Compositio Mathematica **126:** 79–89, 2001.

Quantum Cohomology of a Pfaffian Calabi–Yau Variety: Verifying Mirror Symmetry Predictions

To my parents on their 65th and 70th birthdays

ERIK N. TJØTTA
University of Bergen, Institute of Mathematics, University of Bergen, 5007 Bergen, Norway. e-mail: erik.tjotta@mi.uib.no

(Received: 23 June 1999; accepted: 17 March 2000)

Abstract. We formulate a generalization of Givental–Kim's quantum hyperplane principle. This is applied to compute the quantum cohomology of a Calabi–Yau 3-fold defined as the rank 4 locus of a general skew-symmetric 7×7 matrix with coefficients in $\mathbf{P}^6$. The computation verifies the mirror symmetry predictions of Rødland [25].

Mathematics Subject Classifications (2000). 4N10, 14J30, 14M12.

Key words. Pfaffian, Calabi–Yau, mirror symmetry, quantum cohomology.

0. Introduction

The rank 4 degeneracy locus of a general skew-symmetric 7×7-matrix with $\Gamma(\mathcal{O}_{\mathbf{P}^6}(1))$-coefficients defines a noncomplete intersection Calabi–Yau 3-fold M^3 with $h^{1,1} = 1$. We recall some results of Rødland [25] on the mirror symmetry of M^3: a potential mirror family W_q is constructed as (a resolution of) the orbifold $M_q^3/\mathbf{Z}_7$, where M_q^3 is a one-parameter family of invariants of a natural $\mathbf{Z}_7$-action on the space of all skew-symmetric 7×7-matrices. It is shown that the Hodge diamond of W_q mirrors the one of M^3. Further, at a point of maximal unipotent monodromy*, the Picard–Fuchs operator for the periods is computed to be (with $D = q\mathrm{d}/\mathrm{d}q$):

$$\begin{aligned}&(1 - 289q - 57q^2 + q^3)(1 - 3q)^2 D^4 +\\ &\quad + 4q(3q - 1)(143 + 57q - 87q^2 + 3q^3)D^3 +\\ &\quad + 2q(-212 - 473q + 725q^2 - 435q^3 + 27q^4)D^2 +\\ &\quad + 2q(-69 - 481q + 159q^2 - 171q^3 + 18q^4)D\\ &\quad + q(-17 - 202q - 8q^2 - 54q^3 + 9q^4)\,.\end{aligned} \tag{1}$$

*There are two points with maximal unipotent monodromy. Remarkably, the Picard–Fuchs equation at the other point is the one found in [2] for the mirror of the complete intersection Calabi–Yau 3-fold in $G(2, 7)$.

Mirror symmetry conjectures that this operator is equivalent to the operator

$$D^2\frac{1}{K}D^2, \quad \text{where } K(q) = 14 + \sum_{d\geqslant 1} n_d d^3 \frac{q^d}{1-q^d}, \tag{2}$$

and n_d, the *instanton number of degree d rational curves on* M^3, is defined [20] using Gromov–Witten invariants by

$$\langle p, p, p\rangle_d^{M^3} = \sum_{k|d} k^3 n_k.$$

We shall prove the conjecture.

THEOREM 1. *The differential operators (1) and (2) are equivalent under mirror transformations. That is:*

Let I_0, I_1, I_2, I_3 *be a basis of solutions to (1) with holomorphic solution* $I_0 = 1 + \sum_{d\geqslant 1} a_d q^d$ *and logarithmic solution* $I_1 = \ln(q)I_0 + \sum_{d\geqslant 1} b_d q^d$*. Then*

$$\frac{I_0}{I_0}, \frac{I_1}{I_0}, \frac{I_2}{I_0}, \frac{I_3}{I_0},$$

is a basis of solutions for (2) after change of coordinates $q = \exp(I_1/I_0)$.

Our approach follows closely the work of Givental [14, 15] for complete intersections in toric manifolds, and Batyrev, Ciocan-Fontanine, Kim, Van Straten [1, 2] for complete intersections in partial flag manifolds. It builds on the following three observations:

(i) A well-known construction identifies the degeneracy locus M^3 with the vanishing locus of a section of a vector bundle on a Grassmannian manifold (see Section 2). It is crucial, for us, that this vector bundle decomposes into a direct sum of vector bundles $E \oplus H$, where H is again a direct sum of line bundles.

