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INTERSECTION THEOREMS FOR SYSTEMS OF SETS

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Abstract. Let n and k be positive integers, $k \geq 3$. Denote by $\varphi(n, k)$ the least positive integer such that if F is any family of more than $\varphi(n, k)$ sets, each set with n elements, then some k members of F have pairwise the same intersection. In this paper we obtain a new asymptotic upper bound for $\varphi(n, k)$, k fixed, n approaching infinity.

1. **Introduction.** We shall say, following [2], that k sets form a Δ-system if the sets have pairwise the same intersection. We say a family F does not contain a k element Δ-system if no k sets in F form a Δ-system. Erdös and Rado [2] proved that to each pair of positive integers n, k, $k \geq 3$ there corresponds a least integer $\varphi(n, k)$ so that if F is a family of distinct n-element sets, $|F| > \varphi(n, k)$, then F contains a k-element Δ-system. As the case $k = 3$ is of particular interest, we shall set $\varphi(n) = \varphi(n, 3)$. They showed

$$(1.1)\qquad (k-1)^n \leq \varphi(n, k) \leq n!\,(k-1)^n \left\{1 - \sum_{t=1}^{n-1} \frac{t}{(t+1)!\,(k-1)^t}\right\}$$

We shall restrict our attention to asymptotic results for fixed k. Abbott, Hanson, and Sauer [1] showed

$$(1.2)\qquad \varphi(n) > [\sqrt{10} - o(1)]^n$$

and

$$(1.3)\qquad \varphi(n, k) \leq (n+1)!\left\{\frac{k-1+(k^2+6k-7)^{1/2}}{4}\right\}^n$$

So, in particular,

$$(1.4)\qquad \varphi(n) \leq (n+1)!\left(\frac{1+\sqrt{5}}{2}\right)^n$$

We shall prove:

Theorem 1. *For fixed k, $\varepsilon > 0$ there exists C so that*

$$(1.5)\qquad \varphi(n, k) \leq Cn!\,(1+\varepsilon)^n$$

for all n.

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Our proof shall follow the lines of [1]. In [2] Erdös and Rado ask if $\varphi(n) < K^n$ for some universal constant K. While our efforts were inspired by this question, we cannot resolve it.

2. **The case** $k=3$. Let $\varphi(n)$ be as previously defined. Let $\gamma(n)$ be the least integer so that if F is a family of n element sets, *no two disjoint*, $|F| > \gamma(n)$, then F contains a Δ-system.

We shall make frequent use of the following *reduction principle*: Suppose F does not contain a Δ-system and $X \subseteq A_i \in F$, $1 \le i \le m$. Then $\{A_i - X : 1 \le i \le m\}$ does not contain a Δ-system. (If, say, $A_1 - X$, $A_2 - X$, $A_3 - X$ formed a Δ-system, so would A_1, A_2, A_3 in F.) In particular, setting $X = \{x\}$, if F does not contain a Δ-system at most $\varphi(n-1)$ sets in F can contain a given point x.

LEMMA 1. $\varphi(n) \le n\varphi(n-1) + \gamma(n)$.

Proof. Let $|F| = \varphi(n)$, F not containing a Δ-system. Fix $S \in F$. At most $\varphi(n-1)$ $T \in F$ contain any particular $x \in S$, thus at most $n\varphi(n-1)$ $T \in F$ intersect S. If $T_1, T_2 \in F$, both disjoint from S, then $T_1 \cap T_2 \ne \varphi$, as otherwise S, T_1, T_2 form a Δ-system. Hence at most $\gamma(n)$ $T \in F$ are disjoint from S.

Let $F = \{S_1, \ldots, S_\gamma\}$, $\gamma = \gamma(n)$, be a family of non-disjoint n-sets not containing a Δ-system. Let t be the *average* $|S_i \cap S_j|$, $1 \le i < j \le \gamma$. Formally

$$t = \binom{\gamma}{2}^{-1} \sum_{1 \le i < j \le \gamma} |S_i \cap S_j| \tag{2.1}$$

LEMMA 2.

$$\gamma \le \frac{n}{t}\varphi(n-1).$$

Proof.

$$t = \frac{1}{\gamma} \sum_{I=1}^{\gamma} \left[\frac{1}{\gamma - 1} \sum_{j \ne i} |S_i \cap S_j|\right] \tag{2.2}$$

Hence for some i, say $i = 1$,

$$\frac{1}{\gamma - 1} \sum_{j \ne 1} |S_1 \cap S_j| \ge t. \tag{2.3}$$

For $x \in S_1$, let

$$n(x) = |\{j : x \in S_j, 1 \le j \le \gamma\}|. \tag{2.4}$$

Then

$$\sum_{x \in S_1} n(x) = \sum_{j=1}^{\gamma} |S_1 \cap S_j| = n + \sum_{j \ne 1} |S_1 \cap S_j| \ge n + t(\gamma - 1) \ge t\gamma \tag{2.5}$$

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Hence some $n(x) \geq t\gamma/n$. But, by the Reduction Principle, all $n(x) \leq \varphi(n-1)$.

