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ON DIRECT BIFURCATIONS INTO CHAOS AND ORDER FOR A SIMPLE FAMILY OF INTERVAL MAPS

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We present a simple one-parameter family of interval maps which has a direct bifurcation from order to chaos and then a direct (reverse) bifurcation from chaos back to order.

1. INTRODUCTION

In this note, we present a simple one-parameter family of interval maps which has a direct bifurcation from order to chaos and then another direct bifurcation from chaos back to order. (See also [4, 5].) In fact, for this family of interval maps, the creation of the first non-fixed periodic point is more complicated than we expect. It is the limit point of a series of bifurcations of period $2n$ ($n \geqslant 3$ odd) points. Consequently, the creation of the first non-fixed periodic point is a bifurcation of period 12 points. After the bifurcation into chaos, this family undergoes a series of bifurcations of period $2n$ points with n ($\geqslant 3$ odd) in decreasing order. After the period 6 points are created and live for a while, then, all of a sudden, all chaotic phenomena cease to exist and we have order again. To be more precise, we shall prove the following two results.

THEOREM 1. *Let b be a fixed number in $[3/8, 1/2)$. For $0 \leqslant c \leqslant b$, let*

$$f_c(x) = \begin{cases} 3/4, & 0 \leqslant x \leqslant c, \\ x/(2-4c) + (3-8c)/(4-8c), & c \leqslant x \leqslant 1/2, \\ 1 + (c-1)(2x-1), & 1/2 \leqslant x \leqslant 1, \end{cases}$$

and, for $b \leqslant c \leqslant 1$, let

$$f_c(x) = \begin{cases} 3/4, & 0 \leqslant x \leqslant b, \\ x/(2-4b) + (3-8b)/(4-8b), & b \leqslant x \leqslant 1/2, \\ 1 + (c-1)(2x-1), & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Then the following hold:

(1) *For $c = 0$, f_c has a periodic orbit of least period 4 and no periodic orbit of least period > 4.*

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(2) *For* $0 < c < 1/2$, f_c *has periodic points of least period* 12.

(3) *For* $c = 1/2$, f_c *has infinitely many periodic orbits of least period* 2 *and no periodic orbit of least period* > 2.

(4) *For* $1/2 < c \leqslant 1$, f_c *has exactly one fixed point and no other periodic point.*

REMARKS. (1) Parts (1) and (2) of Theorem 1 imply that $c = 0$ is a bifurcation point of period 12 points for f_c. Consequently, $c = 0$ is a bifurcation point of f_c from order to chaos.

(2) Parts (2)–(4) of Theorem 1 imply that $c = 1/2$ is a bifurcation point of f_c from chaos back to order. Note that the results in the following Theorem 2 are much stronger than Part (2) of Theorem 1.

THEOREM 2. *Let* $g_3(x) = 2x^3 - 4x^2 + 3x - 1/2$ *and, for odd integer* $k \geqslant 3$, *let* $g_{k+2}(x) = x/2 + [(1-x)^2/(1-2x)^2]g_k(x)$. *For every odd integer* $n \geqslant 3$, *let* c_n *denote the unique positive zero of* $g_n(x)$ *in* $[0, 1/2)$. *For any fixed number* b *in* $[3/8, 1/2)$ *and any* $0 \leqslant c \leqslant 1$, *let* $f_c(x)$ *be the continuous map from* $[0, 1]$ *into itself defined as in Theorem 1. Then the following hold:*

(1) $c_3 > c_5 > c_7 > \cdots > 0$ *and* $\lim_{k\to\infty} c_{2k+1} = 0$.

(2) *For every odd integer* $n \geqslant 3$ *and every* $c_n \leqslant c < 1/2$, f_c *has at least one periodic point of least period* $2n$.

REMARKS. (1) We note that, in Theorem 2, the value c_n is a value for which $\{1/2, 1, c_n, 3/4, \ldots\}$ is a period $2n$ orbit of f_{c_n}.

