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Hypercyclic Abelian Groups of Affine Maps on $\mathbb{C}^n$

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Abstract. We give a characterization of hypercyclic abelian group $\mathcal{G}$ of affine maps on $\mathbb{C}^n$. If $\mathcal{G}$ is finitely generated, this characterization is explicit. We prove in particular that no abelian group generated by n affine maps on $\mathbb{C}^n$ has a dense orbit.

1 Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in $\mathbb{C}$ and $\mathrm{GL}(n,\mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is called an *affine map* if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We let $f = (A, a)$, and we call A the *linear part* of f. The map f is invertible if $A \in \mathrm{GL}(n,\mathbb{C})$. Denote by $\mathrm{MA}(n,\mathbb{C})$ the vector space of all affine maps on $\mathbb{C}^n$ and $\mathrm{GA}(n,\mathbb{C})$ the group of all invertible affine maps of $\mathrm{MA}(n,\mathbb{C})$.

Let $\mathcal{G}$ be an abelian affine subgroup of $\mathrm{GA}(n,\mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of $\mathcal{G}$ through v: $\mathcal{G}(v) = \{f(v) : f \in \mathcal{G}\} \subset \mathbb{C}^n$. Denote by $\overline{E}$ the closure of a subset $E \subset \mathbb{C}^n$. The group $\mathcal{G}$ is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $\overline{\mathcal{G}(v)} = \mathbb{C}^n$. For an account of results and bibliography on hypercyclicity, we refer to the book [3] by Bayart and Matheron.

Let $n \in \mathbb{N}_0$ be fixed, denote by:

- $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$;
- $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$ the canonical basis of $\mathbb{C}^{n+1}$ and I_{n+1} the identity matrix of $\mathrm{GL}(n+1,\mathbb{C})$.

For each $m = 1, 2, \dots, n+1$, denote by

- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over $\mathbb{C}$ of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \cdots & a_{m,m-1} & \mu \end{bmatrix}; \tag{1.1}$$

- $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1.1) with $\mu \neq 0$.

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Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $n_1 + \cdots + n_r = n + 1$. In particular, $r \leq n + 1$. Write

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r = 1$, then $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$ and $\eta = (n + 1)$;
- $\mathcal{K}^*_{\eta,r}(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathrm{GL}(n + 1, \mathbb{C})$;
- $u_0 = (e_{1,1}, \dots, e_{r,1}) \in \mathbb{C}^{n+1}$ where $e_{k,1} = (1, 0, \dots, 0) \in \mathbb{C}^{n_k}$, for $k = 1, \dots, r$, so $u_0 \in \{1\} \times \mathbb{C}^n$;
- $p_2 \colon \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, the second projection, defined by $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$;
- $e^{(k)} = (e^{(k)}_1, \dots, e^{(k)}_r) \in \mathbb{C}^{n+1}$ where

$$e^{(k)}_j = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k \\ e_{k,1} & \text{if } j = k \end{cases} \quad \text{for every } 1 \leq j, k \leq r;$$

- $\exp\colon \mathbb{M}_{n+1}(\mathbb{C}) \longrightarrow \mathrm{GL}(n + 1, \mathbb{C})$ is the matrix exponential map; set $\exp(M) = e^M$, $M \in M_{n+1}(\mathbb{C})$.

Define the map $\Phi\colon\ \mathrm{GA}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(n + 1, \mathbb{C})$,

$$f = (A, a) \longmapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}.$$

We have the composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then Φ is an injective homomorphism of groups. Write $G = \Phi(\mathcal{G})$, which is an abelian subgroup of $\mathrm{GL}(n + 1, \mathbb{C})$.

Define the map $\Psi\colon\ \mathrm{MA}(n, \mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C})$,

$$f = (A, a) \longmapsto \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}.$$

We can see that Ψ is injective and linear. Hence $\Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$ is a vector subspace of $M_{n+1}(\mathbb{C})$. We prove (see Lemma 2.8) that Φ and Ψ are related by the following property

$$\exp\Big(\Psi\big(\mathrm{MA}(n, \mathbb{C})\big)\Big) = \Phi\big(\mathrm{GA}(n, \mathbb{C})\big).$$

Let us consider the normal form of $\mathcal{G}$: By Proposition 2.1, there exists a $P \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$ and a partition η of $(n + 1)$ such that

$$G' = P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi\big(\mathrm{GA}(n, \mathbb{C})\big).$$

For such a choice of matrix P, we assume the following:

