Metrical Coordinates in Non-Euclidean Geometry

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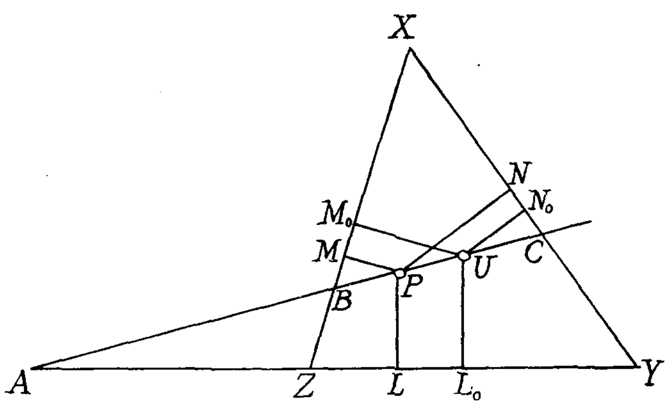
§ 1. The coordinates considered are *linear*, *i.e.* in a plane the equation of a straight line, and in space the equation of a plane, is linear in the coordinates. We shall first consider point-coordinates in plane geometry, taking elliptic geometry as typical, with space-constant unity.

§ 1.1. The general linear or projective coordinates are defined with respect to a fundamental triangle XYZ and a unit-point $U \equiv [1, 1, 1]$. Let $P \equiv [x, y, z]$ be any point; L, M, N the feet of the perpendiculars from P; L_0, M_0, N_0 those from U, on the sides of the triangle. Let PU cut the sides of the triangle in A, B, C. Then[1]

$$\frac{x}{y} = \text{the cross-ratio } (AB, PU)$$
$$= \frac{\sin AP}{\sin BP} \Big/ \frac{\sin AU}{\sin BU}.$$



[1] See, for example, Castelnuovo: *Lezioni di geometria analitica* (6th ed. 1924), p. 229; or the author's *Geometry of n dimensions* (Methuen, 1929), p. 55.

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Now $\qquad \sin LP = \sin AP \, \sin LAP,$
$\sin L_0U = \sin AU \, \sin L_0AU,$

therefore

$$\frac{\sin AP}{\sin AU} = \frac{\sin LP}{\sin L_0U}.$$

Similarly

$$\frac{\sin BP}{\sin BU} = \frac{\sin MP}{\sin M_0U}.$$

Hence

$$\frac{x}{y} = \frac{\sin LP}{\sin L_0U} \bigg/ \frac{\sin MP}{\sin M_0U},$$

and therefore

$$\rho x = \frac{\sin LP}{\sin L_0U}, \quad \rho y = \frac{\sin MP}{\sin M_0U}, \quad \rho z = \frac{\sin NP}{\sin N_0U}, \tag{1.11}$$

where ρ is a factor of proportionality.

§ 1.2. If we assume next that the point-equation of the Absolute is a homogeneous quadratic in x, y, z:

$$(xx) \equiv a_0 x^2 + b_0 y^2 + c_0 z^2 + 2f_0 yz + 2g_0 zx + 2h_0 xy = 0,$$

the distance d between two points (x), (x') is given by

$$\cos d = \frac{(xx')}{\sqrt{(xx)}\sqrt{(x'x')}}. \tag{1 21}$$

Also if the tangential equation of the Absolute is

$$(\xi\xi) \equiv A_0 \xi^2 + \ldots . + 2F_0 \eta\zeta + \ldots . = 0,$$

the angle θ between two lines (ξ), (ξ') is given by

$$\cos\theta = \frac{(\xi\xi')}{\sqrt{(\xi\xi)}\sqrt{(\xi'\xi')}}. \tag{1.22}$$

Further, the distance between the point (x) and the line (ξ) is the complement of the distance between (x) and the absolute pole of (ξ), *i.e.* the point $[x', y', z'] \equiv [A_0\xi + H_0\eta + G_0\zeta, \ldots .]$.