LEMMA 3. *For* $1 \leq s \leq \gamma$,

$$\gamma \leq t\binom{s}{2}\varphi(n-1) + (n-1)^s \varphi(n-s). \tag{2.7}$$

Proof. For $X \subseteq \{1, \ldots, \gamma\}$, $|X| = s$ set

$$g(X) = \sum_{\substack{i,j \in X \\ i<j}} |S_i \cap S_j| \tag{2.8}$$

By linearity of expected value the average $g(X)$ is $t\binom{s}{2}$. Formally

$$\Sigma^* g(X) = \sum_{1 \leq i < j \leq \gamma} |S_i \cap S_j| \binom{\gamma-2}{s-2} = t\binom{\gamma}{2}\binom{\gamma-2}{s-2} = t\binom{\gamma}{s}\binom{s}{2} \tag{2.9}$$

where Σ^* runs over $X \subseteq \{1, \ldots, \gamma\}$, $|X| = s$. Thus some X has

$$g(X) \leq t\binom{s}{2}. \tag{2.10}$$

Renumber so that $X = \{1, \ldots, s\}$ for convenience. Set

$$Y = \bigcup_{1 \leq i < j \leq s} S_i \cap S_j, \quad \text{so} \quad |Y| \leq T\binom{s}{2}. \tag{2.11}$$

For $1 \leq i \leq \gamma$ either

(i) $S_i \cap Y \neq \varphi$. There are at most $|Y|\, \varphi(n-1) \leq t\binom{s}{2}\varphi(n-1)$ such i or,

(ii) $S_i \cap Y = \varphi$. Then there exist (not necessarily unique) $x_1, \ldots, x_s$; $x_j \in S_i \cap (S_j - Y)$ (as $S_i \cap S_j \neq \varphi$ and $S_i \cap Y = \varphi$). These x's are distinct since the $(S_j - Y)$ are disjoint. There are at most $\prod_{j=1}^{s} |S_j - Y| \leq (n-1)^s$ possible sequences and at most $\varphi(n-s)$ sets with the same sequence (i.e. a common s points); thus at most $(n-1)^s \varphi(n-s)$ such i.

We now prove Theorem 1 (for $k=3$) using Lemmas 1, 2, 3. Let C be such that (1.5) holds for $n \leq n_0$ where $n_0 = n_0(\varepsilon)$ shall be determined later. We assume (1.5) holds for all $n' < n$ and proceed by induction. By Lemmas 1, 2

$$\varphi(n) \leq n\varphi(n-1)\left(1 + \frac{1}{t}\right) \tag{2.12}$$

so that if $t \geq \varepsilon^{-1}$ (1.5) follows by induction. We therefore assume $t < \varepsilon^{-1}$. From Lemmas 1, 3

$$\varphi(n) \leq n\varphi(n-1) + t\binom{s}{2}\varphi(n-1) + (n-1)^s \varphi(n-s) \tag{2.13}$$

$$\leq n\varphi(n-1) + \varepsilon^{-1}\binom{s}{2}\varphi(n-1) + (n-1)^s \varphi(n-s). \tag{2.14}$$

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By induction

$$\varphi(n) \le C(1+\varepsilon)^n n!\, \psi(n, \varepsilon, s) \tag{2.15}$$

where

$$\psi(n, \varepsilon, s) = (1+\varepsilon)^{-1} + \varepsilon^{-1}\binom{s}{2}(1+\varepsilon)^{-1}n^{-1} + (1+\varepsilon)^{-s}(n-1)^s/(n)_s \tag{2.16}$$

For ε, s fixed

$$\lim_{n\to\infty} \psi(n, \varepsilon, s) = (1+\varepsilon)^{-1} + (1+\varepsilon)^{-s}. \tag{2.17}$$

Fix $s = s(\varepsilon)$ so that $(1+\varepsilon)^{-1} + (1+\varepsilon)^{-s} < 1$. Then select $n_0 = n_0(\varepsilon, s) = n_0(\varepsilon)$ so that $\psi(n, \varepsilon, s) < 1$ for $n > n_0$. Then by (2.15), our induction is complete.

By a more careful analysis one can show, using only Lemmas 1, 2, 3, that

$$\Phi(N) < n!\, \exp[n^{0.75+o(1)}] \tag{2.18}$$

3. **The general case.** In this section we prove Theorem 1. As the proof is basically a generalization of the case $k=3$, we shall be somewhat sketchy. The term "Δ-system" shall refer to "k-element Δ-system." We note that the reduction principle applies to k-element Δ-systems.

Definition. For $2 \le i \le K$ let $\varphi_i(n, k)$ denote the least integer so that if F is a family of n element sets, no i pairwise disjoint, $|F| > \varphi_i(n, k)$, then F contains a Δ-system.

