THE TERM RANK OF A MATRIX

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1. Introduction. This paper continues a study appearing in **(5)** of the combinatorial properties of a matrix A of m rows and n columns, all of whose entries are 0's and 1's. Let the sum of row i of A be denoted by r_i and let the sum of column i of A be noted by s_i. We call $R = (r_1, \ldots, r_m)$ the *row sum vector* and $S = (s_1, \ldots, s_n)$ the *column sum vector* of A. The vectors R and S determine a class $\mathfrak{A}$ consisting of all (0, 1)-matrices of m rows and n columns, with row sum vector R and column sum vector S. Simple arithmetic properties of R and S are necessary and sufficient for the existence of a class $\mathfrak{A}$ **(1; 5)**.

Let $\delta_i = (1, \ldots, 1, 0, \ldots, 0)$ be a vector of n components, with 1's in the first r_i positions, and 0's elsewhere. A matrix of the form

$$\bar{A} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_m \end{bmatrix}$$

is called *maximal*, and $\bar{A}$ is called the *maximal form* of A. Note that $\bar{A}$ is formed from A by a rearrangement of the 1's in the rows of A. It is clear that for $\bar{A}$ maximal, the class $\mathfrak{A}$ contains the single entry $\bar{A}$.

Consider the 2 by 2 submatrices of A of the types

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An *interchange* is a transformation of the elements of A that changes a minor of type A_1 into type A_2, or vice versa, and leaves all other elements of A unaltered. The interchange theorem **(5)** asserts that if A and A^* are arbitrary in $\mathfrak{A}$, then A is transformable into A^* by a finite sequence of interchanges.

The *term rank* ρ of A is the order of the greatest minor of A with a non-zero term in its determinant expansion **(4)**. This integer is also equal to the minimal number of rows and columns that collectively contain all the non-zero elements of A **(3)**. Let $\tilde{\rho}$ be the minimal and $\bar{\rho}$ the maximal term rank for the matrices in $\mathfrak{A}$. The interchange theorem **(5)** implies the existence of an A in $\mathfrak{A}$ of term rank ρ, an arbitrary integer in the interval $\tilde{\rho} \leqslant \rho \leqslant \bar{\rho}$. In what follows we derive a simple formula for $\bar{\rho}$ and study further combinatorial consequences of the term rank concept.

2. The maximal term rank. Let $\mathfrak{A}$ be the class of all (0, 1)-matrices A of m rows and n columns, with row sum vector R and column sum vector S.

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We suppose throughout that the components of the row sum vector R and column sum vector S of A are positive. This is no genuine restriction on A in the study of term rank. We proceed to evaluate $\bar{\rho}$, the maximal term rank for the matrices in $\mathfrak{A}$.

For this purpose, let $R' = (r_1 - 1, \ldots, r_m - 1)$, where $r_i - 1 \geqslant 0$. Let $\bar{A}'$ be the maximal matrix of m rows and n columns having row sum vector R', and let the column sum vector of $\bar{A}'$ equal

$$\bar{S}' = (\bar{s}_1', \ldots, \bar{s}_n').$$

Note that if $\bar{A}$ is the maximal form of A and if the column sum vector of $\bar{A}$ is $(\bar{s}_1, \ldots, \bar{s}_n)$, then $\bar{s}_i' = \bar{s}_{i+1}$ $(i = 1, \ldots, n-1)$ and $\bar{s}_n' = 0$. Renumber the subscripts of the column sum vector $S = (s_1, \ldots, s_n)$ of A so that

$$s_1 \geqslant \ldots \geqslant s_n,$$

and define the integers $s_i' \geqslant 0$ by

$$s_i' = s_i - 1 \qquad (i = 1, \ldots, n).$$

Finally, let

$$\bar{s}_o' = s_o' = 0.$$

THEOREM 2.1. *Let $\bar{\rho}$ equal the maximal term rank for the matrices in $\mathfrak{A}$. Let M equal the largest integer in the set*

$$\sum_{i=0}^{k} (s_i' - \bar{s}_i') \qquad (k = 0, 1, \ldots, n).$$

Then

$$\bar{\rho} = m - M.$$

Let $A_{\bar{\rho}}$ be the m by n matrix with maximal term rank $\bar{\rho}$. Without loss of generality, we may assume that the row sum vector $R = (r_1, r_2, \ldots, r_m)$ and column sum vector $S = (s_1, s_2, \ldots, s_n)$ of $A_{\bar{\rho}}$ satisfy $r_1 \geqslant \ldots \geqslant r_m$ and $s_1 \geqslant \ldots \geqslant s_n$. We select a specified set of $\bar{\rho}$ 1's of $A_{\bar{\rho}}$ accounting for the maximal term rank and call them the *essential* 1's of $A_{\bar{\rho}}$. All other 1's of $A_{\bar{\rho}}$ are then referred to as *unessential*.

