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KÄHLER IDENTITY ON LEVI FLAT MANIFOLDS AND APPLICATION TO THE EMBEDDING

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Dedication to the memory of Professor Shigeo Nakano

Abstract. It is shown that any compact Levi flat manifold admitting a positive line bundle is embeddable into $\mathbb{P}^n$ by a CR mapping with an arbitrarily high, though finite, order of regularity.

Introduction

For the $\overline{\partial}$ operator on complex manifolds, it is well known that the complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is a real operator on any Kähler manifold, in the sense that the identity $\Box u = \overline{\Box \overline{u}}$ holds there, for any smooth differential form u. Hodge thery and Kodaira-Nakano's vanishing theorem on the $\overline{\partial}$-cohomology groups are famous consequences of this property of complex Laplacian. The purpose of this article is to establish a similar identity for the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on Levi flat CR manifolds and derive some consequences of it. What we shall prove are as follows.

1. An analogue of the Kähler identity on Levi flat CR manifolds which admit "Kähler" metrics.
2. Nakano's formula for $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$
3. Vanishing theorem for the L^2 $\overline{\partial}_b$-cohomology of compact Levi flat manifolds.
4. An embedding theorems for compact Levi flat manifolds.

§1. Levi flat manifold and its normal bundle

Recall that a CR manifold is by definition a $(2n-1)$-dimensinal real manifold of class C^∞, say M, whose complexified tangent bundle $T_M \otimes \mathbb{C}$

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is equipped with a smooth ($= C^\infty$) complex subbundle $T^{1,0}$ ($= T_M^{1,0}$) such that

1. $T^{1,0}$ is of rank $n-1$.
2. $T_x^{1,0} \cap \overline{T_x^{1,0}}$ for any $x \in M$.
3. The set of smooth sections of $T^{1,0}$ is closed under the Lie bracket.

The notion of CR manifold is modeled on real hypersurfaces of complex manifolds. In such a case, $T^{1,0}$ is just the set of holomorphic tangent vectors of the hypersurface.

A CR function on M is by definition a locally integrable function f on M such that $Lf = 0$ for any smooth section L of $\overline{T^{1,0}}$. Here Lf is defined in distribution sense. A CR manifolds M is called a Levi flat manifold if $T^{1,0} + \overline{T^{1,0}}$ is also closed under the Lie bracket. By Frobenius's theorem and by Newlander-Nirenberg's theorem (with parameters), any Levi flat manifold has a system of local coordinates (z,t) with values in $\mathbb{C}^{n-1} \times \mathbb{R}$ such that $\xi(z) = 0$ and $\xi(t) = 0$ for any $\xi \in \overline{T^{1,0}}$. A locally integrable function f on a Levi flat manifold is CR if and only if f is holomorphic with respect to z. Note that a hypersurface of a complex manifold is Levi flat if and only if it locally separate the ambient manifold into pseudoconvex domains. As for nontrivial examples of compact Levi flat hypersurfaces, see [D-O] and [O-S]. In what follows M will represent an oriented and paracompact Levi flat manifold and $\{U_i\}_{i \in j}$ a locally finite open covering of M by coordinate neighbourhoods U_i with local parameters (z^i, t^i) as above. We shall always choose (z^i, t^i) in such a way that they extend to local coordinates on some neighbourhoods of U_i and

$$\frac{\partial t^i}{\partial t^j} \left(= \frac{dt^i}{dt^j} \right) > 0 \quad \text{on } U_i \cap U_j \text{ for any } i, j.$$

Note that t^i depends only on t^j.

A connected submanifold $\mathcal{L} \subset M$ of real codimension one is called Levi leaf of M if every connected component of $\mathcal{L} \cap U_i$ is a fiber of the map $t^i : U_i \to \mathbb{R}$. The decomposition of M into the disjoint union of the Levi leaves will be called the Levi foliation of M. A Levi flat manifold is thus foliated by complex submanifolds of real codimension one. A smooth complex vector bundle on M defined by a system of transition functions e_{ij} with respect to the covering $\{U_i\}$ is called a CR vector bundle if the entries