We observe

$$\varphi_2(n, k) \le \varphi_3(n, k) \le \cdots \le \varphi_k(n, k) = \varphi(n, k). \tag{3.1}$$

For $k=3$, $\varphi_2 = \gamma$, $\varphi_3 = \varphi$ in the notation of §2.

Lemma 5. *For $2 \le i \le k$, $n \ge 1$ there exists t so that*

$$\varphi_i(n, k) \le \frac{n}{t}\varphi(n-1, k) \tag{3.2}$$

and such that for all integral $s \le \varphi_i(n, k)$

$$\varphi_i(n, k) \le t\binom{s}{2}\varphi(n-1, k) + (n-1)^s\varphi(n-s, k) + s\varphi_{i-1}(n, k) \tag{3.3}$$

(*where for $i=2$, $\varphi_1(n, k)$ is interpreted as zero*).

Proof. Let F be a family of $\varphi_i(n, k)$ n-sets, no i pairwise disjoint, not containing a Δ-system. Set t equal the average $|S \cap T|$ where $S, T \in F$, $S \ne T$. Then (3.2) follows as in Lemma 2. For any $s \le \varphi_i(n, k)$ we find (as in Lemma 3)

$S_1, \ldots, S_s \in F$ so that, setting

$$Y = \bigcup_{1 \le \mu < \nu \le s} S_\mu \cap S_\nu \tag{3.4}$$

we have

$$|Y| \le t\binom{s}{2} \tag{3.5}$$

All sets in F either

(i) intersect Y; at most $|Y|\,\varphi(n-1, k) \le t\binom{s}{2}\varphi(n-1, k)$ such sets, or

(ii) are disjoint from Y but intersect $S_1, \ldots, S_s$; at most $(n-1)^s\varphi(n-s, k)$ such sets, or

(iii) are disjoint from S_μ for some $1 \le \mu \le s$. For fixed μ there are at most $\varphi_{i-1}(n, k)$ such sets (as if those sets contained $i-1$ pairwise disjoint sets with S_k there would be i pairwise disjoint sets); at most $s\varphi_{i-1}(n, k)$ such sets.

The remainder of the proof is purely analytic using Lemma 5.

Select $C_2, C_3, \ldots, C_k = C$; $s_2, s_3, \ldots, s_k$ positive integers such that

$$\begin{gathered} 0 < C_{i-1} < [C_i - C(1+\varepsilon)^{-s_i}]/s_i, \qquad 3 \le i \le k \\ 0 < [C_2 - C(1+\varepsilon)^{-s_2}]s_2 \end{gathered} \tag{3.6}$$

(E.g., select $C_k = C$ arbitrarily; having chosen C_i choose s_i so that $C_i - C(1+\varepsilon)^{-s_i} > 0$ and C_{i-1} satisfying (3.6)). Let K be such that

$$\varphi_i(n, k) \le KC_i(1+\varepsilon)^n n! \tag{3.7}$$

for $2 \le i \le k$ and all $n \le n_0(\varepsilon)$ where $n_0(\varepsilon)$ shall be determined. We show (3.7) holds for all n by a double induction on n and i. Assume (3.7) holds for all $n' < n$ and for n with $i' < i$. By (3.2)

$$\varphi_i(n, k) \le K(C/t)n!\,(1+\varepsilon)^{n-1} < KC_i(1+\varepsilon)^n n! \tag{3.8}$$

if $t > C/C_i$. Now assume $t \le C/C_i$. By (3.3), with $s = s_i$

$$\varphi_i(n, k) \le Kn!\,(1+\varepsilon)^n \psi_i(n, s_i, \varepsilon) \tag{3.9}$$

where

$$\psi_i(n, s_i, \varepsilon) = \frac{(C/C_i)(2^{s_i})C}{n} + C(1+\varepsilon)^{-s_i}\frac{(n-1)^{s_i}}{(n)_{s_i}} + s_iC_{i-1} \tag{3.10}$$

(for $i = 2$, $C_1 = 0$). Then

$$\lim_{n\to\infty} \psi_i(n, s_i, \varepsilon) = C(1+\varepsilon)^{-s_i} + s_iC_{i-1} < C_i \tag{3.11}$$

by (3.6). We choose $n_0(\varepsilon)$ so that

$$\psi_i(n, s_i, \varepsilon) < C_i \quad \text{for} \quad 2 \le i \le k, n \ge n_0(\varepsilon). \tag{3.12}$$

(Note that the C_i, s_i depended only on ε.) Then (3.7) holds for n, i by (3.10), (3.12) and (1.5) holds with constant KC.

References

1. H. L. Abbott, D. Hanson, and N. Sauer, *Intersection theorems for systems of sets*, J. Combinatorial Theory **12** (1972), 381–389.

2. P. Erdös and R. Rado, *Intersection theorems for systems of sets*, J. London Math. Soc. **35** (1960), 85–90.

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