We derive two Lemmas.

LEMMA 1. *For $0 \leqslant k \leqslant n$,*

$$\sum_{i=0}^{k} (s_i' - \bar{s}_i') \leqslant m - \bar{\rho}.$$

Let B be formed from $A_{\bar{\rho}}$ by replacing the $\bar{\rho}$ essential 1's of $A_{\bar{\rho}}$ by 0's. We agree to write $A_{\bar{\rho}}$ so that

$$s_1 \geqslant \ldots \geqslant s_n; \quad b_1 \geqslant \ldots \geqslant b_n; \quad b_i = s_i + \epsilon_i - 1.$$

Here s_i and b_i denote the sums of column i of $A_{\bar{\rho}}$ and B, respectively, and column i of $A_{\bar{\rho}}$ contains an essential 1 if and only if $\epsilon_i = 0$. Note $\epsilon_i = +1$ for exactly $n - \bar{\rho}$ values of i.

Let $\bar{B}$ be the maximal form of B, with column sums $\bar{b}_1 \geqslant \ldots \geqslant \bar{b}_n$. Then for each k, $0 \leqslant k \leqslant n$,

$$\sum_{i=0}^{k} s_i' \leqslant \sum_{i=0}^{k} b_i \leqslant \sum_{i=0}^{k} \bar{b}_i.$$

From the definitions of the $\bar{s}_i'$ and the $\bar{b}_i$,

$$\sum_{i=0}^{k} \bar{s}_i' + (m - \bar{\rho}) \geqslant \sum_{i=0}^{k} \bar{b}_i,$$

whence

$$\sum_{i=0}^{k} s_i' \leqslant \sum_{i=0}^{k} \bar{s}_i' + (m - \bar{\rho}).$$

LEMMA 2. *Let f be such that $0 < f < n$ and*

$$\sum_{i=0}^{f} (s_i' - \bar{s}_i') = m - \bar{\rho}.$$

Then the matrix $A_{\bar{\rho}}$ of maximal term rank $\bar{\rho}$ may upon permutations of rows and columns be written in the form

$$A_{\bar{\rho}} = \begin{bmatrix} S & E_1 & * & * \\ E_2 & 0 & 0 & 0 \\ * & 0 & I & 0 \\ * & 0 & 0 & 0 \end{bmatrix}.$$

Here S is a matrix entirely of 1's of size e by f. The matrices E_1 and E_2 are square of orders e and f, respectively, I is an identity matrix of order g, with $\bar{\rho} = e + f + g$, and the 0's denote zero blocks. The $\bar{\rho}$ essential 1's of $A_{\bar{\rho}}$ appear on the main diagonals of E_1, E_2, and I. The degenerate cases $e = 0$ and $g = 0$ are not excluded.

Reading the inequalities of Lemma 1 as equalities, we obtain

$$\sum_{i=0}^{f} s_i' = \sum_{i=0}^{f} b_i = \sum_{i=0}^{f} \bar{b}_i = \sum_{i=0}^{f} \bar{s}_i' + (m - \bar{\rho}).$$

This tells us that the matrix B may be written in the form

$$B = \begin{bmatrix} S & X \\ Y & 0 \end{bmatrix},$$

where S is the e by f matrix of 1's, and where the matrix X has at least one 1 in each row. Now

$$\sum_{i=0}^{f} s_i' = \sum_{i=0}^{f} s_i - f = \sum_{i=0}^{f} b_i$$

implies that essential 1's occur in the first f columns of $A_{\bar{\rho}}$, and they may be placed on the main diagonal of E_2.