(ii) The quantum hyperplane principle of B. Kim [18] extends to relate the E-restricted quantum cohomology with the $E \oplus H$-restricted one. This is formulated as a general principle in Section 1.

(iii) The E-restricted quantum cohomology can be effectively computed using localization techniques and WDVV-relations. An application of the quantum hyperplane principle then yields Theorem 1. The computations are carried out in Section 2.

1. Gromov–Witten Theory

We begin by recalling some basic results on $g = 0$ Gromov–Witten invariants before stating the quantum hyperplane principle. Our approach is the algebraic one following [19]. We refer the reader to [7, 11] for a fuller account and references.

1.1. FROBENIUS RINGS

Let X be a smooth projective variety over $\mathbf{C}$. Unless otherwise specified, we only consider even-dimensional cohomology with rational coefficients. In fact we will work with a further restriction: if E is a vector bundle on X, and Y is the zero-set of a regular section of E, then we are mainly interested in the cohomology classes on Y that are pulled back from X. These are represented by the graded Frobenius ring $A^*(E)$ with

$$A^p(E) := H^{2p}(X, \mathbf{Q})/\mathrm{ann}(E_0) \tag{3}$$

and non-degenerate pairing $\langle \gamma_1, \gamma_2 \rangle^E := \int_X \widetilde{\gamma}_1 \widetilde{\gamma}_2 E_0$, where E_0 is the top Chern class of E, and $\widetilde{\gamma}_i$ denotes a lift of γ_i to $A^*(X)$.

Let $A_1(E, \mathbf{Z})$ be the dual of $A^1(E, \mathbf{Z})/torsion$. We will identify $A_1(E, \mathbf{Z})$ with the image of the natural inclusion

$$A_1(E, \mathbf{Z}) \to A_1(X, \mathbf{Z}). \tag{4}$$

1.2. MODULI SPACE OF STABLE MAPS [11, 20]

Let $(C, s_1, \ldots, s_n)$ be an algebraic curve of arithmetic genus 0 with at worst nodal singularities and n nonsingular marked points. A map $f\colon C \to X$ is stable if all contracted components are stable (i.e. each irreducible component contains at least three *special* points, where special means marked or singular). For $d \in A_1(X, \mathbf{Z})$, let $X_{n,d}$ denote the coarse moduli space (or Deligne-Mumford stack) of stable maps with $f_*[C] = d$. If X is *convex*, that is $H^1(C, f^*TX) = 0$ for all stable maps, then $X_{n,d}$ is an orbifold (only quotient singularities) of complex dimension

$$\dim_{\mathbf{C}} X + \int_d c_1(X) + n - 3\,. \tag{5}$$

Of great importance to the theory are some natural maps on the moduli space of stable maps. For $i = 1, \ldots, n$, let $e_i\colon X_{n,d} \to X$ be the map obtained by evaluating stable maps at s_i, and let $\pi_i\colon X_{n,d} \to X_{n-1,d}$ be the map which forgets the marked point s_i. Also of significance are certain gluing maps which stratify the boundaries of the moduli spaces. In the stack theoretic framework, the diagram

$$\begin{array}{ccc} X_{n+1,d} & \xrightarrow{e_{n+1}} & X \\ {\scriptstyle \pi_{n+1}}\downarrow & & \\ X_{n,d} & & \end{array} \tag{6}$$

along with sections $s_i\colon X_{n,d} \to X_{n+1,d}$ defined by requiring $e_i = e_{n+1} \circ s_i$, is identical to the *universal* stable map.

1.3. GROMOV–WITTEN INVARIANTS [4, 5, 19, 21, 24, 27]

Suppose E is a convex vector bundle (i.e $H^1(C, f^*E) = 0$ for all stable maps). Base change theorems in [16] imply that $\pi_{n+1*}e_{n+1}^*E$ is a vector bundle on $X_{n,d}$ with fibers $H^0(C, f^*E)$. Let $E_{n,d}$ denote the top Chern class of $\pi_{n+1*}e_{n+1}^*E$ and let c_i denote the first Chern class of the line bundle $s_i^*\omega_{\pi_{n+1}}$ on $X_{n,d}$, where $\omega_{\pi_{n+1}}$ is the relative sheaf of differentials. Let c be an indeterminate.