- $v_0 = Pu_0$. So $v_0 \in \{1\} \times \mathbb{C}^n$, since $P \in \Phi\big(\mathrm{GA}(n,\mathbb{C})\big)$.
- $w_0 = p_2(v_0) \in \mathbb{C}^n$. We have $v_0 = (1, w_0)$.
- $\varphi = \Phi^{-1}(P) \in \mathrm{GA}(n,\mathbb{C})$.
- $\mathrm{g} = \exp^{-1}(G) \cap \Big(P\big(\mathcal{K}_{\eta,r}(\mathbb{C})\big)P^{-1}\Big)$. If $G \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$, we have $P = I_{n+1}$ and $\mathrm{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$.
- $\mathrm{g}^1 = \mathrm{g} \cap \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$. This is an additive subgroup of $M_{n+1}(\mathbb{C})$ (because by Lemma 3.2, g is an additive subgroup of $M_{n+1}(\mathbb{C})$).
- $\mathrm{g}^1_u = \{Bu : B \in \mathrm{g}^1\} \subset \mathbb{C}^{n+1}$, $u \in \mathbb{C}^{n+1}$.
- $\mathfrak{q} = \Psi^{-1}(\mathrm{g}^1) \subset \mathrm{MA}(n,\mathbb{C})$. Then $\mathfrak{q}$ is an additive subgroup of $\mathrm{MA}(n,\mathbb{C})$ and we have $\Psi(\mathfrak{q}) = \mathrm{g}^1$. By Corollary 2.12, we have $\exp\big(\Psi(\mathfrak{q})\big) = \Phi(\mathcal{G})$.
- $\mathfrak{q}_v = \{f(v), f \in \mathfrak{q}\} \subset \mathbb{C}^n$, $v \in \mathbb{C}^n$.

For groups of affine maps on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), the study of their dynamics was recently initiated for some classes from a different point of view (see for instance, [2, 4–6]). The purpose here is to give analogous results for linear abelian subgroups of $\mathrm{GL}(n,\mathbb{C})$ [1, Theorem 1.1].

Our main results are the following.

Theorem 1.1 *Let* $\mathcal{G}$ *be an abelian subgroup of* $\mathrm{GA}(n,\mathbb{C})$. *Then the following are equivalent:*

(i) $\mathcal{G}$ *is hypercyclic;*
(ii) *the orbit* $\mathcal{G}(w_0)$ *is dense in* $\mathbb{C}^n$*;*
(iii) $\mathfrak{q}_{w_0}$ *is an additive subgroup dense in* $\mathbb{C}^n$.

In the particular case where $\mathcal{G}$ is an abelian subgroup of $\mathrm{GL}(n,\mathbb{C})$, let $Q \in \mathrm{GL}(n,\mathbb{C})$ such that $Q^{-1}\mathcal{G}Q \subset \mathcal{K}^*_{\eta',r'}(\mathbb{C})$ for some $r' \leq n$ and $\eta' = (n'_1,\dots,n'_{r'}) \in \mathbb{N}_0^{r'}$ with $n'_1 + \cdots + n'_{r'} = n$ (Proposition 2.6). Write

- $u'_0 = (e'_{1,1},\dots,e'_{r',1}) \in \mathbb{C}^n$ where $e'_{k,1} = (1,0,\dots,0) \in \mathbb{C}^{n'_k}$, for $k = 1,\dots,r'$;
- $v'_0 = Qu'_0$;
- $\mathrm{g}' = \exp^{-1}(\mathcal{G}) \cap Q\big(\mathcal{K}_{\eta',r'}(\mathbb{C})\big)Q^{-1}$ and $\mathrm{g}'_{v'_0} = \{f(v'_0), f \in \mathrm{g}'\}$.

Corollary 1.2 ([1, Theorem 1.3]) *Let* $\mathcal{G}$ *be an abelian subgroup of* $\mathrm{GL}(n,\mathbb{C})$. *Under the notations above, the following properties are equivalent:*

(i) $\mathcal{G}$ *is hypercyclic.*
(ii) $\mathrm{g}'_{v'_0}$ *is an additive subgroup dense in* $\mathbb{C}^n$.

For a *finitely generated* abelian subgroup $\mathcal{G} \subset \mathrm{GA}(n,\mathbb{R})$, let us introduce the following property. Consider the following rank condition on a collection of affine maps $f_1,\dots,f_p \in \mathcal{G}$. Let $f'_1,\dots,f'_p \in \mathfrak{q}$ be such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1,\dots,p$. We say that $f_1,\dots,f_p$ satisfy the *property* $\mathcal{D}$ if for every $(s_1,\dots,s_p;t_2,\dots,t_r) \in \mathbb{Z}^{p+r-1}\backslash\{0\}$,

$$\operatorname{rank}\begin{bmatrix} \mathrm{Re}\big(f'_1(w_0)\big) & \cdots & \mathrm{Re}\big(f'_p(w_0)\big) & 0 & \cdots & 0 \\ \mathrm{Im}\big(f'_1(w_0)\big) & \cdots & \mathrm{Im}\big(f'_p(w_0)\big) & 2\pi p_2(e^{(2)}) & \cdots & 2\pi p_2(e^{(r)}) \\ s_1 & \cdots & s_p & t_2 & \cdots & t_r \end{bmatrix} = 2n+1.$$

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For $r = 1$, this means that for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \backslash \{0\}$,

$$\operatorname{rank} \begin{bmatrix} \operatorname{Re}\big(f_1'(w_0)\big) & \cdots & \operatorname{Re}\big(f_p'(w_0)\big) \\ \operatorname{Im}\big(f_1'(w_0)\big) & \cdots & \operatorname{Im}\big(f_p'(w_0)\big) \\ s_1 & \cdots & s_p \end{bmatrix} = 2n + 1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = \operatorname{Re}(v) + i \operatorname{Im}(v)$ where $\operatorname{Re}(v)$ and $\operatorname{Im}(v) \in \mathbb{R}^n$. The next result can be stated as follows.