(2) Since the map $[(1-x)/(1-2x)]^2$ is strictly increasing on $[0, 1/2)$, it follows by induction that each $g_k(x)$, $k \geqslant 3$ odd, is also strictly increasing on $[0, 1/2)$. So, each $g_k(x)$, $k \geqslant 3$ odd, has a unique (positive) zero in $[0, 1/2)$.

2. PROOFS OF THEOREMS 1 AND 2

For the proofs of Theorems 1 and 2, we need the following two well-known results:

LEMMA 1. (Sharkovskii's theorem [**1-3, 6-8, 10-13**]). Rearrange the set of positive integers according to the following order: $3 \to 5 \to 7 \to \cdots \to 2.3 \to 2.5 \to 2.7 \to \cdots \to 2^k.3 \to 2^k.5 \to 2^k.7 \to \cdots \to 2^j \to 2^{j-1} \to \cdots \to 2^3 \to 2^2 \to 2 \to 1$. Assume that f is a continuous map from $[0, 1]$ into itself which has a periodic point of least period m. Then f also has a periodic point of least period n for every n with $m \to n$.

LEMMA 2. ([**9**]). *Let* f *be a continuous map from* $[0, 1]$ *into itself and let* $n \geqslant 3$ *be an odd integer. Assume that, for some* $x_0 \in [0, 1]$, *we have either* $f^n(x_0) \leqslant x_0 < f(x_0)$ *or* $f(x_0) < x_0 \leqslant f^n(x_0)$. *Then* f *has periodic points of least period* n.

PROOF OF THEOREM 1: Parts (1), (3) and (4) are quite obvious. So we only give a proof of Part (2). It is clear that $f_c^4(1/2) = (1+c)/2 > 1/2$, $f_c^5(1/2) = c^2 - c + 1 > 1/2$, and $f_c^6(1/2) = 2c^3 - 4c^2 + 3c$. Since $f_c^6(1/2)$ is a strictly increasing map of c, there is a unique value $a \approx .221855$ such that $f_a^6(1/2) = 1/2$. So, for $a \leqslant c < 1/2$, we have $f_c^2(1/2) = c < 1/2 \leqslant f_c^6(1/2)$. By Lemma 2, f_c^2 has a period 3 point and so, by Lemma 1, f_c has a period 6 point for $a \leqslant c < 1/2$.

Now assume that $0 < c < a$. Then $f_c^6(1/2) < 1/2$ so, $f_c^8(1/2) = -(c-1/2)^3 + (c-1/2)^2 + (c-1/2)/2 + 1/2 + 1/[16(c-1/2)]$. Then,

$$\frac{\partial}{\partial c}\left[f_c^8\left(\frac{1}{2}\right)\right] = -3\left(c-\frac{1}{2}\right)^2 + 2\left(c-\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{[16(c-1/2)^2]}$$
$$= -3c^2 + 5c - \frac{5}{4} - \frac{1}{[16(c-1/2)^2]} < -3c^2 + 5c - \frac{5}{4} < 0$$

for, say, $$0 < c < \frac{(5-\sqrt{10})}{6} \approx .306.$$

That is, $f_c^8(1/2)$ is a decreasing map of c for $0 < c < a$. In particular, $f_c^8(1/2) < f_0^8(1/2) = 1/2$ for $0 < c < a$. Consequently, for $0 < c < a$, we obtain that

$$f_c^{10}\left(\frac{1}{2}\right) - \frac{1}{2} = \frac{[(c-1/2)^3]}{2} - \frac{3[(c-1/2)^2]}{4} + \left(c-\frac{1}{2}\right) + \frac{1}{8} - \frac{1}{[32(c-1/2)]} + \frac{1}{[64(c-1/2)^2]}$$
$$= \left\{\frac{1}{64}\left[\left(c-\frac{1}{2}\right)^{-2}\right]\right\}\left[32\left(c-\frac{1}{2}\right)^5 - 48\left(c-\frac{1}{2}\right)^4 + 64\left(c-\frac{1}{2}\right)^3 + 8\left(c-\frac{1}{2}\right)^2 - 2\left(c-\frac{1}{2}\right) + 1\right].$$

