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CO-ABSOLUTELY CO-PURE MODULES

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B. Maddox [**15**] defined absolutely pure modules and derived some interesting properties of these modules. C. Megibben [**17**] continued the study of these modules and found more interesting properties. We introduce in this paper co-absolutely co-pure modules as dual to absolutely pure modules. We first prove that over a commutative classical ring these modules are precisely the flat modules. As a biproduct we get a projective characterization of flat modules over a commutative co-noetherian ring. Secondly, over a quasi-Frobenius ring R, co-absolutely co-pure right R-modules turn out to be projective modules. Finally we get a characterization of almost Dedekind domains in terms of co-absolutely co-pure modules.

Throughout this paper by a ring R we mean an associative ring R with identity and by an R-module M we mean an unitary right R-module M.

Before defining a co-absolutely co-pure module we recall:

Definition 1. (i) An R-module M is said to be *finitely embedded* [**22**] (later called by Jans [**14**] *co-finitely generated*) if $E(M)=E(S_1)\oplus\cdots\oplus E(S_n)$ for some simple R-modules $S_1,\dots,S_n$ (here $E(X)$ denotes the injective hull of an R-module X).

(ii) An R-module M is said to be *co-free* [**10**, Definition 6] if M is isomorphic to $\Pi\{E(S_\alpha)\colon S_\alpha$ is a simple R-module, $\alpha\in\Lambda\}$ for some index set Λ.

(iii) An R-module M is said to be *co-finitely related* [**10**, Definition 14] if there is an exact sequence $0\to M\to N\to K\to 0$ of R-modules where N is co-finitely generated, co-free and K is co-finitely generated.

(iv) A short exact sequence $0\to A\to B\to C\to 0$ of R-modules is said to be *co-pure* [**11**, Definition 3] if every co-finitely related R-module is injective relative to this sequence.

(v) A ring R is said to be *right co-noetherian* [**14**, p. 588] if every homomorphic image of a co-finitely generated R-module is co-finitely generated.

(vi) A submodule A of an R-module B is said to be *pure* [**4**, p. 383] if for every left R-module M, the induced map $A\otimes_R M\to B\otimes_R M$ of abelian groups is a monomorphism.

More generally, a monomorphism $f\colon A\to B$ of R-modules is said to be *pure* if for any left R-module M, the induced map $f\otimes 1_M\colon A\otimes_R M\to B\otimes_R M$ is a monomorphism. We then say that a short exact sequence $0\to A\xrightarrow{f} B\xrightarrow{g}\to C\to 0$ of R-modules is pure if f is a pure monomorphism.

(We remark that Warfield [**24**, Proposition 3] has proved that a short exact sequence $0\to A\to B\to C\to 0$ of R-modules is pure if and only if every finitely presented R-module is projective relative to this sequence. Using this characterization of pure short exact sequences we dually defined co-purity (iv) noting that "co-finitely related" is the dual of "finitely presented".)

(vi) An R-module A said to be *absolutely pure* [**15**, **17**] if A is a pure submodule of every R-module in which it is contained (or equivalently, every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is pure).

Dually we define:

Definition 2. An R-module C is said to be *co-absolutely co-pure* (c.c. in short) if every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is co-pure.

Remark 3. Clearly every projective R-module is c.c. But the converse need not be true.

Example. The additive group $\mathbb{Q}$ of rational numbers being flat as a Z-module it is c.c. as a Z-module by [**21**, Proposition I.11.1] since purity and co-purity are equivalent for Z-modules [**11**, Theorem 20]. But $\mathbb{Q}$ is not projective as a Z-module.

Proposition 4. *For a ring R the following conditions are equivalent*:

(i) *every R-module is c.c.*;

(ii) *every co-finitely related R-module is injective*;

(iii) *every short exact sequence of R-modules is co-pure*;

(iv) *R is a right V-ring.*

Proof. (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) follow from [**11**, Proposition 5]. (i)$\Leftrightarrow$(iii) are obvious.

Now we derive a few equivalent conditions for the class of c.c. modules.

