \mid |M|$, then $p^\alpha - 1 \equiv 0$ (mod $D(q)$) if and only if $p^\alpha = q^{22}$ or q^{44};*
(c) *If $p \in \pi_1(M)$ and $p^\alpha \mid |M|$, then $p^\alpha + 1 \equiv 0$ (mod $D(q)$) if and only if $p^\alpha = q^{11}$, q^{33} or q^{55}.*

Proof

(a) We know that

$$\begin{aligned}|M| =& q^{55}(q+1)^{10}(q-1)^5(q^2-q+1)^3(q^2+1)^2(q^4-q^3+q^2-q+1)^2\\ &\times (q^2+q+1)(1-q+q^2-q^3+q^4-q^5+q^6)(q^4+1)(q^6-q^3+1)\\ &\times (q^4+q^3+q^2+q+1)\times\frac{(q^{11}+1)}{k(q+1)}.\end{aligned}$$

By easy calculations we determine the greatest common divisors of any two factors of $|M|$. For example, $(q-1, q+1) \mid 2$, $(q+1, q^2-q+1) \mid 3$, $(q+1, q^2+1) \mid 2$, $(q+1, q^4-q^3+q^2-q+1) \mid 5$, $(q+1, q^6-q^5+q^4-q^3+q^2-q+1) \mid 7$, $(q+1, q^4+1) \mid 2$, $(q+1, q^6-q^3+1) \mid 3$ and $q+1$ is coprime with respect to other factors of $|M|$. So if $p^\alpha \mid |M|$ and $p \in \pi_1$, then one of the following occurs: p^α is a divisor of q^{55}, $2^8 3^4 5^2 7(q+1)^{10}$, $2^{13}5^2 3(q-1)^5$, $3^{11}(q^2-q+1)^3$, $2^{16}(q^2+1)^2$, $5^{10}(q^4-q^3+q^2-q+1)^2$,

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$3^5(q^2+q+1)$, $7^{10}(1-q+q^2-q^3+q^4-q^5+q^6)$, $2^{17}(q^4+1)$, $3^{13}(q^6-q^3+1)$ or $5^5(q^4+q^3+q^2+q+1)$. Therefore, (a) follows.

(b) Now let there exist $p \in \pi_1(M)$, $p^\alpha \,\big|\, |M|$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$. It is obvious that $p^\alpha > D(q)$.

For $q \leq 32$ numerical calculations show that there is no p^α such that (b) holds. Hence let $q > 32$. We consider the following possible cases:

(1) If $p^\alpha \mid 2^8 3^4 5^2 7 (q+1)^{10}$, then we consider the following subcases:

- (1.1) Let $p \neq 2,3,5,7$ and $p^\alpha \,\big|\, (q+1)^{10}$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$. Then $p^\alpha - 1 = sD(q)$ for some $s > 0$. But $(q+1)^{10}/20 < D(q)$, which implies that $p^\alpha = (q+1)^{10}/t$, where $st \leq 20$. Now numerical calculation shows that these equations have no solution and hence there can not exist any p, α such that the above relations holds.
- (1.2) If $p = 2$, then $2^\alpha \big| 2^8(q+1)^{10}$. Since $2^8(q+1)^{10}/4000 < D(q)$ for $q > 32$, we have $2^8(q+1)^{10}/t - 1 = sD(q)$, where $st \leq 4000$. Now by using mathematical software (for example Maple), we can check all of these equations and see that there exists no α such that (b) holds.
- (1.3) If $p = 3, 5$ or 7, then we get a contradiction similar to subcase (1.2).

(2) If $p^\alpha \,\big|\, 2^{13}5^2 3(q-1)^5$, then p^α divides $2^{13}(q-1)^5$, $5^2(q-1)^5$ or, $3(q-1)^5$. But in each case $p^\alpha < D(q)$ which implies that $p^\alpha - 1 \not\equiv 0 \pmod{D(q)}$.

(3) If $p^\alpha \,\big|\, 3^{11}(q^2-q+1)^3$, $2^{16}(q^2+1)^2$, $5^{10}(q^4-q^3+q^2-q+1)^2$, $3^5(q^2+q+1)$, $7^{10}(1-q+q^2-q^3+q^4-q^5+q^6)$, $2^{17}(q^4+1)$, $3^{13}(q^6-q^3+1)$ or, $5^5(q^4+q^3+q^2+q+1)$, then in each case $p^\alpha < D(q)$ which implies that $p^\alpha - 1 \not\equiv 0 \pmod{D(q)}$.