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of e_{ij} are smooth CR functions on $U_i \cap U_j$. Notions as equivalence, direct sum, tensor product, etc. are naturally carried over to CR vector bundles on CR manifolds. What is particular for Levi flat manifolds is that $T^{1,0}$ has a natural structure of a CR vector bundle. Moreover,

$$T_M \otimes \mathbb{C}/(T^{1,0} + \overline{T^{1,0}})$$

has also a canonical structure of a CR vector bundle for the Levi flat manifold M. In fact, $T_M \otimes \mathbb{C}/(T^{1,0} + \overline{T^{1,0}})$ has a system of smooth local frames $\partial/\partial t^i$, for which the transition function are $\partial t^i/\partial t^j$, which are CR. We shall call $T_M \otimes \mathbb{C}/(T^{1,0} + \overline{T^{1,0}})$ the normal bundle of the Levi foliation, or simply the normal bundle of M and denote it by N_M. Obviously N_M is topologically trivial and the powers N_M^α are well defined by transitions $(\partial t^i/\partial t^j)^\alpha$ for $\alpha \in \mathbb{R}$, but nothing more is known in general. (for some special cases, see [LN], [O-1], [O-3]).

§2. The Kähler identity

Let us recall the definition of the $\overline{\partial}_b$ operator. Since $T^{1,0}$ and $\overline{T^{1,0}}$ are subbundles of $T_M \otimes \mathbb{C}$, there are canonical projections from $\bigwedge(T_M \otimes \mathbb{C})^*$ to $\bigwedge^p (T^{1,0})^* \otimes \bigwedge^q \overline{(T^{1,0})}^*$. We put

$$\overline{\partial}_b u = \pi_{p,q+1} \circ d \circ \pi_{p,q}^{-1}(u).$$

Well-definedness of $\overline{\partial}_b$ is clear. ∂_b is defined similarly by $\overline{\partial}_b = \pi_{p+1,q} \circ d \circ \pi_{p,q}^{-1}$.

Let E be a CR line bundle over M. We denote by $C^{p,q}(M, E)$ the set of smooth sections of $\bigwedge^p (T^{1,0})^* \otimes \bigwedge^q \overline{(T^{1,0})}^* \otimes E$ and put

$$C_0^{p,q}(M, E) = \{u \in C^{p,q}(M, E) \mid \operatorname{supp} u \subset M\}.$$

$\overline{\partial}_b$ naturally acts on $C^{p,q}(M, E)$. Given a Riemann metric g on M and a fiber metric h of E, the vector space $C_0^{p,q}(M, E)$ is naturally equipped with an inner product by integration. The completion of $C_0^{p,q}(M, E)$ is denoted by $L^{p,q}(M, E)$. Elements of $L^{p,q}(M, E)$ will be referred to as square integrable (p, q) forms with values in E. The operator $\overline{\partial}_b$ acts on $L^{p,q}(M, E)$ in distribution sense. Particularly, if $E = (N_M^*)^{1/2}$, the inner product of $C_0^{p,q}(M, (N_M^*)^{1/2})$ with respect to g and the fiber metric of $(N_M^*)^{1/2}$ induced from g depends only on the restriction $\hat{g} = g|T^{1,0} + \overline{T^{1,0}}$, since the fiber metric and the factor of the volume form cancel each other in this case. Therefore the (formal) adjoint of

$$\overline{\partial}_b : C^{p,q}(M, (N_M^*)^{1/2}) \longrightarrow C^{p,q+1}(M, (N_M^*)^{1/2})$$

depends only on $\hat{g}$. We note that $N_M = \overline{N_M}$, so that ∂_b acts on $C^{p,q}(M, (N_M^*)^{1/2})$, too. If $\hat{g}$ induces a Hermitian metric on each leaf (We say then g is a Hermitian metric), let ω be the fundamental form of $\hat{g}$ and let $e(\omega)$ be the exterior multiplication by ω. Note that $e(\omega)$ acts on $C^{p,q}(M, E)$ for any CR vector bundle E. Let Λ be the adjoint of $e(\omega)$. It is clear that Λ depends only on $\hat{g}$. Let $*$ be the Hodge's star operator with respect to $\hat{g}$. Then the inner product of $u, v \in C^{p,q}(M, (N_M^*)^{1/2})$ is expressed as

$$(u, v) = \int_M u_i \wedge \overline{*v_i} \wedge dt^i$$

where $u = u_i(dt^i)^{1/2}$ and $v = v_i(dt^i)^{1/2}$. Hence the adjoint $\overline{\partial}_b^*$ and ∂_b^* of $\overline{\partial}_b$ and ∂_b are respectively expressed as

$$\overline{\partial}_b^* = -\overline{*}\,\overline{\partial}_b\,\overline{*}$$

and

$$\partial_b^* = -\overline{*}\,\partial_b\,\overline{*}.$$

Therefore, if g induces a Kähler metric on each Levi leaf, the identities

$$\overline{\partial}_b^* = \sqrt{-1}\,[\partial_b, \Lambda] \tag{1}$$

and

$$\partial_b^* = \sqrt{-1}\,\left[\overline{\partial}_b, \Lambda\right] \tag{2}$$

hold similarly as in the case of Kähler manifolds. We put $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$ and $\overline{\Box}_b = \partial_b\partial_b^* + \partial_b^*\partial_b$. The we obtain from (1) and (2),

$$\Box_b = \overline{\Box}_b \tag{3}$$

provided that $\Box_b$ acts on $C^{p,q}(M, (N_M^*)^{1/2})$ and g is Kählerian in the above sense.