Proposition 5. *The following conditions are equivalent for an R-module C*:

(i) *C is c.c.*;

(ii) *there is a co-pure short exact sequence* $0 \to K \xrightarrow{i} P \to C \to 0$ *of R-modules with P projective*;

(iii) $\mathrm{Ext}_R^1(C, M) = 0$ *for every co-finitely related R-module M.*

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii): For any co-finitely related R-module M we have the induced exact sequence

$$\mathrm{Hom}_R(P, M) \xrightarrow{i^*} \mathrm{Hom}_R(K, M) \to \mathrm{Ext}_R^1(C, M) \to \mathrm{Ext}_R^1(P, M) = 0$$

of abelian groups (where i^* is the map induced by i) the last group being zero as P is projective. By the co-purity of the exact sequence in (ii), i^* is an epimorphism. Hence $\mathrm{Ext}_R^1(C, M) = 0$.

(iii)$\Rightarrow$(i): Let $0 \to A \xrightarrow{i} B \to C \to 0$ be a short exact sequence of R-modules. For any co-finitely related R-module M, we have the induced exact sequence

$$\mathrm{Hom}_R(B, M)\xrightarrow{i^*}\mathrm{Hom}_R(A, M)\to\mathrm{Ext}^1_R(C, M)=0$$

of abelian groups where the last group is zero by hypothesis. Then i^* is an epimorphism showing the co-purity of the given sequence. Thus C is c.c.

Corollary 6. *The class of c.c. modules is closed under taking arbitrary direct sums and direct summands.*

Before stating the next proposition we recall [**18**, p. 136] that if ε is a class of short exact sequences of R-modules, an R-module M is said to be *ε-projective* if M is projective relative to each member of ε.

Proposition 7. *The following conditions are equivalent for an R-module C:*

(i) *C is c.c.;*

(ii) *C is ε-projective where ε is the class of all short exact sequences $0\to X\to Y\to Z\to 0$ of R-modules where X is co-finitely related;*

(iii) *C is ε-projective where ε is the class of all short exact sequences $0\to X\to Y\to Z\to 0$ of R-modules where X is co-finitely related and Y is injective;*

(iv) *C is ε-projective where ε is the class of all short exact sequences $0\to X\to Y\to Z\to 0$ of R-modules where Y is co-finitely generated, injective and Z is co-finitely generated.*

Proof. (i)$\Rightarrow$(ii): Let $0\to X\to Y\xrightarrow{g}Z\to 0$ be any short exact sequence of R-modules with X co-finitely related. This yields us the exact sequence

$$\mathrm{Hom}_R(C, Y)\xrightarrow{g_*}\mathrm{Hom}_R(C, Z)\to\mathrm{Ext}^1_R(C, X)=0$$

of abelian groups. Since X is co-finitely related, the last group is zero by Proposition 5. Hence g_* is an epimorphism.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are obvious.

(iv)$\Rightarrow$(i): Let M be any co-finitely related R-module. Then we have an exact sequence $0\to M\to N\xrightarrow{g}K\to 0$ of R-modules with N co-finitely generated, co-free (so injective) and K co-finitely generated. This yields us the exact sequence

$$\mathrm{Hom}_R(C, N)\xrightarrow{g_*}\mathrm{Hom}_R(C, K)\to\mathrm{Ext}^1_R(C, M)\to\mathrm{Ext}^1_R(C, N)=0$$

of abelian groups. The last group is zero since N is injective and g_* is an epimorphism by (iv). Hence $\mathrm{Ext}^1_R(C, M)=0$. Thus C is c.c. by Proposition 5.

Corollary 8. *If R is a right co-noetherian ring, then the c.c. R-modules are precisely* *the ε-projective R-modules where ε is the class of short exact sequences* $0 \to X \to Y \to Z \to 0$ *of R-modules where Y is artinian and injective.*

Proof. The proof follows from (i)⇔(iv) of the above Proposition 7 and the facts that over a right co-noetherian ring R, the co-finitely generated R-modules are precisely the artinian R-modules ([**22**], Proposition 2*) and every homomorphic image of a co-finitely generated R-modules is co-finitely generated.

Before stating the next corollary we recall [**12**, Definition 1] that an R-module A is said to be *co-finitely projective* if it is ε-projective where ε is the class of all short exact sequences $0 \to X \to Y \to Z \to 0$ of R-modules where Z is co-finitely generated.