(4) If $p^\alpha \,\big|\, q^{55}$, then we consider two subcases, namely $k=1$, $k=11$. Since the proofs are similar we state only the case $k=1$.

We can see easily that $q = p^n$ for some $n > 0$. First we prove that if $p^\beta \,\big|\, q^{11}$ and $p^\beta + 1 \equiv 0 \pmod{D(q)}$, then $p^\beta = q^{11}$. We have

$$p^\beta + 1 = s.D(q) = s.\frac{q^{11}+1}{q+1} = s(q^{10} - \cdots + q^2 - q + 1),$$

and $1 \leq s \leq q+1$. Also since $q \,\big|\, p^\beta$ we have $q \,\big|\, s-1$ which implies that $q \leq s-1$. Therefore, $q+1 = s$ and hence $p^\beta = q^{11}$.

Now we prove that if $p^\alpha \,\big|\, q^{22}$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$, then $p^\alpha = q^{22}$. If we assume that $p^\alpha \leq q^{11}$ and $p^\alpha + 1 = s \cdot D(q)$, then $s < q+1$. Since $q \,\big|\, p^\alpha$ we have $q \,\big|\, s+1$ and hence $q \leq s+1$. Thus $s = q$ or $s = q-1$. But easy calculation shows that $p^\alpha - 1 \neq s \cdot D(q)$, which is a contradiction. Therefore, $p^\alpha > q^{11}$ and hence $p^\alpha = q^{11}p^m$ for some $m > 0$. Now we have

$$p^\alpha - 1 = q^{11}p^m - 1 = p^m(q^{11}+1) - p^m - 1.$$

Therefore, $D(q) \,\big|\, p^m + 1$ which implies that $p^m = q^{11}$, by the above statement and hence $p^\alpha = q^{22}$. If $p^\alpha > q^{22}$ and $p^\alpha | q^{55}$, then by a similar method we conclude that $p^\alpha = q^{44}$.

(c) Similar to part (b), we conclude that p^α must be equal to q^{11}, q^{33} or q^{55} and the proof is complete. ■

Remark For convenience let $X = \{q^{11}, q^{33}, q^{55}\}$ and $Y = \{q^{22}, q^{44}\}$.

Lemma 2.11 *Let G be a finite group, $M = PSU_{11}(q)$ with $OC(G) = OC(M)$. Then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof G is not a Frobenius group otherwise by Lemma 2.4, $OC(G) = \{|H|, |K|\}$ where K and H are the Frobenius kernel and the Frobenius complement of G, respectively. Since $|H| \bigm| (|K|-1)$, we have $|H| < |K|$. So $|H| = \frac{q^{11}+1}{(q+1)(11,q+1)}$, $|K| = |G|/|H|$. There exists a prime p such that $p^\alpha \bigm| 3(q-1)^5$. If P is a p-Sylow subgroup of K, then since K is nilpotent, $P \lhd G$ and hence $D(q) \bigm| |P|-1$ by Lemma 2.6, which implies that $p^\alpha \in Y$ by Lemma 2.10(b). Then $3(q-1)^5 \geq q^{22}$ which is a contradiction. Therefore, G is not a Frobenius group.

Let G be a 2-Frobenius group. By Lemma 2.5, there is a normal series $1 \unlhd H \unlhd K \unlhd G$ such that $|K/H| = \frac{q^{11}+1}{(q+1)(11,q+1)} < 2^8(q+1)^{10}$ and $|G/K| < |K/H|$. Thus there exists a prime p such that $p \bigm| 2^8(q+1)^{10}$ and $p \bigm| |H|$. If P is a p-Sylow subgroup of H, since H is nilpotent, P must be a normal subgroup of K with $P \subseteq H$ and $|K| = \frac{q^{11}+1}{k(q+1)}|H|$. Therefore, $\frac{q^{11}+1}{k(q+1)} \bigm| (|P|-1)$ by Lemma 2.6 and hence $q^{22} \bigm| |P|$, which is impossible since $|P| \leq 2^8(q+1)^{10}$. Therefore, G is not a 2-Frobenius group. ■