Now $\qquad (xx') = x'(a_0 x + h_0 y + g_0 z) + \ldots .$
$= \Delta(\xi x + \eta y + \zeta z),$

where Δ is the discriminant of (xx). Also

$$(x'x') = \Delta\{\xi(A_0\xi + H_0\eta + G_0\zeta) + \ldots .\} = \Delta(\xi\xi).$$

Hence we see that

$$\sin p = \frac{\Delta^{\frac{1}{2}}(\xi x + \eta y + \zeta z)}{\sqrt{(xx)}\sqrt{(\xi\xi)}}. \tag{1.23}$$

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§ 1.3. The distance from $[x, y, z]$ to the line $x=0$, *i.e.* the line $[1, 0, 0]$, is then given by

$$\sin LP = \frac{\Delta^{\frac{1}{2}} x}{\sqrt{(xx)} \, . \, A_0^{\frac{1}{2}}}.$$

Comparing this with (1.11), we have

$$A_0 = k / \sin^2 p_0,$$

where $p_0 = L_0 U$ and $k = \Delta / \rho^2 (xx)$.

Again, if a, b, c are the lengths of the sides of the triangle of reference, we have

$$\cos a = \frac{f_0}{\sqrt{(b_0 c_0)}}.$$

Therefore

$$k / \sin^2 p_0 = A_0 = b_0 c_0 - f_0^2 = b_0 c_0 \sin^2 a.$$

Hence

$$a_0 = k' \sin^2 a \sin^2 p_0,$$
$$f_0 = k' \sin q_0 \sin r_0 \sin b \sin c \cos a,$$

where $k' = a_0 b_0 c_0 / k$.

The point-equation of the Absolute is therefore

$$x^2 \sin^2 p_0 \sin^2 a + \ldots + 2yz \sin q_0 \sin r_0 \sin b \sin c \cos a + \ldots = 0 \qquad (1.31)$$

and the tangential equation is

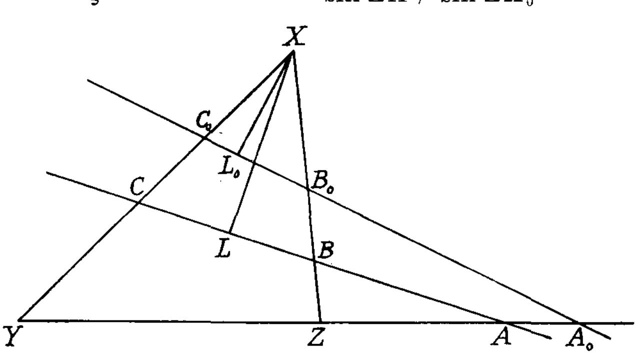
$$\xi^2 \sin^2 q_0 \sin^2 r_0 + \ldots + 2\eta\zeta \sin^2 p_0 \sin q_0 \sin r_0 (\cos b \cos c - \cos a) + \ldots = 0$$

or $$\xi^2/\sin^2 p_0 + \ldots - 2 \cos A \; \eta\zeta/\sin q_0 \sin r_0 - \ldots = 0. \qquad (1.32)$$

where A, B, C are the angles of the triangle of reference.

§ 1.4. Similarly if the line (ξ) cuts the sides of the triangle of reference in A, B, C, and the unit-line cuts the sides in A_0, B_0, C_0, then

$$\frac{\eta}{\zeta} = (YZ, AA_0) = \frac{\sin YA}{\sin ZA} \Big/ \frac{\sin YA_0}{\sin ZA_0}.$$



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Let L, M, N be the feet of the perpendiculars from X, Y, Z on the line (ξ), and L_0, M_0, N_0 those of the unit-line. Then we find

$$\rho\xi = \frac{\sin XL}{\sin XL_0}, \quad \text{etc.} \tag{1.41}$$

The unit-line is determined by the unit-point, being polar and pole with regard to the triangle of reference, and we find

$$\rho' \sin XL_0 = \sin p_0 \sin A, \quad \text{etc.}$$

Hence *the general point-coordinates are certain multiples l, m, n of the sines of the distances of the point from the sides of the triangle, and the related line-coordinates are multiples $l/\sin a$, $m/\sin b$, $n/\sin c$ of the sines of the distances of the line from the vertices.*

§ 1.5. *The Analogue of Trilinears.*

We consider now the special systems of metrical coordinates which correspond to trilinears, areals, and Cartesians.