The equation

$$\sum_{i=0}^{f} s_i = \sum_{i=0}^{f} \bar{s}_i' + m - \bar{\rho} + f$$

implies that there are $m - \bar{\rho} + f$ rows of $A_{\bar{\rho}}$ in which 0's occur in each of the columns $f + 1, \ldots, n$. Let $e' \leqslant e$ essential 1's of $A_{\bar{\rho}}$ occur in rows $1, \ldots, e$ of $A_{\bar{\rho}}$, and let g essential 1's occur in rows $e + f + 1, \ldots, m$ of $A_{\bar{\rho}}$. Then $e' + f + g = \bar{\rho}$ and $m - \bar{\rho} + f + g = m - e$, whence $e' = e$. Hence essential 1's occur in the first e rows of $A_{\bar{\rho}}$, and these may be placed on the main diagonal of E_1.

To prove Theorem 2.1 it suffices to establish the existence of a $k = f$ for which equality holds in Lemma 1. The theorem is valid for m by 1 and 1 by n matrices. The induction hypothesis asserts the statement of the theorem for all matrices of size $m - 1$ by n', with $1 \leqslant n' \leqslant n$, and we shall prove the theorem for matrices of size m by n. Moreover, if $\bar{\rho} = m$, then

$$s_0' - \bar{s}_0' = m - \bar{\rho} = 0.$$

Also, if $\bar{\rho} = n$, then

$$\sum_{i=0}^{n} (s_i' - \bar{s}_i') = \sum_{i=0}^{n} s_i - n - \left(\sum_{i=0}^{n} s_i - m\right) = m - \bar{\rho}.$$

Since the theorem is valid in each of these cases, we may assume that $\bar{\rho} < m$ and $\bar{\rho} < n$.

In $A_{\bar{\rho}}$ suppose that $s_i > s_j$. Then we may normalize the first row of $A_{\bar{\rho}}$ in one of two ways. Either $a_{1i} = 1$ or, in the other case, $a_{1i} = 0$ and $a_{1j} = 0$ or 1, with $a_{1j} = 1$ an essential 1 of $A_{\bar{\rho}}$. For otherwise we must have $a_{1i} = 0$ and $a_{1j} = 1$, an unessential 1 of $A_{\bar{\rho}}$. But then there exists an unessential 1 of $A_{\bar{\rho}}$ such that $a_{ui} = 1$ and $a_{uj} = 0$. We may then perform an interchange that does not affect the term rank and obtain $a_{1i} = 1$ and $a_{1j} = 0$. We agree to normalize the first row of $A_{\bar{\rho}}$ to fulfill this requirement.

Now delete row 1 from the normalized $A_{\bar{\rho}}$ of maximal term rank $\bar{\rho}$. Also delete any zero columns from the resulting $(m - 1)$-rowed matrix. We then obtain a matrix C of $m - 1$ rows and n' columns, $1 \leqslant n' \leqslant n$. Let C belong to the class $\mathfrak{C}$. The maximal term rank for the matrices in $\mathfrak{C}$ equals $\bar{\rho}$ or $\bar{\rho} - 1$.

Suppose there exists a C' of term rank $\bar{\rho}$ in $\mathfrak{C}$. To C' we may adjoin $n - n'$ columns of 0's and the first row of $A_{\bar{\rho}}$, and thereby obtain a matrix $A' = [a_{rs}']$ in the class $\mathfrak{A}$. Now if $a_{1i}' = 1$, where column i does not contain an essential 1 of C', then this contradicts the maximality of $\bar{\rho}$ in $\mathfrak{A}$. Suppose then that $a_{1i}' = 0$ for each column i that does not contain an essential 1 of C'. Since $r_1 \geqslant r_j$, we may perform an interchange involving row 1 and some other row of A' to obtain $a_{1i}' = 1$ for some column i not containing an essential 1 of C'. This again contradicts the maximality of $\bar{\rho}$ in $\mathfrak{A}$. Hence we conclude that $\bar{\rho} - 1$

is the maximal term rank for the matrices in $\mathfrak{C}$. This term rank is attained by C. The $\bar{\rho} - 1$ essential 1's of C plus one essential 1 from the first row of $A_{\bar{\rho}}$ comprise the $\bar{\rho}$ essential 1's of $A_{\bar{\rho}}$.