Lemma 2.12 *Let G be a finite group. If the order components of G are the same as those of $M = PSU_{11}(q)$, then G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover, the odd order component of M is equal to one of those of K/H, and in particular, $t(\Gamma(K/H)) \geq 2$.*

Proof The first part of the Lemma follows from the above lemmas since the prime graph of M has two components. For primes p and q, if K/H has an element of order pq, then G has one. Hence, by the definition of prime graph component, the odd order component of G must be an odd order component of K/H. ■

3 Proof of the Main Theorem

By Lemma 2.12, G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, $t(\Gamma(K/H)) \geq 2$ and the odd order component of M is an odd order component of K/H. We now proceed the proof of the main theorem in the following steps:

Step 1 If $K/H \cong A_n$ where $n = p,\ p+1,\ p+2$ and $p \geq 5$ is a prime number. Then we have two cases:

Case 1.1: $k = 1$. In this case p or $p-2$ is equal to $\frac{q^{11}+1}{q+1}$. If $p = \frac{q^{11}+1}{q+1}$, then $p - 1 = q(q-1)(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$ and

$$(1) \qquad p-2 = q^{10}-q^9+q^8-q^7+q^6-q^5+q^4-q^3+q^2-q-1.$$

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But easy calculation shows that $(p-2,|G|)\,\big|\,3^5\times 5^2\times 7\times 43$ and hence $p-2\,\big|\,3^5\times 5^2\times 7\times 43$. So $p=3,5,7,\dots$. But $D(2)=683$, $D(3)=44287$, $D(4)=838861$ and hence equation (1) is not satisfied in each case. If $p-2=q^{10}-q^9+\cdots-q+1$, then we proceed similarly for $p-4$ since $p>5$.

Case 1.2: $k=11$. Then p or $p-2$ is equal to $\frac{q^{11}+1}{11(q+1)}$ and $p-2$ or $p-4$ must be equal to $\frac{q^{10}-q^9+\cdots-q-21}{11}$, respectively. Now we proceed similarly to the last case and get a contradiction.

Step 2 If K/H is a sporadic simple group, then $D(q)$ must be equal to 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71, which has no solution, since $D(2)=683$. Therefore, K/H is a simple group of Lie type.

Step 3 If $K/H\cong E_6(q')$, then $D(q)=(q'^6+q'^3+1)/(3,q'-1)$ and hence $q'^9\in Y$, which implies that $q'^9=q^{22}$ or q^{44}. But $q'^{36}>q^{55}$ which is a contradiction, by Lemma 2.10(a).

Step 4 If $K/H\cong {}^2E_6(q')$, then $D(q)=(q'^6-q'^3+1)/(3,q'+1)$ and hence $q'^9\in X$, which implies that $q'^9=q^{11}$, q^{33} or q^{55}. If $q'^9=q^{33}$ or q^{55} then $q'^{36}>q^{55}$ which is a contradiction. If $q'^9=q^{11}$ then the equations $(q'^3+1)(3,q'+1)=(q+1)(11,q+1)$, and $q'^9=q^{11}$ have no common solution in $\mathbb{Z}$, which is a contradiction.

Step 5 If $K/H\cong A_r(q')$, then we distinguish the following 6 cases:

Case 5.1: $K/H\cong A_{p'-1}(q')$ where $(p',q')\neq(3,2),(3,4)$. Then $q'^{p'}-1\equiv 0(\text{mod } D(q))$ from which by Lemma 2.10(b) we have $q'^{p'}\in Y$. This implies that $q'^{p'}=q^{22}$ or q^{44}. Now if $p'>5$, then $\frac{q'^{p'(p'-1)}}{2}>q^{55}$, which is impossible by Lemma 2.10(a). If $p'=3$ and $q'^3=q^{22}$, then

$$(q^{11}-1)(q+1)(11,q+1)=(q'-1)(3,q'-1),\qquad q'^3=q^{22}.$$

But these equations have no common solution in $\mathbb{Z}$, and hence this case is also impossible. If $p'=3$ and $q'^3=q^{44}$ or if $p'=5$, then we get a contradiction similarly.

Case 5.2: $K/H\cong A_{p'}(q')$ where $(q'-1)\,\big|\,(p'+1)$. Then $q'^{p'}\in Y$, which implies that $q'^{p'}=q^{22}$ or q^{44}. But if $p'>3$, then $q'^{\frac{p'(p'+1)}{2}}>q^{55}$, which is impossible. If $p'=3$, then $q'-1\,\big|\,4$, which implies that $q'\leq 5$. But $q^{22}|q'^3$ and $q>1$ which is impossible.