Let $C = c - 1/2$. Then it is easy to see that the map $32C^5 - 48C^4 + 64C^3 + 8C^2 - 2C + 1$ has a unique negative zero at approximately $C_0 \approx -.307141$ or, equivalently, at $c_0 \approx .192859$. Therefore, we easily obtain that $f_c^{10}(1/2) < 1/2$ for $0 < c < c_0 \approx .192859$ and $f_c^{10}(1/2) \geqslant 1/2$ for $c_0 \leqslant c < a$. So, assume that $c_0 \leqslant c < a$. Then $f_c^{10}(1/2) \geqslant 1/2 > f_c^2(1/2)$. By Lemma 2, f_c^2 has a period 5 point and so, by Lemma 1, f_c has a period 10 point for $c_0 \leqslant c < a$.

Finally assume that $0 < c < c_0$. Then $f_c^6(1/2) < 1/2$, $f_c^8(1/2) < 1/2$ and $f_c^{10}(1/2) < 1/2$. So

$$f_c^{12}\left(\frac{1}{2}\right) = -\frac{[(c-1/2)^3]}{4} + \frac{[(c-1/2)^2]}{2} + \frac{5(c-1/2)}{16} + \frac{11}{16} + \frac{3}{[64(c-1/2)]} - \frac{1}{[64(c-1/2)^2]} + \frac{1}{[256(c-1/2)^3]}.$$

Let $C = c - 1/2$ and let $h(C) = -C^3/4 + C^2/2 + 5C/16 + 11/16 + 3/(64C) - 1/(64C^2) + 1/(256C^3) - 1/2$. Then

$$\begin{aligned} h(C) &= \left[-\frac{1}{(256C^3)}\right][64C^6 - 128C^5 - 80C^4 - 48C^3 - 12C^2 + 4C - 1] \\ &= \left[-\frac{1}{(256C^3)}\right](2C+1)(32C^5 - 80C^4 - 24C^2 + 6C - 1) \leqslant 0 \text{ when } -\frac{1}{2} < C < 0. \end{aligned}$$

Consequently, $f_c^{12}(1/2) < 1/2$ when $0 < c < c_0$. So, for $0 < c < c_0$, we have $f_c^{12}(1/2) < 1/2 < f_c^4(1/2)$. By Lemma 2, f_c^4 has a period 3 point and so, by Lemma 1, f_c has a period 12 point for $0 < c < c_0$.

By Lemma 1, we obtain that f_c has a periodic point of least period 12 for every $0 < c < 1/2$. This proves Part (2). The proof of Theorem 1 is now complete. ∎

PROOF OF THEOREM 2: By assumption, we have, for $n \geqslant 3$ odd,

$$\begin{aligned} g_{n+2}(x) &= \frac{x}{2} + \left(\frac{1-x}{1-2x}\right)^2 g_n(x) \\ &= \frac{x}{2} + \frac{x}{2}\left(\frac{1-x}{1-2x}\right)^2 + \left(\frac{1-x}{1-2x}\right)^4 \left[\frac{x}{2} + \left(\frac{1-x}{1-2x}\right)^2 g_{n-4}(x)\right] \\ &= \qquad \cdots \\ &= \frac{x}{2} + \frac{x}{2}\left(\frac{1-x}{1-2x}\right)^2 + \frac{x}{2}\left(\frac{1-x}{1-2x}\right)^4 + \cdots + \left(\frac{1-x}{1-2x}\right)^{2k} g_{n-2k+2}(x) \\ &= \frac{x}{2}\,\frac{[(1-x)/(1-2x)]^{2k} - 1}{[(1-x)/(1-2x)]^2 - 1} + \left(\frac{1-x}{1-2x}\right)^{2k} g_{n-2k+2}(x) \end{aligned}$$