A system of *E-restricted Gromov–Witten invariants* for X is the family of multilinear functions $\langle\ \rangle_d^E$ on $A^*(E)[[c]]^{\otimes n}$, defined for all $n \geqslant 0$ and $d \in A_1(E, \mathbf{Z})$ by

$$\langle P_1\gamma_1, \ldots, P_n\gamma_n\rangle_d^E := \int_{[X_{n,d}]} \prod_{i=1}^{n} P_i(c_i)e_i^*(\widetilde{\gamma}_i)E_{n,d}\,, \tag{7}$$

where $\gamma_i \in A^*(E)$, $P_i \in \mathbf{Q}[[c]]$, and $[X_{n,d}]$ is the *virtual fundamental class* of dimension (5).

There is an exact sequence of vector bundles

$$0 \to \ker \to \pi_{n+1*}e_{n+1}^*E \to e_i^*E \to 0, \tag{8}$$

where the right-hand map is obtained by evaluating sections at the ith marked point. This implies that $E_{n,d}$ is divisible by E_0 in $A^*(X_{n,d})$, hence the invariants (7) are independent of the chosen lifts $\widetilde{\gamma}_i$.

When X is convex, then $[X_{n,d}]$ is simply the fundamental class of $X_{n,d}$. If $Y \subset X$ is cut out by a regular section of E, then

$$j_* \sum_{i_*d'=d} [Y_{n,d'}] = E_{n,d} \cdot [X_{n,d}]\,, \tag{9}$$

where the map $j\colon Y_{n,d'} \to X_{n,d}$ is induced from the inclusion map $i\colon Y \to X$.

Let $\{\Delta_i\}$, $\{\Delta^i\}$ denote a pair of homogeneous bases of $A^*(E)$ such that $\langle\Delta_i, \Delta^j\rangle^E = \delta_i^j$, and let $T_i \in A^*(E)[[c]]$. The natural maps on the moduli space of stable maps respect the virtual classes, hence induce important relations on GW-invariants. Among these are:

Divisor equation. For $p \in A^1(E)$ we have

$$\begin{aligned}&\langle p, T_1, \ldots, T_n\rangle_d^E\\ &\quad= \left(\int_d \widetilde{p}\right)\langle T_1, \ldots, T_n\rangle_d^E + \sum_{i=1}^{n}\langle T_1, \ldots, pT_i/c, \ldots, T_n\rangle_d^E.\end{aligned}$$

WDVV-relation. Denote*

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\!\!\succ\!\!\!\prec\!\!\begin{pmatrix} T_4 \\ T_3 \end{pmatrix}_d := \sum_{d_1+d_2=d} \langle T_1, T_2, \Delta_i\rangle_{d_1}^E \langle\Delta^i, T_3, T_4\rangle_{d_2}^E.$$

*We use the Einstein summation convention.

https://doi.org/10.1023/A:1017585802461 Published online by Cambridge University Press

Then,

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\!\!\asymp\!\!\begin{pmatrix} T_4 \\ T_3 \end{pmatrix}_d = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\!\!\asymp\!\!\begin{pmatrix} T_4 \\ T_3 \end{pmatrix}_d.$$

Topological recursion relation (TRR).

$$\langle T_1, T_2, T_3\rangle_d^E = \sum_{d_1+d_2=d} \langle T_1/c, \Delta_i\rangle_{d_1}^E \langle \Delta^i, T_2, T_3\rangle_{d_2}^E.$$

The above equations are subject to some restrictions: If $d = 0$ we must assume that $n \geqslant 3$ in the divisor equation. Further, all undefined correlators appearing above are set to 0, except for $\gamma_1, \gamma_2 \in A^*(E)$ we define

$$\langle \gamma_1/c, \gamma_2\rangle_0^E := \langle \gamma_1, \gamma_2\rangle^E.$$

Let $\{p_i\}$ be a *nef* (i.e. pairs nonnegatively with all effective curve classes in $A_1(E, \mathbf{Z})$) basis for $A^1(E, \mathbf{Z})/torsion$, and let $\{q_i\}$ be formal homogeneous parameters such that $\sum_i \deg(q_i)p_i = c_1(X) - c_1(E)$ modulo $\operatorname{ann}(E_0)$. The WDVV-relations imply the associativity of the *quantum product* defined by

$$\Delta_i *_E \Delta_j := \sum_{d,k} \langle \Delta_i, \Delta_j, \Delta_k\rangle_d^E q^d \Delta^k,$$

where $q^d = \prod_i q_i^{\int_d \widetilde{p}_i}$. Note that the product is homogeneous with the chosen grading.