In trilinears the unit-point is the centre of the inscribed circle of the triangle of reference, so that $p_0 = q_0 = r_0$, and the point-equation of the Absolute is

$$x^2 \sin^2 a + \ldots . + 2yz \sin b \sin c \cos a + \ldots . = 0, \tag{1.51}$$

where a, b, c are the sides of the triangle of reference. The coordinates x, y, z are

$$\rho x = \sin LP, \quad \rho y = \sin MP, \quad \rho z = \sin NP. \tag{1.52}$$

The line-equation of the Absolute is

$$\xi^2 + \eta^2 + \zeta^2 - 2\eta\zeta \cos A - 2\zeta\xi \cos B - 2\xi\eta \cos C = 0. \tag{1.53}$$

§ 1.6. *The Analogue of Areals.*

In areals the point-coordinates are

$$\rho x = \sin LP \sin a, \quad \rho y = \sin MP \sin b, \quad \rho z = \sin NP \sin c. \tag{1.61}$$

The point-equation of the Absolute is

$$x^2 + y^2 + z^2 + 2yz \cos a + 2zx \cos b + 2xy \cos c = 0 \tag{1.62}$$

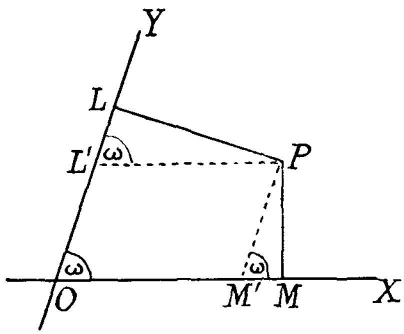
and the line-equation is

$$\xi^2 \sin^2 A + \ldots . - 2\eta\zeta \sin B \sin C \cos A - \ldots . = 0. \tag{1.63}$$

If XU, YU, ZU cut the opposite sides of the triangle in D, E, F, it is readily found that $\cos YD = \cos \frac{1}{2}a = \cos DZ$. Hence D, E, F are the mid-points of the sides, and U is the centroid of the triangle.

§ 1.7. *Cartesian coordinates.*

For the analogue of Cartesian coordinates we modify the areal system by choosing the side XY as the absolute pole of the opposite vertex Z. Then $b = c = \frac{1}{2}\pi = B = C$, and $a = A$. We shall change the notation, calling Z the origin O, and let A, the angle between the axes, be denoted by ω.



Then

$$\rho' x = \sin LP, \quad \rho' y = \sin MP, \quad \rho' z = \cos OP \sin \omega$$

or, putting $\rho' = \rho \sin \omega$,

$$\rho x = \sin LP/\sin \omega, \quad \rho y = \sin MP/\sin \omega, \quad \rho z = \cos OP. \tag{1.71}$$

Now if $L'P$ and $M'P$ are drawn so that angle $PL'L = \omega = PM'M$,

then $$\rho x = \sin L'P, \quad \rho y = \sin M'P, \quad \rho z = \cos OP. \tag{1.72}$$

The point-equation of the Absolute is

$$x^2 + y^2 + z^2 + 2xy \cos \omega = 0 \tag{1.73}$$

and the line-equation

$$\xi^2 + \eta^2 + \zeta^2 \sin^2 \omega - 2\xi\eta \cos \omega = 0. \tag{1.74}$$

When $\omega = \frac{1}{2}\pi$ we have the coordinates of Weierstrass, the analogue of rectangular cartesians.

§ 2. *Metrical coordinates in three dimensions.* The extension to three dimensions is perhaps not quite obvious owing to the variety of angular magnitudes: edges of the tetrahedron, dihedral angles, and face-angles.

§ 2.1. As in plane geometry, if L, M, N are the feet of the perpendiculars from P on the faces of the tetrahedron of reference, L_0, M_0, N_0 those for the unit-point U, we have

$$\rho x = \sin LP/\sin L_0 U = \sin LP/\sin p_0, \text{ etc.} \tag{2.11}$$

Let the point-equation of the Absolute be

$$(xx) \equiv a_0 x^2 + b_0 y^2 + c_0 z^2 + d_0 w^2 + 2f_0 yz + 2g_0 zx + 2h_0 xy + 2l_0 xw + 2m_0 yw + 2n_0 zw = 0.$$

Let the edges of the tetrahedron be $XY = a_{12}$, etc., the dihedral angle between the faces $x = 0$ and $y = 0$, *i.e.* at the edge ZW, be α_{12}, etc.

The tangential equation of the Absolute is

$$(\xi\xi) = A_0 \xi^2 + \ldots. + 2F_0 \eta\zeta + \ldots. + 2L_0 \xi\omega + \ldots. = 0,$$

where, as usual, capital letters denote the cofactors of the corresponding small letters in the determinant Δ.