§3. Nakano's identity and its application

In what follows we assume that g is Kählerian . Let E be any CR vector bundle of rank r over M equipped with a smooth Hermitian metric $h \in C^\infty(M, \mathrm{Hom}(E, \overline{E^*}))$. Let $\overline{\partial}_{b,h}^*$ be the adjoint of $\overline{\partial}_b : C^{p,q}(M, E) \to C^{p,q+1}(M, E)$ with respect to g and h. As well as $\overline{\partial}_b^*$, $\overline{\partial}_{b,h}^*$ depends only on g and h if $\overline{\partial}_b$ acts on $C^{p,q}(M, E \otimes (N_M^*)^{1/2})$. In such a case, $\overline{\partial}_{b,h}^*$ is locally expressed as

$$\overline{\partial}_{b,h}^* u = -(h^{-1}\overline{*}\,\overline{\partial}_b\overline{*}\,u_i)(dt^i)^{1/2}$$

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if $u = u_i(dt^i)^{1/2}$. We put further $\partial_{b,h} = h^{-1}\circ\partial_b\circ h$. $\partial_{b,h}$ acts on $C^{p,q}(M, E\otimes (N_M^*)^{1/2})$ as well as on $C^{p,q}(M,E)$. Note that the adjoint of

$$\partial_{b,h} : C^{p,q}(M, E\otimes (N_M^*)^{1/2}) \longrightarrow C^{p+1,q}(M, E\otimes (N_M^*)^{1/2})$$

is $\partial_b^* = -\overline{*}\,\partial_b\overline{*}$. We put $\Box_{b,h} = \overline{\partial}_b\overline{\partial}_{b,h}^* + \overline{\partial}_{b,h}^*\overline{\partial}_b$ and $\Box_{b,h} = \partial_b\partial_{b,h}^* + \partial_{b,h}^*\partial_b$. Then (1) and (3) are generalized respectively to the identities

$$\overline{\partial}_{b,h}^* = \sqrt{-1}\,[\partial_{b,h}, \Lambda] \tag{4}$$

and

$$\Box_{b,h} = \overline{\Box}_{b,h} + \left[\sqrt{-1}(\partial_{b,h}\overline{\partial}_b + \overline{\partial}_b\partial_{b,h}), \Lambda\right] \tag{5}$$

on $C^{p,q}(M, E\otimes (N_M^*)^{1/2})$. Note that the operator $\partial_{b,h}\overline{\partial}_b + \overline{\partial}_b\partial_{b,h}$ is equal to the multiplication by a section of $(T^{0,1})^*\otimes\overline{(T^{1,0})}^*\otimes \mathrm{Hom}(E,E)$, say Θ_h. Θ_h is called the curvature form of h. Denoting the (exterior) multiplication of Θ_h by $e(\Theta_h)$, (5) will be written as

$$\Box_{b,h} = \overline{\Box}_{b,h} + \left[\sqrt{-1}\,e(\Theta_h), \Lambda\right] \tag{6}$$

this is an analogue of well known Nakano's equality on Kähler manifolds. As in the complex manifolds case, Θ_h and (E,h) will be said to be positive if the matrix $(\Theta^{\nu}_{\mu\alpha\overline{\beta}})$ appearing in the local expression $\sum_{\alpha,\beta}\Theta^{\nu}_{\mu\alpha\overline{\beta}}\,dz^\alpha\wedge d\overline{z}^\beta$ of Θ_h satisfies

$$\sum_{\alpha,\beta,\kappa,\mu,\nu} h_{\kappa\overline{\nu}}\Theta^{\kappa}_{\mu\alpha\overline{\beta}}\xi^{\alpha\mu}\overline{\xi^{\beta\nu}} > 0$$

for any $(\xi^{\alpha\mu}\in\mathbb{C}^{r(n-1)})\backslash\{0\}$.