Case 5.3: $K/H\cong A_1(q')$, where $4\mid(q'+1)$. If $D(q)=\frac{q'-1}{2}$, then $q'\in Y$, which implies that $q'=q^{22}$ or q^{44}. But then $2=(q^{11}-1)(q+1)(11,q+1)$, and it is impossible, since $q>1$. If $D(q)=q'$, then we consider two cases:

Case 5.3.a: If $k=1$ then $q'=(q^{11}+1)/(q+1)$ and since $q'+1\,\big|\,|K/H|=|A_1(q')|$, we have $q'+1\,\big|\,|G|$. But $(q'+1,|G|)\,\big|\,2^{18}\times 3^5\times 19\times 43$. Since $|K/H|\,\big|\,|G|$ and $q'+1\,\big|\,2^{18}\times 3^5\times 19\times 43$, the only possible case is $q=2$ and $K/H=A_1(683)$. Hence $|G/K|\cdot|H|=2^3\times 3^2\times 11\times 19\times 31$. Since $|\operatorname{Out}(A_1(683))|=1$ and by Lemma 2.9(2), $|G/K|\,\big|\,|\operatorname{Out}(A_1(683)|$ we conclude that $|H|=2^3\times 3^2\times 11\times 19\times 31$. Let P be the 3-Sylow subgroup of H. Since H is nilpotent, $P\lhd G$ and hence $683=D(2)\,\big|\,(|P|-1)=8$, by Lemma 2.6, which is a contradiction.

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Case 5.3.b: If $D(q) = q'$ and $k = 11$, then $q' + 1 = (q^{11} + 1)/(11(q + 1)) + 1$ and we get a contradiction similarly.

Case 5.4: $K/H \cong A_1(q')$ where $4 \mid (q' - 1)$. Since the possibility $D(q) = q'$ was discussed in case 5.3, we assume that $D(q) = \frac{q'+1}{2}$. Then $q' \in X$, which implies that $q' = q^{11}$, q^{33} or q^{55}. Obviously $q' = q^{11}$ implies that $q = 1$, therefore, $q' = q^{33}$ or q^{55}. If $q' = q^{33}$, then $k(q^{22} - q^{11} + 1)(q + 1) = 2$ which is impossible. If $q' = q^{55}$, then we proceed similarly.

Case 5.5: $K/H \cong A_1(q')$ where $4 \mid q'$. If $D(q) = q' - 1$, then $q' \in Y$, which implies that $q' = q^{22}$ or q^{44}. But for example if $q' = q^{22}$, then $1 = (q^{11} - 1)(q + 1)(11, q + 1)$ which is impossible. If $D(q) = q' + 1$, then $q' \in X$, which implies that $q' = q^{11}$, q^{33} or q^{55}. Now proceed similarly to Case 5.4.

Case 5.6: $K/H \cong A_2(2)$ or $K/H \cong A_2(4)$. Then $D(q)$ must be equal to $3, 5, 7, 9$ which is impossible, since $D(q) > 683$.

Step 6 If $K/H \cong B_r(q')$ or $C_r(q')$ or $D_r(q')$, by a similar method we get contradictions. For example, suppose $K/H \cong B_r(q')$, then we consider two cases:

Case 6.1: $K/H \cong B_m(q')$ where $m = 2^k \geq 4$ and q' is odd. Then $q'^m \in X$, which implies that $q'^m = q^{11}$, q^{33} or q^{55}. If $m = 4$ and $q'^4 > q^{11}$ or if $m > 4$, then $q'^{m^2} \mid |K/H|$ and hence $q'^{m^2} > q^{55}$, which is a contradiction. If $q'^m = q^{11}$ and $m = 4$, *i.e.*, $q'^4 = q^{11}$, then $2 = (q + 1)(11, q + 1)$ which is a contradiction, since $q > 1$.

Case 6.2: $K/H \cong B_p(3)$. Then $3^p \in Y$ and therefore $3^p = q^{22}$ or q^{44} which is a contradiction, since p is a prime number and can not be equal to 22 or 44.