Corollary 9. *Every co-finitely projective R-module is c.c.*

Remark 10. The converse of the Corollary 9 need not be true.

Example. We have observed in Remark 3 that the additive group $\mathbb{Q}$ of rational numbers is c.c. as a Z-module. But $\mathbb{Q}$ is not co-finitely projective as a Z-module by [**12**, Proposition 6].

We now compare co-absolute co-purity with flatness and projectivity.
For the next proposition we recall the following from [**23**].

(i) An R-module M is said to be *linearly compact* if every family of cosets in M with finite intersection property has non-empty intersection.

(ii) A commutative ring R is said to be *classical* if $E(S)$ is linearly compact for every simple R-module S.

Proposition 11. *Over a commutative classical ring every flat module is c.c.*

Proof. The proof follows from [**21**, Proposition I.11.1] and the fact that purity implies co-purity for a commutative classical ring [**13**, Corollary 16].

Remark 12. In general a flat module need not be c.c.

Example. Since there are Von Neumann regular rings which are not V-rings (for example, the ring of linear operators of an infinite dimensional vector space), by Proposition 4 and [7, Theorem 11.24] there is a flat module which is not c.c.

Proposition 13. *For a commutative ring R, co-purity implies purity.*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a co-pure short exact sequence of R-modules. To prove the purity of this short exact sequence we need only prove, by [**4**, p. 384], that the induced sequence $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ of R-modules (note that R is commutative) is exact for every finitely presented R-module M. Since, by [**10**, Proposition 2], the family $\{E(S){:}S$ a simple R-module$\}$ is a family of co-generators for mod-R, the category of all R-modules and all R-homomorphisms, it suffices to prove

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that the induced sequence

$$0 \to \mathrm{Hom}_R(C \otimes_R M, E(S)) \to \mathrm{Hom}_R(B \otimes_R M, E(S)) \to \mathrm{Hom}_R(A \otimes_R M, E(S)) \to 0$$

of R-modules is exact for every finitely presented R-module M and for every simple R-module S. Since $\mathrm{Hom}_R(M, E(S))$ is co-finitely related whenever M is finitely presented and S is simple the exactness of the last sequence follows from the co-purity of the given short exact sequence and from the adjoint isomorphism Hom and $\otimes$. This proves the proposition.

Corollary 14. *Over a commutative ring R every c.c. R-module is flat.*

Proof. Follows from Proposition 13 and [**21**, Proposition I.11.1].

Remark 15. In general a c.c. module need not be flat.

Example. Cozzens [5] has constructed the ring $R = k[x, D]$ of all differentiable polynomials in an indeterminate x with coefficients in an universal field k with a derivation D (here the multiplication is given by $ax = xa + D(a)$, $a \in k$). Cozzens has proved that R is a right V-domain (that is, a right V-ring which is a domain) and not a field. Then by [7, Theorem 11.24] and by Proposition 4 there is a c.c. module which is not flat.

From Proposition 11 and Corollary 14 we have:

Corollary 16. *If R is a commutative classical ring then the c.c. R-modules are precisely the flat R-modules.*

Since a commutative co-noetherian ring is classical [**22**, Theorem 2], and [**23**, Proposition 4.1] we have, from Corollaries 8 and 16, the following projective characterization of flat modules over a commutative co-noetherian ring.

Proposition 17. *If R is a commutative co-noetherian ring, the flat R-modules are precisely the ε-projective R-modules where ε is the class of all short exact sequences $0 \to A \to B \to C \to 0$ of R-modules with B artinian, injective.*

We next prove that the c.c. modules are projective over a quasi-Frobenius ring.

Proposition 18. *Let R be a right co-noetherian, right perfect ring* [**1**]. *Then an R-module is c.c. if and only if it is projective.*

We need the following lemma for the proof of this proposition.

Lemma 19. *If R is a right co-noetherian ring then an R-module cannot contain a non-zero co-pure small submodule.*

Proof. Let K be a small co-pure submodule of an R-module A. Suppose $K \neq 0$. Let $0 \neq x \in K$ and let L be a submodule of K maximal with respect to $x \notin L$. Then K/L is subdirectly irreducible and hence is co-finitely generated. Since R is right co-noetherian, K/L is co-finitely related. As K is co-pure in A, K/L is a direct summand of A/L by [**11**, Proposition 11]. Now K/L is small in A/L as K is small in A. So $K/L=0$, that is, $L=K$, a contradiction since $x\in K\backslash L$. Thus $K=0$.