We permute the columns of C so that $c_1 \geqslant c_2 \geqslant \ldots \geqslant c_{n'}$ and apply the induction hypothesis to C. Then there exists an f, $0 \leqslant f \leqslant n'$, such that

$$\sum_{i=0}^{f} c_i' = \sum_{i=0}^{f} \bar{c}_i' + (m - \bar{\rho}).$$

We may suppose that $0 < f < n'$. For if $f = 0$, then $\bar{\rho} = m$ and the theorem is valid. Also if $f = n'$, then $\bar{\rho} = n' + 1$. This implies that $n' < n$. If $n' = n - 1$, then $\bar{\rho} = n$ and the theorem is valid. Thus if $f = n'$, we may suppose that $n' \leqslant n - 2$. But in this case the last $n - n' \geqslant 2$ columns of $A_{\bar{\rho}}$ must have 1's in the first row, and only one of them can be essential. By the normalization process applied to $A_{\bar{\rho}}$, every column of $A_{\bar{\rho}}$ headed by 0's must have column sum equal to 1 and these columns occupy the last of the first n' positions in $A_{\bar{\rho}}$. If such columns exist we may take a smaller value of f in C. If all of the columns of $A_{\bar{\rho}}$ are headed by 1's, the theorem is valid for $A_{\bar{\rho}}$ with $f = n'$.

Thus we may suppose that $0 < f < n'$, and upon permutations of rows and columns, we may write the matrix C in the form given by Lemma 2:

$$C = \begin{bmatrix} S & D_1 & * & * \\ D_2 & 0 & 0 & 0 \\ * & 0 & I & 0 \\ * & 0 & 0 & 0 \end{bmatrix}.$$

Here S is the matrix of 1's of size e by f, and the orders of D_1, D_2, and I total $\bar{\rho} - 1$. The $\bar{\rho} - 1$ essential 1's of C appear on the main diagonals of D_1, D_2, and I. The matrix I need not appear, but we may assume that $e \neq 0$. For if $e = 0$, we again obtain $\bar{\rho} - 1 = n'$.

We restore now to C the $n - n'$ zero columns, and finally a row of r_1 1's and $n - r_1$ 0's. We thereby obtain $\tilde{A}$, where $\tilde{A} = [\tilde{a}_{rs}]$ is the same as $A_{\bar{\rho}}$ apart from possible row and column permutations. Suppose that $\tilde{a}_{1i} = 1$ $(i = 1, \ldots, f)$. Then

$$\sum_{i=0}^{f} (s_i' - \bar{s}_i') = m - \bar{\rho},$$

and the theorem follows.

Suppose that on the other hand some $\tilde{a}_{1j} = 0$, where $1 \leqslant j \leqslant f$. If we permute the first f columns of $\tilde{A}$, then we may assume that $\tilde{a}_{1i} = 1$ $(i = 1, \ldots, h)$ and that $\tilde{a}_{1j} = 0$ $(j = h + 1, \ldots, f)$. The case $h = 0$ is not to be excluded. If $h = 0$, then $\tilde{a}_{1j} = 0$ $(j = 1, \ldots, f)$. Now there must exist an essential 1 of the form $\tilde{a}_{1u} = 1$ for some u, where u satisfies $e + f + 1 \leqslant u \leqslant n$. If there does not exist an unessential 1 of the form $\tilde{a}_{1v} = 1$, where v satisfies $f + 1 \leqslant v \leqslant n$, then

$$\sum_{i=0}^{f} (s_i' - \bar{s}_i') = m - \bar{\rho},$$

and the theorem is valid. Suppose then that one or more unessential 1's exist of the form $\tilde{a}_{1v} = 1$, where v satisfies $f + 1 \leqslant v \leqslant n$. We assert that then an unessential 1 cannot occur in the intersection of rows $e + 2, \ldots, m$ and columns $h + 1, \ldots, f$ of $\tilde{A}$. For suppose that an unessential 1 appears in this position. Then by our normalization process, for each v associated with the unessential 1's of the form $\tilde{a}_{1v} = 1, f + 1 \leqslant v \leqslant n$, we must have $\tilde{a}_{jv} = 1$ $(j = 1, \ldots, e + 1)$. Furthermore, there must exist in each of these columns an essential 1 of the form $\tilde{a}_{tv} = 1$, for some t satisfying $e + f + 2 \leqslant t \leqslant m$. All of the remaining entries of these columns must be 0. But consider now row 1 and row 2 of $\tilde{A}$. A 1 in row 1 may appear directly above a 0 in row 2 only in the column of the essential 1 of the form $\tilde{a}_{1u} = 1$. However, a 0 in row 1 must appear directly above a 1 in row 2 in at least two columns. But this contradicts the fact that the number of 1's in row 1 of $\tilde{A}$ is greater than or equal to the number of 1's in row 2 of $\tilde{A}$. Thus an unessential 1 cannot occur in the intersection of rows $e + 2, \ldots, m$ and columns $h + 1, \ldots, f$ of $\tilde{A}$. Hence it follows that