In particular, $g_{2m+1}(0) = g_3(0) = -1/2$ and

$$g_{2m+1}(x) = \frac{x}{2}\,\frac{[(1-x)/(1-2x)]^{2m} - 1}{[(1-x)/(1-2x)]^2 - 1} + \left(\frac{1-x}{1-2x}\right)^{2m} g_3(x).$$

Since $(1 - c_{2m+1})/(1 - 2c_{2m+1}) > 1$, it is clear that the zeros of g_{2m+1} tend to 0^+ as m tends to infinity.

On the other hand, if $x = c_n$, where $n \geqslant 3$ is odd, then $g_n(c_n) = 0$ and so $g_{n+2}(c_n) = c_n/2 > 0$. But $g_{n+2}(0) = -1/2 < 0$. So $0 < c_{n+2} < c_n$. This proves Part (1).

For the proof of Part (2), we note that g_3 is strictly increasing and has a unique zero at $c_3 \approx .221855$. Furthermore, for $c_3 \leqslant c < 1/2$, we have $f_c^6(1/2) = g_3(c) + 1/2 \geqslant 1/2$. By Lemmas 1 and 2, f_c has at least one period 6 orbit for $c_3 \leqslant c < 1/2$. Let

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$a = \max\{0 < c < c_3 \mid f_c^8(1/2) = 1/2\}$. Then

$$f_a^8\left(\frac{1}{2}\right) = \frac{1}{2},\ f_a^9\left(\frac{1}{2}\right) = 1,\ f_a^{10}\left(\frac{1}{2}\right) = a < \frac{1}{2}.$$

On the other hand,

$$f_{c_3}^6\left(\frac{1}{2}\right) = \frac{1}{2},\ f_{c_3}^7\left(\frac{1}{2}\right) = 1,\ f_{c_3}^8\left(\frac{1}{2}\right) = c_3 < \frac{1}{2},\ f_{c_3}^9\left(\frac{1}{2}\right) = \frac{3}{4},$$

and

$$f_{c_3}^{10}\left(\frac{1}{2}\right) = \frac{(1+c_3)}{2} > 1/2.$$

So, if $c_5 = \min\{0 < c < c_3 \mid f_c^{10}(1/2) \geqslant 1/2 \text{ on } (c, c_3)\}$, then $c_5 > a$ and hence, for $c_5 \leqslant c < c_3$, we have $f_c^6(1/2) < 1/2$, $f_c^7(1/2) > 1/2$, $f_c^8(1/2) < 1/2$, and $f_c^9(1/2) > 1/2$. So, by direct computation,

$$f_c^{10}\left(\frac{1}{2}\right) = g_5(c) + \frac{1}{2} = \frac{c}{2} + \left[\frac{(1-c)^2}{(1-2c)^2}\right]\left[f_c^6\left(\frac{1}{2}\right) - \frac{1}{2}\right] + \frac{1}{2} \geqslant \frac{1}{2}$$

for $c_5 \leqslant c < c_3$. It then follows from Lemmas 1 and 2 and the above that f_c has periodic points of least period 10 for $c_5 \leqslant c < 1/2$.

Assume that $c_3 > c_5 > c_7 > \cdots > c_{2k+1} > 0$ are defined with the following properties:

(a) For each $2 \leqslant i \leqslant k$, $c_{2i+1} = \min\{0 < s < c_{2i-1} \mid f_c^{2(2i+1)}(1/2) \geqslant 1/2 \text{ on } (s, c_{2i-1})\}$.