1.4. QUANTUM HYPERPLANE PRINCIPLE

Let $\hbar$ be a formal homogeneous variable of degree 1. Let e_{1*}^E be the map induced from the push-forward e_{1*} by passing to the quotient (3). Use $E'_{1,d}$ to denote the top Chern class of the kernel in (8), thus $E_{1,d} = E_0 E'_{1,d}$. Consider the following degree 0 vector in $A^*(E)[[q, \hbar^{-1}]]$:

$$\begin{aligned} J_E :&= \mathrm{e}^{p\ln(q)/\hbar} \sum_d q^d e_{1*}^E \left(\frac{E'_{1,d}}{\hbar(\hbar - c_1)} \right) \\ &= \mathrm{e}^{p\ln(q)/\hbar} \sum_d q^d \left\langle \frac{\Delta_i}{\hbar(\hbar - c)} \right\rangle_d^E \Delta^i, \end{aligned} \tag{10}$$

where $p \ln q = \sum_i \ln(q_i)p_i$, and the convention $\langle \Delta_i/(\hbar(\hbar - c))\rangle_0^E \Delta^i = 1$ is used. Suppose $H = \oplus L_i$ is a sum of convex line bundles on X. If $c_1(X) - c_1(E \oplus H)$ is nef, the quantum hyperplane principle suggests an explicit relationship between J_E and $J_{E\oplus H}$ via the following adjunct in $A^*(E \oplus H)[[q, \hbar^{-1}]]$:

$$I_E^H := \mathrm{e}^{p\ln(q)/\hbar} \sum_d q^d H_d e_{1*}^{E\oplus H} \left(\frac{E'_{1,d}}{\hbar(\hbar - c_1)} \right), \tag{11}$$

https://doi.org/10.1023/A:1017585802461 Published online by Cambridge University Press

where

$$H_d := \prod_i \prod_{m=1}^{\int_d c_1(L_i)} (c_1(L_i) + m\hbar),$$

and the q_i's are regraded so that $\sum_i \deg(q_i)p_i = c_1(X) - c_1(E \oplus H)$ modulo ann(E_0). Hence, by assumption, all $\deg(q_i) \geqslant 0$. The precise statement is:

The vectors I_E^H and $J_{E\oplus H}$ coincide up to a mirror transformation *of the following type:*

(i) *multiplication by a* $\exp(b/\hbar)$ *where a and b are homogeneous q-series of degree* 0 *and* 1 *respectively,*

(ii) *coordinate changes* $\ln q_i \mapsto \ln q_i + f_i$, *where f_i are homogeneous q-series of degree* 0 *without constant term.*

Further, the mirror transformation is uniquely determined by the first two coefficients in the $\hbar^{-1}$-Taylor expansion of I_E^H and $J_{E\oplus H}$.

THEOREM 2. *If X is a homogenous space and E is equivariant with respect to a maximal torus action on X, then the quantum hyperplane conjecture as formulated above is true.*

Proof. The proof in [18] for $\mathrm{rank}(E) = 0$ extends with minor modifications to the general case. □

Remark 1. An early version of this principle appeared in [3]. Givental formulated and proved the $\mathrm{rank}(E) = 0$ case for toric manifolds [14, 15]. The above formulation when $\mathrm{rank}(E) = 0$ is due to B. Kim [18]. Extending the conjecture of B. Kim we expect that the principle holds for more general X. In [26] the conjecture was tested on a nonconvex, nontoric manifold.

Remark 2. An analogue generalization of the hyperplane principle in [13, 17] for *concavex H* can be formulated with E convex/concave. The proof in [18] extends to cover these cases when X is homogenous. See also [22].