THEOREM 1. (analogue of Nakano's vanishing theorem) *Let M be a compact Levi flat CR manifold of dimension $2n-1$, and let (E,h) be a Hermitian CR vector bundle over M whose curvature form Θ_h is positive. Then, for any $\overline{\partial}_b$-closed square integrable $(n-1,q)$ form v with values in $E\otimes(N_M^*)^{1/2}$, with $q>0$, there exists a square integrable $(n-1,q-1)$ form u with values in $E\otimes(N_M^*)^{1/2}$ such that $\overline{\partial}_b u = v$.*

Proof. Since Θ_h is positive, m carries a Kählerian metric. Hence, from (6) we obtain

$$\left\|\overline{\partial}_b u\right\|^2 + \left\|\overline{\partial}_{b,h}^* u\right\|^2 \geq \left(\sqrt{-1}\,[e(\Theta_h),\Lambda]u, u\right) \tag{7}$$

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for any $u \in C^{p,q}(M, E \otimes (N_M^*)^{1/2})$. Here $\| \ \|$ denotes the L^2 norm. From the positivity of Θ_h again, there exists a positive constant c such that

$$\|\overline{\partial}_b u\|^2 + \|\overline{\partial}_{b,h}^* u\|^2 \geq c\|u\|^2 \tag{8}$$

holds for any $u \in C^{n-1,q}(M, E \otimes (N_M^*)^{1/2})$ with $q > 0$. Since the estimate (8) carries over to the space $L^{n-1,q}(M, E \otimes (N_M^*)^{1/2}) \cap \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_{b,h}^*$ as in the case of the $\overline{\partial}$ operator, the conclusion is reached by the well known application of Hahn-Banach's theorem. ∎

Similarly we obtain the following.

THEOREM 2. (analogue of Akizuki-Nakano's vanishing theorem) *Let M be as in Theorem 1 and let (B, a) be a CR Hermitian line bundle whose curvature form v with values in $B \otimes (N_M^*)^{1/2}$, with $p + q > n - 1$, there exists a square integrable $(p, q-1)$ form u with values in $B \otimes (N_M^*)^{1/2}$ such that $\overline{\partial}_b u = v$.*

To discuss the regularity of the solutions the equation $\overline{\partial}_b u = v$, estimates for the Sobolev norms are needed. From the formula (6), we can easily deduce the following. The argument may well be omitted since it is standard and routine.

PROPOSITION 1. *Let M be a compact Levi flat minifold and let (B, a) be a positive CR line bundle over M. Then, for any metrized CR vector bundle (E, h) over M and for any positive integer m, there exists an integer $k_0 = k_0(m)$ such that*

$$\|\Box_{b,h\otimes a^k} v\|_m \geq \|\overline{\partial}_{b,h\otimes a^k}^* v\|_m$$

holds for any $v \in C^{p,q}(M, E \otimes B^{\otimes k})$ if $q \geq 1$ and $k \geq k_0$. Here $\| \ \|_m$ denotes a Sobolev norm of order m.

Proof. With respect to a local coordinate (z, t) we have

$$\frac{\partial}{\partial t} \circ \left(\overline{\partial}_{b,h\otimes a^k}^*\right) = \left(\overline{\partial}_{b,h\otimes a^k}\right) \circ D_k + A_k.$$

Here D_k and A_k are respectively of the first and the zeroth order in t. Therefore, if the support of an element v of $C^{p,q}(M, E \otimes B^{\otimes k})$ is contained in a local coordinate neighbourhood, for any fixed $m \in \mathbb{N}$ we have

$$\left\|\frac{\partial^m}{\partial t^m} \Box_{b,h\otimes a^k} v\right\| + \|\Box_{b,h\otimes a^k} v\| \geq \left\|\frac{\partial^m}{\partial t^m} \overline{\partial}_{b,h\otimes a^k}^* v\right\| + \|\overline{\partial}_{b,h\otimes a^k}^* v\|$$

if k is sufficciently large. ∎

By a partition of unity, the required estimate follows from this, if we replace k by a larger number if necessary.

Consequently, similarly as the global embedding theorem of Boutet be Monvel for strongly pseudoconvex CR manifolds, we obtain the following by solving the $\overline{\partial}_b$-equations with L^2 estimates.

Theorem 3. *Let M be a compact Levi flat manifold equipped with a positive CR line bundle (B, a). Then, for any $m \in \mathbb{N}$ there exists a $k_0 \in \mathbb{N}$ such that one can find CR sections $s_0, \ldots, s_N$ of B^k, of class C^m, for any $k \geq k_0$, such that the ratio $(s_0 : \cdots : s_N)$ embeds M into $\mathbb{P}^N$.*

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