Step 7 If $K/H \cong F_4(q')$, then we consider 2 cases:

Case 7.1: If $D(q) = q'^4 - q'^2 + 1$, then $q'^6 \in X$, which implies that $q'^6 = q^{11}$, q^{33} or q^{55}. If $q'^6 > q^{11}$, then $q'^{24} > q^{55}$ which is a contradiction. If $q'^6 = q^{11}$, then $q'^2 + 1 = (q + 1)(11, q + 1)$. But these equations have no common solution in $\mathbb{Z}$.

Case 7.2: If $D(q) = q'^4 + 1$, then $q'^4 \in X$, which implies that $q'^4 = q^{11}$, q^{33} or q^{55}. But then $q'^{24} > q^{55}$ which is impossible.

Step 8 If $K/H \cong E_7(2)$ or $E_7(3)$ or ${}^2E_6(2)$ or ${}^2F_4(2)'$, then $D(q)$ must be equal to $13, 17, 19, 73, 127, 757, 1093$ which is impossible.

Step 9 If $K/H \cong G_2(q')$, then we consider 3 cases:

Case 9.1: $K/H \cong G_2(q')$ where $2 < q' \equiv 1 \pmod 3$. Then $D(q) = q'^2 - q' + 1$ and hence $q'^3 \in X$, which implies that $q'^3 = q^{11}$, q^{33} or q^{55}. If $q'^3 = q^{11}$, then $q' + 1 = (q + 1)(11, q + 1)$. But these equations have no common solution in $\mathbb{Z}$. If $q'^3 = q^n$ where $n = 33$ or 55, then we get a contradiction similarly.

Case 9.2: $K/H \cong G_2(q')$ where $2 < q' \equiv -1 \pmod 3$. Then $D(q) = q'^2 + q' + 1$ and hence $q'^3 = q^{22}$ or q^{44}. Now we can proceed similarly to 9.1 and get contradictions.

Case 9.3: $K/H \cong G_2(q')$ where $3 \mid q'$. Then $q'^2 \pm q' + 1 = D(q)$. This is similar to cases 9.1 and 9.2.

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Step 10 If $K/H \cong {}^3D_4(q')$, then $D(q) = q'^4 - q'^2 + 1$, and hence $q'^6 = q^{11}, q^{33}$ or q^{55}. If $q'^6 > q^{11}$, then $q'^{12} > q^{55}$ which is a contradiction by Lemma 2.10(a). If $q'^6 = q^{11}$, then $q'^2 + 1 = (q+1)(11, q+1)$, which have no a common solution in $\mathbb{Z}$.

Step 11 If $K/H \cong E_8(q')$ or $K/H \cong {}^2G_2(q')$ where $q' = 3^{2r+1}$, then we get a contradiction similarly. For example if $K/H \cong {}^2G_2(q')$ then $D(q) = q' \pm \sqrt{3q'} + 1$. Thus $q'^3 \in X$ and we get a contradiction similar to the last steps.

Step 12 If $K/H \cong {}^2F_4(q')$ where $q' = 2^{2r+1} > 2$, then $D(q) = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$. Therefore, $q'^6 + 1 \equiv 0 \pmod{D(q)}$ and hence $q'^6 \in X$. Now we get a contradiction similar to the last step.

Step 13 If $K/H \cong {}^2B_2(q')$ where $q' = 2^{2t+1} > 2$, then if $D(q) = q' - 1$ we get $q' \in Y$ and if $D(q) = q' \pm \sqrt{2q'} + 1$, we get $q'^2 + 1 \equiv 0 \pmod{D(q)}$. Therefore, $q'^2 \in X$. Now we proceed similar to the last steps and get contradictions.

Step 14 If $K/H \cong {}^2D_r(q')$, then we consider 6 cases:

Case 14.1: $K/H \cong {}^2D_r(q')$ where $r = 2^t \geq 4$. Then $q'^r \in X$. If $r = 4$ and $q'^4 = q^{11}$, then $(2, q'+1) = k(q+1)$ which is impossible. Also in other cases if $r > 4$ or if $r = 4$ and $q'^4 > q^{11}$, then since $r - 1 \geq 3$, G has a subgroup of size $q^n > q^{55}$ which is a contradiction by Lemma 2.10(a).

Case 14.2: $K/H \cong {}^2D_r(2)$ where $r = 2^t+1 \geq 5$. Then $2^{r-1} \in X$. But $r-1 = 2^t \geq 4$ and $11 \nmid 2^t$, which is a contradiction.