1.5. DIFFERENTIAL EQUATIONS

The vector J_E encodes all the E-restricted one-point GW-invariants. Reconstruction using TRR [23] shows that these are determined by two-point GW-invariants without c's. This is organized nicely in terms of differential equations [8, 15]. Consider the *quantum differential equation*

$$\hbar q_k \frac{\mathrm{d}}{\mathrm{d}q_k} T = p_k *_E T, \quad k = 1, \ldots, \mathrm{rank}(A^1(E)), \tag{12}$$

where T is a series in the variables $\ln q_i$ and $\hbar^{-1}$ with coefficients from $A^*(E)$. The WDVV-relations imply that the system is solvable. An application of the divisor equation and TRR [23, 26] shows that $J_E = \langle S, 1\rangle^E$ for fundamental solution S of (12). In particular, if $c_1(X) - c_1(E)$ is *positive* the hyperplane principle, when true, yields an algebraic representation of the (*quantum*) $\mathcal{D}$-module generated by $J_{E\oplus H}$.

Remark 3. A useful application of Theorem 2 is the following: for partial flag manifolds $F = F(n_1, \dots, n_r, n)$ with universal sub-bundles U_{n_i} and quotient bundles Q_{n_i}, one may consider vector bundles E that are direct sums of bundles of type

$$\wedge^{p_1} U^{\vee}_{n_{i_1}} \otimes \wedge^{p_2} Q_{n_{i_2}} \otimes S^{p_3} U^{\vee}_{n_{i_3}} \otimes S^{p_4} Q_{n_{i_4}}.$$

If $c_1(F) - c_1(E)$ is positive, the E-restricted quantum cohomology can in principle be computed using localization techniques, and the theorem will yield the quantum D-module for the nef (and, in particular, the Calabi–Yau) cases of type $E \oplus H$, with H decomposable.

2. The Pfaffian Variety

Let V be a vector space of dimension 7 and consider the projective space $P = \mathbf{P}(\wedge^2 V)$ with universal 7×7 skew-symmetric linear map $\alpha\colon V_P^{\vee}(-1) \to V_P$, where V_P denotes the trivial vector bundle on P with fiber V. Define $M \subset P$ as the locus where $\operatorname{rank}\alpha \leqslant 4$. The scheme structure is determined by the Pfaffians of the diagonal 6×6-minors. The variety is locally Gorenstein of codimension 3 in P with canonical sheaf $\mathcal{O}_M(-14)$. Its singular locus, which is the rank 2 degeneracy locus of α, is of codimension 7 in M [6]. This implies that the intersection $M^k = M \cap \mathbf{P}^{k+3}$ with a general linear sub-space $\mathbf{P}^{k+3}$ in P is of dimension k, has canonical sheaf $\mathcal{O}_{M^k}(3-k)$, and is smooth when $k \leqslant 6$.

We recall a classic construction for degeneracy loci (see, for instance, [12], Example 14.4.11). Let $G = \mathrm{Grass}_4(V)$ be the Grassmannian of 4-planes in V with universal exact sequence

$$0 \to U \to V_G \to Q \to 0.$$

Pulling everything back to $PG = P \times G$, we regard $\wedge^2 U(1)$ as a sub-bundle of $\wedge^2 V_{PG}(1)$, where the twists are with respect to $\mathcal{O}_P(1)$. The map α induces a regular section $\overline{\alpha}$ of the convex rank 15 quotient bundle on PG

$$A := \wedge^2 V_{PG}(1) / \wedge^2 U(1).$$

LEMMA 1. *The zero-scheme $V(\overline{\alpha}) \subset PG$ projects birationally onto M. Moreover, the projection is isomorphic over the nonsingular locus of M.*

For a general linear sub-space $\mathbf{P}^{k+3}$ in P, let $(A^k, \overline{\alpha}_k)$ denote the pull-back of the pair $(A, \overline{\alpha})$ to $\mathbf{P}^{k+3} \times G$. By Lemma 1, $V(\overline{\alpha}_k)$ projects isomorphically to M^k for $k \leqslant 6$.