$$\sum_{i=0}^{h} s_i' - \sum_{i=0}^{h} \bar{s}_i' = m - \bar{\rho}.$$

Note that the degenerate case $h = 0$ gives $\bar{\rho} = m$. This completes the proof.

3. Applications. In the following applications we continue to require positive components for the vectors R and S that determine the class $\mathfrak{A}$. A (0, 1)-matrix $A = [a_{rs}]$ may be regarded as an incidence matrix distributing n elements $x_1, \ldots, x_n$ into m sets $S_1, \ldots, S_m$. Here $a_{ij} = 1$ or 0 according as x_j is or is not in S_i. From this approach the term rank of a matrix generalizes the concept of a system of distinct representatives for subsets $S_1, \ldots, S_m$ of a finite set **(2)**. The subsets $S_1, \ldots, S_m$ possess a system of distinct representatives if and only if the term rank of the associated incidence matrix satisfies $\rho = m$. In this case we say A possesses a system of distinct representatives.

THEOREM 3.1. *There exists an A in $\mathfrak{A}$ possessing a system of distinct representatives if and only if*

$$\sum_{i=0}^{k} (s_i' - \bar{s}_i') \leqslant 0 \qquad (k = 0, 1, \ldots, n).$$

This is the special case of Theorem 2.1 with $\bar{\rho} = m$.

For a (0, 1)-matrix A, let $N_0(A)$ denote the number of 0's in A and let $N_1(A)$ denote the number of 1's in A.

THEOREM 3.2. *Let A be in $\mathfrak{A}$ and let $\bar{\rho} < m, n$. Then upon permutations of rows and columns, A may be reduced to the form*

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}.$$

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Here W is of size e by f $(0 < e < m, 0 < f < n)$ and $N_0(W) + N_1(Z) = \bar{\rho} - (e+f)$. For $A_{\bar{\rho}}$, we have $N_0(W) = 0$ and $N_1(Z) = \bar{\rho} - (e+f)$.

In the equation

$$\sum_{i=0}^{f} (s_i' - \bar{s}_i') = m - \bar{\rho},$$

we have $0 < f < n$, for otherwise $\bar{\rho} = m$ or $\bar{\rho} = n$. Also for the matrix $A_{\bar{\rho}}$ of Lemma 2, $0 < e < m$ and

$$\sum_{i=0}^{e} r_i + \sum_{i=0}^{f} s_i + \bar{\rho} - (e+f) - ef = N_1(A_{\bar{\rho}}).$$

But

$$\sum_{i=0}^{e} r_i + \sum_{i=0}^{f} s_i = N_1(X) + N_1(Y) + 2N_1(W)$$

and

$$N_1(W) + N_1(X) + N_1(Y) + N_1(Z) = N_1(A_{\bar{\rho}}).$$

Hence

$$ef - N_1(W) + N_1(Z) = \bar{\rho} - (e+f)$$

and

$$N_0(W) + N_1(Z) = \bar{\rho} - (e+f).$$

Let $A = [a_{rs}]$ be in $\mathfrak{A}$. Suppose an element $a_{uv} = 1$ of A is such that no sequence of interchanges applied to A replaces $a_{uv} = 1$ by 0. Then $a_{uv} = 1$ is called an *invariant* 1 of A. An analogous definition holds for an invariant 0.

THEOREM 3.3. *Let a_{uv} be an invariant 1 of A. If $A' = [a_{rs}']$ is in $\mathfrak{A}$, then a_{uv}' is an invariant 1 of A'.*

For if for some $A^* = [a_{rs}^*]$ in $\mathfrak{A}$, $a_{uv}^* = 0$, then transforming A into A^* by interchanges contradicts the hypothesis that $a_{uv} = 1$ is an invariant 1 of A. Thus all or none of the matrices in $\mathfrak{A}$ contains an invariant 1, and we refer to $\mathfrak{A}$ as being with or without an invariant 1.