Then, as before, we have

$$A_0 = k/\sin^2 p_0, \text{ etc.}$$

and also

$$\cos a_{12} = h_0/\sqrt{(a_0 b_0)}, \text{ etc.}$$

Hence

$$\frac{k}{\sin^2 p_0} = A_0 = \begin{vmatrix} b_0 & f_0 & m_0 \\ f_0 & c_0 & n_0 \\ m_0 & n_0 & d_0 \end{vmatrix} = b_0 c_0 d_0 \begin{vmatrix} 1 & \cos a_{23} & \cos a_{24} \\ \cos a_{23} & 1 & \cos a_{34} \\ \cos a_{24} & \cos a_{34} & 1 \end{vmatrix}$$

$$= b_0 c_0 d_0 S_1^2, \quad \text{say.}$$

Then

$$a_0 = k' \sin^2 p_0 \, S_1^2, \ldots.$$
$$f_0 = k' \cos a_{23} \sin q_0 \sin r_0 \, S_2 S_3, \ldots.$$
$$l_0 = k' \cos a_{14} \sin p_0 \sin s_0 \, S_1 S_4, \ldots.$$

Hence the point-equation of the Absolute is

$$x^2 \sin^2 p_0 S_1^2 + \ldots. + 2yz \sin q_0 \sin r_0 S_2 S_3 \cos a_{23} + \ldots. + 2xw \sin p_0 \sin s_0 S_1 S_4 \cos a_{14} + \ldots. = 0. \quad (2.12)$$

For the tangential equation we have already found $A_0, \ldots., D_0$. To find F_0 we have

$$-\cos \alpha_{23} = F_0/\sqrt{(B_0 C_0)}.$$

Hence the tangential equation is

$$\xi^2/\sin^2 p_0 + \ldots - 2\eta\zeta \cos \alpha_{23}/\sin q_0 \sin r_0 - \ldots - 2\xi\omega \cos \alpha_{14}/\sin p_0 \sin s_0 - \ldots = 0. \quad (2.13)$$

§ 2.2. *Trilinears.*

Taking U as the centre of the inscribed sphere, we have $p_0 = q_0 = r_0 = s_0$, and

$$\rho x = \sin LP, \quad \text{etc.} \quad (2.21)$$

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The point-equation of the Absolute is

$$x^2 S_1^2 + \,.\,.\; 2yz\; S_2 S_3 \cos a_{23} + \,.\,. + 2xw\; S_1 S_4 \cos a_{14} + \,.\,. = 0 \quad (2.22)$$

and the tangential equation is

$$\xi^2 + \,.\,. - 2\eta\zeta \cos \alpha_{23} - \,.\,. - 2\xi\omega \cos \alpha_{14} - \,.\,. = 0. \quad (2.23)$$

§ 2.3. *Areals.*

Choosing U so that

$$S_1 \sin p_0 = S_2 \sin q_0 = S_3 \sin r_0 = S_4 \sin s_0,$$
$$\rho x = S_1 \sin LP, \text{ etc.}, \quad (2.31)$$

then the point-equation of the Absolute is

$$x^2 + \,.\,. + 2yz \cos a_{23} + \,.\,. + 2xw \cos a_{14} + \,.\,. = 0 \quad (2.32)$$

and the tangential equation

$$\xi^2 S_1^2 + \,.\,. - 2\eta\zeta\; S_2 S_3 \cos \alpha_{23} - \,.\,. - 2\xi\omega\; S_1 S_4 \cos \alpha_{14} - \,.\,. = 0. \quad (2.33)$$

We find that the plane UXY bisects the edge ZW, etc., so that U is the centroid of the tetrahedron.

§ 2.4. *Cartesians.*

Modifying the "areal" system by choosing XYZ as the absolute polar of W (or the origin O), we have $a_{14} = a_{24} = a_{34} = \frac{1}{2}\pi = \alpha_{14} = \alpha_{24} = \alpha_{34}$; write also for the plane-angles between the axes OX, OY, OZ, $a_{23} = \lambda$, $a_{31} = \mu$, $a_{12} = \nu$.