We are now set to compute the quantum $\mathcal{D}$-module of the Calabi–Yau variety M^3 using Theorem 2. This can in principle be done from any of the A^k-restricted $(k \geqslant 4)$ GW-theories. We provide details for the case $E = A^6$.

https://doi.org/10.1023/A:1017585802461 Published online by Cambridge University Press

First, we need to determine the cohomology ring $A^*(E)$. Pull-backs to $\mathbf{P}^9 \times G$ of the Chern classes $p = c_1(\mathcal{O}_P(1))$ and $\gamma_i = c_i(Q)$, $i = 1, 2, 3$, generate the $\mathbf{Q}$-algebra $A^*(\mathbf{P}^9 \times G)$, and induce generators of $A^*(E)$.

LEMMA 2.

(i) *The* $\mathbf{Q}$*-algebra* $A^*(E)$ *is generated by* p *and* γ_2 *with top degree monomial values as follows:*

$$p^6 = 14, \qquad p^4\gamma_2 = 28, \qquad p^2\gamma_2^2 = 59, \qquad \gamma_2^3 = 117.$$

In particular the Betti numbers of $A^*(E)$ *are* [1, 1, 2, 2, 2, 1, 1].

(ii) *We have the relation* $\gamma_1 = 2p$.

Proof. Computed using Schubert.⋆ □

Our choice of basis $\{\Delta_i\}$ for $A^*(E)$ is the following:

$$\{1, p, p^2, \gamma_2, p^3, p\gamma_2, p^4, p^2\gamma_2, p^5, p^6\}.$$

Rather than to work directly with the solutions (10) and (11) we prefer to work with their governing differential equations. The quantum differential equation (12) is determined by the two-point numbers $\langle \Delta_i, \Delta_j \rangle_d^E$ for all d in $A_1(E, \mathbf{Z}) \simeq \mathbf{Z}$. A simple dimension count shows that these are 0 unless

$$\operatorname{codim}(\Delta_i) + \operatorname{codim}(\Delta_j) = 5 + 3d\,. \tag{13}$$

LEMMA 3. *Values of* $d \geqslant 1$ *GW-invariants satisfying (13) are as follows:*

$$\begin{array}{lll} \langle p^2, p^6 \rangle_1^E = 238, & \langle p\gamma_2, p^5 \rangle_1^E = 2044, & \langle p^2\gamma_2, p^2\gamma_2 \rangle_1^E = 6617, \\ \langle \gamma_2, p^6 \rangle_1^E = 504, & \langle p^4, p^4 \rangle_1^E = 1568, & \\ \langle p^3, p^5 \rangle_1^E = 980, & \langle p^4, p^2\gamma_2 \rangle_1^E = 3220, & \langle p^5, p^6 \rangle_2^E = 9800 \end{array}$$

Proof. It follows from Lemma 2 that the map (4) identifies d with the curve class $(d, 2d)$ in $A_1(\mathbf{P}^9 \times G, \mathbf{Z})$, so the GW-invariants are integrals over $(\mathbf{P}^9 \times G)_{(d,2d)}$.

Localization. Consider the standard action of $T = (\mathbf{C}^*)^{10} \times (\mathbf{C}^*)^7$ on $\mathbf{P}^9 \times G$. Since the integrands are polynomials in Chern classes of equivariant vector bundles with respect to the induced T-action on $(\mathbf{P}^9 \times G)_{(d,2d)}$, they may be evalued using Bott's residue formula (see [10, 20]). As there are only finitely many fixed points and curves in $\mathbf{P}^9 \times G$, the formulae involved are similar to the ones found in [20]. Details are left to the reader. The two-point integrals with $d = 1$ were evaluated in this manner.

⋆A MAPLE package for enumerative geometry written by S. Katz and S.-A. Strømme. Software and documentation available at http://www.math.okstate.edu/katz/schubert.html.

https://doi.org/10.1023/A:1017585802461 Published online by Cambridge University Press