THEOREM 3.4. *Let A contain an invariant 1. Then by permutations of rows and columns, A may be reduced to the form*

$$\begin{bmatrix} S & X \\ Y & 0 \end{bmatrix}.$$

Here S is the matrix of 1's and contains the invariant 1 of A.

For by permutations of rows and columns we may reduce A to the following form:

$$A^* = \left[\begin{array}{c|c|c|c|c}
1 & 1 \quad \cdots & & 1 & 0 \quad \cdots \quad 0 \\
1 & S & S^* & C_0 & R_1 \\ \cline{2-5}
\cdot & \bar{S} & * & M & \\ \cline{2-4}
1 & R_0 & N & 0 & 0 \\ \hline
0 & & & & \\
\cdot & C_1 & 0 & & 0 \\
0 & & & &
\end{array}\right].$$

Here the 1 in the (1, 1) position of A^* is the invariant 1. The block in the lower right hand corner is then composed entirely of 0's. We permute rows so that R_1 contains at least one 1 in each row, and then permute columns so that C_1 contains at least one 1 in each column. The intersection of the rows of A^* containing R_1 and the columns of A^* containing C_1 is S, a matrix of 1's. We now permute columns so that S^* is a matrix of 1's and C_0 contains at least one 0 in each column. Next we permute rows so that $\bar{S}$ is a matrix of 1's and R_0 contains at least one 0 in each row. The intersection of the columns of A^* containing C_0 and the rows of A^* containing R_0 is a zero matrix. If one or more of S^*, C_0, $\bar{S}$, R_0 do not appear, the theorem follows. If all appear, we replace M by a matrix of the form

$$\begin{bmatrix} R_1^* \\ 0 \end{bmatrix}$$

and N by a matrix of the form $[C_1^* \; 0]$, where R_1^* has at least one 1 in each row and C_1^* has at least one 1 in each column, and then continue as before. This procedure must terminate, and upon termination we obtain the matrix of the theorem.

Note that X and Y may contain further invariant 1's and the normalizing procedure may be applied to each of these blocks separately. Also, if A, X, and Y are of term ranks ρ, ρ_x, and ρ_y, respectively, and if S has size e' by f', then

$$\rho = \rho_x + \rho_y + \min\,(e' - \rho_x, f' - \rho_y),$$

whence

$$\rho = \min\,(e' + \rho_y, f' + \rho_x).$$

THEOREM 3.5. *If* $\mathfrak{A}$ *is without an invariant* 1 *and if* $\bar{\rho} < m, n$, *then the minimal term rank* $\tilde{\rho}$ *for the matrices in* $\mathfrak{A}$ *must satisfy* $\tilde{\rho} < \bar{\rho}$.

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In the matrix $A_{\bar{\rho}}$ of Theorem 3.2, the 1 in the (1, 1) position is not invariant. But by Theorem 3.2, $N_0(W) + N_1(Z) = \bar{\rho} - (e + f)$. This means that there are matrices in $\mathfrak{A}$ with fewer than $\bar{\rho} - (e + f)$ 1's in Z. Hence $\tilde{\rho} < \bar{\rho}$.

Note that Theorem 3.5 is not necessarily valid for $\bar{\rho} = m$. For we may let $m = n$, and let $\mathfrak{A}$ be the class of all (0, 1)-matrices with exactly k 1's in each row and column, $1 \leqslant k < m$. Then $\mathfrak{A}$ is without an invariant 1, but $\tilde{\rho} = \bar{\rho} = m$ **(3)**. Also Theorem 3.5 need not hold for a class $\mathfrak{A}$ with an invariant 1. For example, let A be maximal. Then A is the only matrix in $\mathfrak{A}$, and we must have $\tilde{\rho} = \bar{\rho}$.

In conclusion, a deeper insight into the structure of $\tilde{\rho}$ would be of considerable interest. An arithmetic formula for $\tilde{\rho}$ analogous to the formula for $\bar{\rho}$ given in §2 would be especially desirable.

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