Proof of the proposition. We need only prove the "only if" part. Let C be a c.c. R-module. Since R is right perfect, C has a projective cover, say, $C=P/K$ where P is a projective R-module and K is a small submodule of P. But K is co-pure in P as C is c.c. Then $K=0$ by the above lemma. Thus $C=P$ is projective.

We recall [**8**, p. 204] that a ring R is said to be *quasi-Frobenius* (QF) if R is both left and right artinian and R is right self-injective.

Lemma 20. *Every QF ring is right co-noetherian.*

Proof. Let R be a QF ring. To prove that R is right co-noetherian we need only prove that the injective hull of every simple R-module is artinian. Let S be a simple R-module. Since $E(S)$ is projective, by [**8**, Theorem 24.20], $E(S)$ is contained in a direct sum of cyclic R-modules. Then $E(S)$ is finitely generated by [**8**, Proposition 20.14]. Since R is right artinian it follows that $E(S)$ is also artinian. Thus R is right co-noetherian.

Since every QF ring (more generally, any left or right artinian ring) is right perfect [**1**, Theorem P] we have the following proposition as a consequence of Proposition 18 and Lemma 20.

Proposition 21. *Over a quasi-Frobenius ring R, the c.c. R-modules are precisely the projective R-modules.*

We are not able to characterize the rings for which every c.c. R-module is projective. However we have:

Proposition 22. (i) *If R is a commutative perfect ring then every c.c. R-module is projective.*

(ii) *If R is a commutative classical ring such that every c.c. R-module is projective then R is artinian.*

Proof. (i) follows from [**1**, Theorem P] and Corollary 14.

(ii) follows from [**1**, Theorem P] and [**23**, Proposition 4.6].

We next investigate the rings for which the co-absolute co-purity is a hereditary property.

Prior to this, we derive some results for flat modules. We recall [**2**, p. 122] that if M is an R-module and n is a non-negative integer then we define n to be the weak dimension of M (notation: $n=\text{w.dim}\,M$) if n is the largest integer such that $\text{Tor}_n^R(M,N)\neq 0$ for some left R-module N. Our weak dimension is the flat dimension of Rotman [**20**, p. 180, Exercise 9.13]. We define the *right global weak dimension of R* (notation: $\text{r.gl.w.dim}\,R$) to be the supremum of the weak dimensions of all R-modules.

Similarly we define the weak dimension of a left R-module and the left global weak dimension of R. Since, by [**20**, Theorem 9.16], r.gl.w.dim R = l.gl.w.dim R we denote this common value by gl.w.dim R and call it the *global weak dimension of* R.

Proposition 23. *For a ring R the following conditions are equivalent.*

(i) *Every submodule of a flat R-module is flat.*

(ii) *Every right ideal of R is flat as an R-module.*

(iii) *gl.w.*dim $R \leqq 1$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii): Let I be any right ideal of R. Then the natural short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ of R-modules gives us the exact sequence

$$\mathrm{Tor}_2^R(R, A) \rightarrow \mathrm{Tor}_2^R(R/I, A) \rightarrow \mathrm{Tor}_1^R(I, A)$$

of abelian groups, for any left R-module A. In this latter exact sequence the first and the last abelian groups are zero by [**20**, Theorem 8.7] as both R and I are flat R-modules. Hence $\mathrm{Tor}_2^R(R/I, A) = 0$ for any left R-module A so that w.dim$R/I \leqq 1$. Hence, by [**20**, Theorem 9.18], w.gl.dim $R \leqq 1$.

(iii)⇒(i): Let M be a flat R-module and let N be a submodule of M. Then, for any left R-module A, we have the exact sequence

$$\mathrm{Tor}_2^R(M/N, A) \rightarrow \mathrm{Tor}_1^R(N, A) \rightarrow \mathrm{Tor}_1^R(M, A)$$

of abelian groups. Now the first member of this sequence is zero as, by hypothesis, w.gl.dim $R \leqq 1$ and the last member is zero as M is flat [**20**, Theorem 8.7]. Thus $\mathrm{Tor}_1^R(N, A) = 0$ for any left R-module A. So N is flat by [**20**, Theorem 8.8].