Reconstruction. The number $\langle p^5, p^6\rangle_2^E$ is however more easily obtained from $d = 1$ numbers using the following WDVV-relations:

$$\left(\begin{smallmatrix} p \\ p^2\gamma_2 \end{smallmatrix} \asymp \begin{smallmatrix} p^5 \\ \gamma_2 \end{smallmatrix}\right)_2, \left(\begin{smallmatrix} p \\ p^4 \end{smallmatrix} \asymp \begin{smallmatrix} p^5 \\ \gamma_2 \end{smallmatrix}\right)_2, \left(\begin{smallmatrix} p \\ p^3 \end{smallmatrix} \asymp \begin{smallmatrix} p^6 \\ \gamma_2 \end{smallmatrix}\right)_2, \left(\begin{smallmatrix} p \\ p\gamma_2 \end{smallmatrix} \asymp \begin{smallmatrix} p^6 \\ \gamma_2 \end{smallmatrix}\right)_2. \tag{14}$$

In fact, a further analysis⋆ shows that the 3-point $d = 1$ numbers appearing in the equations (14) are in turn determined by the 2-point $d = 1$ numbers above. □

Remark. Employing a description in [9] of the class in $\mathrm{Grass}_2(\wedge^2 V)$ of lines on M, the $d = 1$ GW-invariants which only involve powers of p can be computed without using Bott's formula.

Consider the differential equation (12), with invariants as in Lemma 3 and $\hbar = 1$. Denote $q = q_1$ and $D = q\mathrm{d}/\mathrm{d}q$. By reduction we find the (order 10, degree 5)-differential equation $P(D) = \sum_d q^d P_d(D) = 0$, with P_d as below, for $J_E(\hbar = 1)$.

$$\begin{aligned}
P_0 &= 3D^7(D-1)^3,\\
P_1 &= D^3(194D^7 - 776D^6 + 1072D^5 - 1405D^4 - 1716D^3 - 1272D^2 -\\
&\quad - 414D - 51),\\
P_2 &= 343D^{10} - 1715D^9 + 3185D^8 - 58593D^7 - 55484D^6 - 460D^5 + 10697D^4 +\\
&\quad + 1850D^3 - 896D^2 - 480D - 96,\\
P_3 &= -99127D^7 + 22736D^5 - 11772D^4 - 34797D^3 - 31654D^2 -\\
&\quad - 13495D - 2175,\\
P_4 &= -19551D^4 - 39102D^3 - 31360D^2 - 11524D - 1430,\\
P_5 &= 343(D+1).
\end{aligned}$$

Let $H = 3\mathcal{O}_P(1)$ and assume $\hbar = 1$. The adjunct I_E^H is obtained by correcting the q^d-coefficients of J_E with the class $H_d = \prod_{m=1}^{d}(p+m)^3$. A reformulation of this transformation on the corresponding differential equations takes the same form⋆⋆. That is, the differential operator

$$\sum_{d=0}^{5} q^d P_d \prod_{m=1}^{d}(D+m)^3 \tag{17}$$

annihilates I_E^H. Recall that the commutation rule is $Dq - qD = q$. If we factor out the "trivial" term $D^3(D-1)^3$ from the left of (15) and re-organize the terms we recover the Picard–Fuchs operator (1).

⋆For instance, using Farsta, a computer program written by A. Kresch, available at http://www.math.upenn.edu/kresch/computing/farsta.html.

⋆⋆This is a general principle when $rank A^1(E) = 1$. It is easily proved using recursion formulas for solutions of differential equations [3, 26].

Proof of Theorem 1. From (9) it follows that $\langle p, p, p\rangle_d^{M^3} = \langle p, p, p\rangle_d^{A^3}$. Using (12) it is straightforward to check that $D^2 1/KD^2$ is the differential operator governing J_{A^3} (see, for instance, [26]). The rest follows from Theorem 2. □

The first five curve numbers are

$$n_1 = 588,\ n_2 = 12103,\ n_3 = 583884,\ n_4 = 41359136,\ n_5 = 3609394096.$$

Acknowledgements

I would like to thank S.-A. Strømme for bringing observation (i) to my attention. Thanks also to G. Jamet, B. Kim, E. Rødland and D. van Straten for helpful discussions. Thanks to J. Stienstra for spotting a crucial misprint in an earlier version. An extra thank you to E. Rødland again for letting me use his Maple scripts.⋆ I am grateful for the hospitality of l'Institut de Mathematiques de Jussieu where this work was done. The author has been supported by a grant from the Research Council of Norway.