The functions S_1^2, S_2^2, S_3^2 simplify to $\sin^2\lambda$, $\sin^2\mu$, $\sin^2\nu$: hence

$$\rho x = \sin LP \sin \lambda, \quad \rho y = \sin MP \sin \mu, \quad \rho z = \sin NP \sin \nu,$$
$$\rho w = \cos OP \,.\, S_4. \quad (2.41)$$

The point-equation of the Absolute becomes

$$x^2 + y^2 + z^2 + w^2 + 2yz \cos \lambda + 2zx \cos \mu + 2xy \cos \nu = 0, \quad (2.42)$$

and the tangential equation

$$\xi^2 \sin^2 \lambda + \eta^2 \sin^2 \mu + \zeta^2 \sin^2 \nu + \omega^2 S_4^2 - 2\eta\zeta \sin \mu \sin \nu \cos \alpha_{23}$$
$$- 2\zeta\xi \sin \nu \sin \lambda \cos \alpha_{31} - 2\xi\eta \sin \lambda \sin \mu \cos \alpha_{12} = 0. \quad (2.43)$$

Let L, M, N be the angles which each coordinate-axis makes with the opposite coordinate-plane. Draw a sphere with centre O. The coordinate planes cut this in a spherical triangle $X'\,Y'\,Z'$ whose sides are λ, μ, ν, angles, α_{23}, α_{31}, α_{12}, and altitudes, L, M, N.

Now we have

$$\sin L = \sin \mu \sin \alpha_{12},$$

and

$$\cos \nu = \cos \lambda \cos \mu + \sin \lambda \sin \mu \cos \alpha_{12}.$$

Therefore, eliminating a_{12}, we obtain

$$\sin^2 L = \sin^2 \mu \left\{ 1 - \frac{(\cos \nu - \cos \lambda \cos \mu)^2}{\sin^2 \lambda \sin^2 \mu} \right\},$$

so that

$$\begin{aligned} \sin^2 L \sin^2 \lambda &= \sin^2 \lambda \sin^2 \mu - (\cos \nu - \cos \lambda \cos \mu)^2 \\ &= 1 - \cos^2 \lambda - \cos^2 \mu - \cos^2 \nu + 2 \cos \lambda \cos \mu \cos \nu \\ &= S_4^2. \end{aligned}$$

Hence $$\sin L \sin \lambda = \sin M \sin \mu = \sin N \sin \nu = S_4. \tag{2.44}$$

The point-coordinates are

$$\rho' x = \sin LP \sin \lambda, \ \ldots, \ \rho' w = S_4 \cos OP,$$

or, using (2.44),

$$\rho x = \sin LP / \sin L, \ \ldots, \ \rho w = \cos OP. \tag{2.45}$$

§ 2.5. If α, β, γ are the angles which OP makes with the coordinate-planes, then

$$\sin LP = \sin OP \sin \alpha, \text{ etc.}$$

$$\cos OP = w/\sqrt{(xx)}.$$

Therefore

$$w^2 = (x^2 + y^2 + z^2 + w^2 + 2yz \cos \lambda + 2zx \cos \mu + 2xy \cos \nu) \cos^2 OP,$$

$$\rho'^2 w^2 \sin^2 OP = \sin^2 OP (\sin^2 \alpha \sin^2 \lambda + \sin^2 \beta \sin^2 \mu + \sin^2 \gamma \sin^2 \nu + 2 \sin \beta \sin \gamma \sin \mu \sin \nu \cos \lambda + \ldots) \cos^2 OP.$$

Hence the direction-angles α, β, γ are connected by the identity

$$\sin^2 \alpha \sin^2 \lambda + \ldots + 2 \sin \beta \sin \gamma \sin \mu \sin \nu \cos \lambda + \ldots = S_4^2. \tag{2.51}$$

When the coordinates are rectangular, $\lambda = \mu = \nu = \frac{1}{2}\pi$ and $S_4 = 1$; then

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 1.$$

This indicates that, contrary to the usual custom, the direction-angles should be defined as the angles which the radius-vector makes with the coordinate-planes, instead of with the coordinate-axes. The "direction-sines" $\sin \alpha$, ..., are of course a system of "trilinear" coordinates in the spherical geometry about the point O.

§ 3.1. *Extension to n dimensions.* In the extension to n dimensions we shall alter the notation. Let the simplex of reference be X_0, X_1, ..., X_n; let the coordinates be x_0, ..., x_n; let the perpendicular from unit-point on to the coordinate-plane $x_r = 0$ be p_r, and that from P be $L_r P$.

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Then

$$\rho x_r = \frac{\sin L_r P}{\sin p_r}. \qquad (3.11)$$

Let the lengths of the edges of the simplex be $X_r X_s = k_{rs}$, and the dihedral angle between the primes $x_r = 0$ and $x_s = 0$ be κ_{rs}.