Corollary 24. *If R is a right semi-hereditary ring then the flatness is a hereditary property for R-modules.*

Proof. Follows from [3, Theorem 4.1] and Proposition 23 by noting the left–right symmetry of global weak dimension of R.

We now return to c.c. modules.

Proposition 25. *For a ring R the following conditions are equivalent:*

(i) *every submodule of a c.c. R-module is c.c.;*

(ii) *every right ideal of R is c.c. as an R-module;*

(iii) *every quotient of an injective R-module by a co-finitely related submodule is injective.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii): Let Q be an injective R-module and let A be a co-finitely related submodule of Q. Let $f: I \to Q/A$ be any R-homomorphism of a right ideal I of R into Q/A. Since I is c.c., by (ii), there is a homomorphism $g: I \to Q$, by Proposition 7, such that $\eta g = f$ where $\eta: Q \to Q/A$ is the canonical map. Now g will be given by multiplication by an element $x \in Q$ as Q is injective. Since f is then given by multiplication by $\eta(x) \in Q/A$ the injectivity of Q/A follows.

(iii)$\Rightarrow$(i): Let B be a c.c. R-module and let A be a submodule of B. Let Q be an injective R-module and let K be a co-finitely related submodule of Q. Let $f: A \to Q/K$ be any homomorphism. Since Q/K is injective, by (iii), f has an extension g to B. Then by Proposition 7 and by the co-absolute co-purity of B there is an $h: B \to Q$ such that $\eta h = g$ where $\eta: Q \to Q/K$ is the canonical map. If $k = h|A: A \to Q, k$ has the property that $\eta k = f$ proving, by Proposition 7, the co-absolute co-purity of A.

Corollary 26. (i) *If R is a right hereditary ring then every submodule of a c.c. R-module is c.c.*

(ii) *If R is a commutative classical semi-hereditary ring then every submodule of a c.c. R-module is c.c.*

Proof. (i) follows from Proposition 25.

(ii) follows from the Corollaries 16 and 24.

We recall [**23**, p. 126] that a commutative ring R is said to be a *valuation ring* if the ideals of R are totally ordered under inclusion. A valuation ring R is said to be *almost maximal* if every proper homomorphic image of R, as an R-module, is linearly compact and R is said to be *maximal* if R is linearly compact as an R-module.

Since every almost maximal valuation domain is classical [**23**, Proposition 4.4] and semi-hereditary [**6**, Theorem 2], we have:

Corollary 27. *For an almost maximal valuation domain R, every submodule of a c.c. R-module is c.c.*

We do not know whether, in general, for a right semi-hereditary ring R, every submodule of a c.c. R-module is c.c.

Proposition 28. *For a commutative co-noetherian ring R the following conditions are equivalent:*

(i) *every submodule of a c.c. R-module is c.c.;*

(ii) *every submodule of a flat R-module is flat;*

(iii) $w.gl.\dim R \leq 1$;

(iv) *$R_{\mathscr{M}}$ is a discrete valuation ring for each maximal ideal $\mathscr{M}$ of R.*

Proof. (i)$\Leftrightarrow$(ii) follow from Corollary 16.

(ii)$\Leftrightarrow$(iii) follow from Proposition 23.

(iii)$\Leftrightarrow$(iv) follow from [**6**, Proposition 11] and the facts that for a co-noetherian ring R, $R_{\mathscr{M}}$ is a noetherian ring for each maximal ideal $\mathscr{M}$ of R [**22**, Theorem 2] and a noetherian valuation domain is a discrete valuation ring (in [**6**] a valuation ring is assumed to be a domain).

A commutative integral domain R is said to be an *almost Dedekind domain* [9, p. 434] if $R_{\mathcal{M}}$ is a noetherian valuation ring i.e., a discrete valuation ring for each maximal ideal $\mathcal{M}$ of R.

We now have:

Corollary 29. *A commutative co-noetherian domain is an almost Dedekind domain if and only if every submodule of a c.c. R-module is c.c.*

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