References

1. Batyrev, V. V., Ciocan-Fontanine, I., Kim, B. and van Straten, D.: Mirror symmetry and toric degenerations of partial flag manifolds, math.AG/9803108.
2. Batyrev, V. V., Ciocan-Fontanine, I., Kim, B. and van Straten, D.: Conifold transitions and mirror symmetry for Calabi–Yau complete intersections in Grassmannians, *Nuclear Phys. B* **514**(3) (1998), 640–666.
3. Batyrev, V. V. and Duco van Straten, D.: Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties, *Comm. Math. Phys.* **168**(3) (1995), 493–533.
4. Behrend, K.: Gromov–Witten invariants in algebraic geometry, *Invent. Math.* **127**(3) (1997), 601–617.
5. Behrend, K. and Fantechi, B.: The intrinsic normal cone, *Invent. Math.* **128**(1) (1997), 45–88.
6. Buchsbaum, D. A. and Eisenbud, D.: Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99**(3) (1977), 447–485.
7. Cox, D. A. and Katz, S.: *Mirror Symmetry and Algebraic Geometry*, Amer. Math. Soc., Providence, RI, 1999.
8. Dubrovin, B.: Geometry of 2D topological field theories, In: *Integrable Systems and Quantum Groups (Montecatini Terme, 1993)*, Springer, Berlin, 1996, pp. 120–348.
9. Ellingsrud, G., Peskine, C. and Strømme, S.A.: On the family of lines on a variety defined by linear 6×6 Pfaffians, Unpublished.
10. Ellingsrud, G. and Strømme, S. A.: Bott's formula and enumerative geometry, *J. Amer. Math. Soc.* **9**(1) (1996), 175–193.

⋆http://www.math.uio.no/~einara/Maple/Maple.html.

11. Fulton, W. and Pandharipande, R.: Notes on stable maps and quantum cohomology, In: *Algebraic Geometry – (Santa Cruz 1995)*, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96.
12. William Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 2nd edn, 1998.
13. Givental, A.: Elliptic Gromov–Witten invariants and the generalized mirror conjecture, In: *Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997)*, World Scientific, River Edge, NJ, 1998, pp. 107–155.
14. Givental, A.: A mirror theorem for toric complete intersections, In: *Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996)*, Birkhäuser, Boston, MA, 1998, pp. 141–175.
15. Givental, A. B.: Equivariant Gromov–Witten invariants, *Internat. Math. Res. Notices*, 1996 (13) 613–663.
16. Hartshorne, R.: *Algebraic Geometry*, Graduate Texts in Math. 52, Springer-Verlag, New York, 1977.
17. Kim, B.: Quantum hyperplane section principle for concavex decomposable vector bundles, math.AG/9905179.
18. Kim, B.: Quantum hyperplane section theorem for homogeneous spaces. To appear in *Acta Math.*
19. Kontsevich, M. and Manin, Yu.: Gromov–Witten classes, quantum cohomology, and enumerative geometry, *Comm. Math. Phys.* **164**(3) (1994), 525–562.
20. Kontsevich, M.: Enumeration of rational curves via torus actions, In: *The Moduli Space of Curves (Texel Island, 1994)*, Birkhäuser, Boston, MA, 1995, pp. 335–368.
21. Li, J. and Tian, G.: Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties, *J. Amer. Math. Soc.* **11**(1) (1998), 119–174.
22. Lian, B. H., Liu, K. and Yau, S-T.: Mirror principle. I, *Asian J. Math.* **1**(4) (1997), 729–763.
23. Pandharipande, R.: Rational curves on hypersurfaces (after A. Givental), *Astérisque*, (252): Exp. No. 848, 5, 307–340, 1998. Séminaire Bourbaki, Vol. 1997/98.
24. Ruan, Y. and Tian, G.: A mathematical theory of quantum cohomology, *J. Differential Geom.* **42**(2) (1995), 259–367.
25. Rødland, E. A.: The Pfaffian Calabi–Yau, its mirror, and their link to the Grassmannian $G(2,7)$, *Compositio Math.* **122** (2000), 135–149.
26. Tjøtta, E.: Rational curves on the space of determinantal nets of conics, PhD thesis, University of Bergen, 1997, math.AG/9802037.
27. Witten, E.: Two-dimensional gravity and intersection theory on moduli space, In: *Surveys in Differential Geometry (Cambridge, MA, 1990)*, Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.