Case 14.3: $K/H \cong {}^2D_p(3)$ where $5 \leq p \neq 2^r + 1$. Then $3^p = q^{11}, q^{33}$ or q^{55} and since p is an odd prime number, $q = 3$ and $p = 11$. Then $3^{p(p-1)} > q^{55}$ which is a contradiction.

Case 14.4: $K/H \cong {}^2D_r(3)$ where $r = 2^t + 1 \neq p, t \geq 2$. Then $3^{r-1} \in X$, hence $3^{r-1} = q^{11}, q^{33}$ or q^{55}. Since $r > 5$, we have $3^{r(r-1)} > q^{55}$ and hence G has a subgroup of size $q^n > q^{55}$ which is a contradiction by Lemma 2.10(a).

Case 14.5: $K/H \cong {}^2D_p(3)$ where $p = 2^t + 1, t \geq 2$. Then $3^{p-1} = q^{11}, q^{33}$ or q^{55}. Therefore, $11 \,\big|\, p - 1 = 2^t$ which is a contradiction.

Case 14.6: $K/H \cong {}^2D_{p+1}(2)$ where $p = 2^r - 1, r \geq 2$. Then similar to (14.4) and (14.5) we get a contradiction.

Step 15 If $K/H \cong {}^2A_r(q')$, then we consider 3 cases:

Case 15.1: $K/H \cong {}^2A_3(2)$ or $K/H \cong {}^2A_5(2)$. Then $D(q)$ must be equal to 5, 7, 11 which is impossible.

Case 15.2: $K/H \cong {}^2A_{p'}(q')$ where $(q'+1)|(p'+1)$ and $(p', q') \neq (3,3), (5,2)$. Then $q'^{p'} = q^{11}, q^{33}$ or q^{55}. Let $q'^{p'} > q^{11}$. If $p' > 3$, then $q'^{\frac{p'(p'+1)}{2}} > q^{55}$, which is impossible. If $p' = 3$, then $q' = 3$ but $(p', q') \neq (3,3)$. If $q'^{p'} = q^{11}$ and $p' > 5$ we do similarly. Also if $p' = 3$ or 5 and $q'^{p'} = q^{11}$, then $q' < 10$, which is impossible.

Case 15.3: $K/H \cong {}^2A_{p'-1}(q')$. Then $q'^{p'} = q^{11}, q^{33}$ or q^{55}. If $p' > 11$, then $q'^{\frac{p'(p'-1)}{2}} > q^{55}$, which is impossible. If $p' = 3, 5, 7$, then

$$(q'+1)(p', q'+1) = (q+1)(11, q+1), \qquad q'^{p'} = q^{11}.$$

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But these equations have no common solution in $\mathbb{Z}$. If $p' = 11$, then $q = q'$. Thus $|G| = |PSU_{11}(q)| = |K/H| = |K|/|H|$ which implies that $|H| = 1$ and $|K| = |G| = |PSU_{11}(q)|$. Therefore, $K = PSU_{11}(q)$ and hence $G = PSU_{11}(q)$.

The proof of the main theorem is now complete. ■

Remark 3.1 It is a well known conjecture of J. G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$. We can give a positive answer to this conjecture for the groups under discussion.

Corollary 3.2 *Let G be a finite group with* $Z(G) = 1$, $M = PSU_{11}(q)$ *with* $N(G) = N(M)$, *then* $G \cong M$.

Proof By Lemma 2.8 if G and M are two finite groups satisfying the conditions of Corollary 3.2, then $OC(G) = OC(M)$. So the main theorem implies this corollary. ■

Remark 3.3 Wujie Shi and Bi Jianxing in [22] put forward the following conjecture:

Conjecture *Let G be a group, M a finite simple group. Then* $G \cong M$ *if and only if*

(i) $|G| = |M|$, *and*
(ii) $\pi_e(G) = \pi_e(M)$, *where* $\pi_e(G)$ *denotes the set of orders of elements in G.*

This conjecture is valid for sporadic simple groups [20], groups of alternating type [24], and some simple groups of Lie types [21–23]. As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.

Corollary 3.4 *Let G be a finite group and* $M = PSU_{11}(q)$. *If* $|G| = |M|$ *and* $\pi_e(G) = \pi_e(M)$, *then* $G \cong M$.

Proof The assumption implies that $OC(G) = OC(M)$, then the corollary follows by the main theorem. ■

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