Let the point-equation and tangential-equation of the Absolute be

$$(xx) \equiv \Sigma\Sigma\, a_{rs} x_r x_s = 0,$$
$$(\xi\xi) \equiv \Sigma\Sigma\, A_{rs} \xi_r \xi_s = 0.$$

Then

$$\sin L_r P = \frac{\Delta^{\frac{1}{2}} x_r}{A_{rr}^{\frac{1}{2}} \sqrt{(xx)}}.$$

Hence

$$A_{rr} = k/\sin^2 p_r.$$

Also

$$\cos k_{rs} = \frac{a_{rs}}{\sqrt{(a_{rr} a_{ss})}}.$$

Expressing A_{rr} as the cofactor of a_{rr} in Δ, we have

$$A_{rr} a_{rr} = a_{00} a_{11} \ldots a_{nn} S_{n,r}^2,$$

where

$$S_{n,r}^2 \equiv \begin{vmatrix} 1 & \cos k_{12} & \ldots & \cos k_{1,r-1} & \cos k_{1,r+1} & \ldots & \cos k_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cos k_{n1} & \cos k_{n2} & \ldots & \cos k_{n,r-1} & \cos k_{n,r+1} & \ldots & 1 \end{vmatrix}.$$

Then

$$\left.\begin{aligned} a_{rr} &= k' \sin^2 p_r S_{n,r}^2, \\ a_{rs} &= k' \sin p_r \sin p_s S_{n,r} S_{n,s} \cos k_{rs} \end{aligned}\right\}. \qquad (3.12)$$

§ 3.2. If U is the centroid of the simplex (areals)

$$\sin p_r \,.\, S_{n,r} = \text{const}. \qquad (3.21)$$

and

$$\rho x_r = S_{n,r} \sin L_r P; \qquad (3.22)$$

and the point-equation of the Absolute becomes

$$\Sigma x_r^2 + 2 \Sigma x_r x_s \cos k_{rs} = 0. \qquad (3.23)$$

The tangential equation is

$$\Sigma S_{n,r}^2 \xi_r^2 - 2 \Sigma \xi_r \xi_s S_{n,r} S_{n,s} \cos \kappa_{rs} = 0. \qquad (3.24)$$

§ 3.3. To obtain the Cartesian system take the prime $x_0 = 0$ as the absolute polar of X_0 (or O), so that $k_{0r} = \frac{1}{2}\pi = \kappa_{0r}$, where k_{rs} is the angle between the lines OX_r and OX_s. The point-equation of the Absolute is then

$$x_0^2 + \Sigma x_r^2 + 2 \Sigma x_r x_s \cos k_{rs} = 0 \quad (r \neq s = 1, 2, \ldots, n). \qquad (3.31)$$

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For the tangential equation the coefficient of ξ_0^2 is $S_{n,0}^2$. That of ξ_1^2 is

$$\begin{vmatrix} 1 & \cos k_{23} & \dots & \cos k_{2n} & 0 \\ \cos k_{32} & 1 & \dots & \cos k_{3n} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \cos k_{n2} & \cos k_{n3} & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix} = S_{n-1,1}^2.$$

The tangential equation is then

$$S_{n,0}^2 \xi_0^2 + \Sigma S_{n-1,r}^2 \xi_r^2 - 2 \Sigma \xi_r \xi_s S_{n-1,r} S_{n-1,s} \cos \kappa_{rs} = 0. \qquad (3.32)$$

The point-coordinates are

$$\left. \begin{aligned} \rho x_0 &= \frac{\sin L_0 P}{\sin p_0} = S_{n,0} \cos OP, \\ \rho x_r &= S_{n-1,r} \sin L_r P. \end{aligned} \right\} \qquad (3.33)$$

Let θ_r be the angle which OX_r makes with the opposite coordinate-prime. Draw a hypersphere round O and we obtain a spherical simplex of $n-1$ dimensions whose edges are the angles k_{rs} and altitudes θ_r. Applying to this the formula for distance in $n-1$ dimensions we have

$$\sin \theta_r = \frac{\Delta_{n-1}^{\frac{1}{2}}}{S_{n-1,r}},$$

and $\Delta_{n-1} = S_{n,0}^2$.

Hence dividing the coordinates by $S_{n,0}$, we have

$$\left. \begin{aligned} \rho x_0 &= \cos OP, \\ \rho x_r &= \frac{S_{n-1,r} \sin L_r P}{S_{n,0}} = \sin L_r P \sin \theta_r \end{aligned} \right\}. \qquad (3.34)$$

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