(b) For $c_{2i+1} \leqslant c < c_{2i-1}$, $2 \leqslant i \leqslant k$, we have $f_c^{2(2i+1)}(1/2) = g_{2i+1}(c) + 1/2 = c/2 + [(1-c)^2/(1-2c)^2][f_c^{2(2i-1)}(1/2) - 1/2] + 1/2 \geqslant 1/2$.

(c) For $c_{2i+1} \leqslant c < 1/2$, $1 \leqslant i \leqslant k$, f_c has periodic points of least period $2(2i+1)$.

Note that since, for each odd $m \geqslant 3$, $g_{m+2}(x) = x/2 + [(1-x)^2/(1-2x)^2]g_m(x)$, we see that $g_{m+2}(x) \geqslant 0$ whenever $g_n(x) \geqslant 0$ and $0 < x < 1/2$. Consequently, the c_{2i+1}'s defined here are exactly the same as those defined in Theorem 2. Now since

$f_{c_{2k+1}}^{2(2k+1)}(1/2) = 1/2$, hence

$$f_{c_{2k+1}}^{4k+3}\left(\frac{1}{2}\right) = 1,\ f_{c_{2k+1}}^{4k+4}\left(\frac{1}{2}\right) = c_{2k+1} < \frac{1}{2},\ f_{c_{2k+1}}^{4k+5}\left(\frac{1}{2}\right) = \frac{3}{4},$$

and
$$f_{c_{2k+1}}^{4k+6}\left(\frac{1}{2}\right) = \frac{(1+c_{2k+1})}{2} > \frac{1}{2}.$$

If
$$d = \max\{0 < c < c_{2k+1} \mid f_c^{4k+4}\left(\frac{1}{2}\right) = \frac{1}{2}\},$$

then
$$f_d^{4k+4}\left(\frac{1}{2}\right) = \frac{1}{2},\ f_d^{4k+5}\left(\frac{1}{2}\right) = 1,$$

and
$$f_c^{4k+6}\left(\frac{1}{2}\right) = d < \frac{1}{2}.$$

So
$$f_d^{4k+6}\left(\frac{1}{2}\right) = d < \frac{1}{2} < f_{c_{2k+1}}^{4k+6}\left(\frac{1}{2}\right).$$

Thus, if
$$c_{2k+3} = \min\{0 < s < c_{2k+1} \mid f_c^{2(2k+3)}\left(\frac{1}{2}\right) \geqslant \frac{1}{2} \text{ on } (s,\ c_{2k+1})\},$$

then $d < c_{2k+3} < c_{2k+1}$. Therefore, for $c_{2k+3} \leqslant c < c_{2k+1}$, we have

$$f_c^{2(2k+1)}\left(\frac{1}{2}\right) < \frac{1}{2},\ f_c^{4k+3}\left(\frac{1}{2}\right) > \frac{1}{2},\ f_c^{4k+4}\left(\frac{1}{2}\right) < \frac{1}{2},\ \text{and } f_c^{4k+5}\left(\frac{1}{2}\right) > \frac{1}{2}.$$

So, by direct computation, we obtain that

$$f_c^{2(2k+3)}\left(\frac{1}{2}\right) = g_{2k+3}(c) + \frac{1}{2} = \frac{c}{2} + \left[\frac{(1-c)^2}{(1-2c)^2}\right]\left[f_c^{2(2k+1)}\left(\frac{1}{2}\right) - \frac{1}{2}\right] + \frac{1}{2} \geqslant \frac{1}{2}.$$

By Lemmas 1 and 2, f_c has periodic points of least period $2(2k+3)$ for $c_{2k+3} \leqslant c < c_{2k+1}$. Since f_c has periodic points of least period $2(2k+1)$ for $c_{2k+1} \leqslant c < 1/2$, we obtain that, by Lemma 1, f_c has periodic points of least period $2(2k+3)$ for $c_{2k+3} \leqslant c < 1/2$. Part (2) now follows from induction on $k \geqslant 1$.

This completes the proof of Theorem 2. □

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