Theorem 1.3 *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GA}(n, \mathbb{C})$ generated by $f_1, \dots, f_p$ and let $f_1', \dots, f_p' \in \mathfrak{q}$ be such that $e^{\Psi(f_1')} = \Phi(f_1), \dots, e^{\Psi(f_p')} = \Phi(f_p)$. Then the following are equivalent:*

(i) *$\mathcal{G}$ is hypercyclic;*
(ii) *the maps $\varphi^{-1} \circ f_1 \circ \varphi, \dots, \varphi^{-1} \circ f_p \circ \varphi$ in $\mathrm{GA}(n, \mathbb{C})$ satisfy the property $\mathcal{D}$;*
(iii)

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{Z} f_k'(w_0) + 2i\pi \sum_{k=2}^{r} \mathbb{Z}\big(p_2(Pe^{(k)})\big) & \text{if } r \geq 2, \\ \sum_{k=1}^{p} \mathbb{Z} f_k'(w_0) & \text{if } r = 1, \end{cases}$$

is an additive subgroup dense in $\mathbb{C}^n$.

Corollary 1.4 *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GA}(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Let $P \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$ such that $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$ where $1 \leq r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$. If $\mathcal{G}$ is generated by $2n - r + 1$ commuting invertible affine maps, then it has no dense orbit.*

Corollary 1.5 *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GA}(n, \mathbb{C})$. If $\mathcal{G}$ is generated by n commuting invertible affine maps, then it has no dense orbit.*

2 Normal Form of Abelian Affine Groups

The aim of this section is to prove the following proposition.

Proposition 2.1 *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GA}(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$, for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.*

The group $G' = P^{-1}GP$ is called the *normal form* of G. In particular, we have $Pu_0 = v_0 \in \{1\} \times \mathbb{C}^n$. Denote by $\mathcal{L}_{\mathcal{G}}$ the set of the linear parts of all elements of $\mathcal{G}$. Then $\mathcal{L}_{\mathcal{G}}$ is an abelian subgroup of $\mathrm{GL}(n, \mathbb{C})$. A subset $F \subset \mathbb{C}^n$ is called *G-invariant* (resp. *$\mathcal{L}_{\mathcal{G}}$-invariant*) if $A(F) \subset F$ for any $A \in G$ (resp. $A \in \mathcal{L}_{\mathcal{G}}$). To prove Proposition 2.1, we need the following results.

Lemma 2.2 *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GA}(n, \mathbb{C})$, $n \geq 1$ and $G = \Phi(\mathcal{G})$. Then there exist an integer $p \in \mathbb{N}$, $0 \leq p \leq n$ and $Q \in \mathrm{GL}(n, \mathbb{C})$ such that*

(i) *$\mathbb{C}^n = E \oplus H$ where $E = Q(\mathbb{C}^p \times \{0_{\mathbb{C}^{n-p}}\})$ and $H = Q(\{0_{\mathbb{C}^p}\} \times \mathbb{C}^{n-p})$ are $\mathcal{L}_{\mathcal{G}}$-invariant;*
(ii) *if $E \neq \{0\}$, then for every $A \in \mathcal{L}_{\mathcal{G}}$, $A_{/E}$ has 1 as the only eigenvalue;*

(iii) *if* $E \neq \{0\}$, $H \neq \{0\}$ *and* $P_1 = \mathrm{diag}(1, Q)$, *then for every* $f = (A, a) \in \mathcal{G}$, *one has*

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix}$$

where $A_1 = A_{/E} \in \mathbb{T}_p^*(\mathbb{C})$, $A_2 = A_{/H} \in \mathcal{K}^*_{\eta'',r''}(\mathbb{C})$ *for some* $r'' \leq n - p$ *and* $\eta'' \in \mathbb{N}_0^{r''}$, $a_1 \in \mathbb{C}^p$ *and* $a_2 \in \mathbb{C}^{n-p}$;

(iv) *if* $H = \{0\}$, *then for every* $f = (A, a) \in \mathcal{G}$, *one has* $P_1^{-1}\Phi(f)P_1 \in \mathbb{T}^*_{n+1}(\mathbb{C}) \cap \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$.

Proof Apply Proposition 2.6 to the group $\mathcal{L}_\mathcal{G}$; there exists $Q \in \mathrm{GL}(n, \mathbb{C})$ such that $Q^{-1}\mathcal{L}_\mathcal{G}Q$ is a subgroup of $\mathcal{K}^*_{\eta',r'}(\mathbb{C})$ for some $r' \leq n$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$ such that $n'_1 + \cdots + n'_{r'} = n$. Hence for every $A \in \mathcal{L}_\mathcal{G}$, we have $Q^{-1}AQ = \mathrm{diag}(A'_1, \dots, A'_{r'})$ with $A'_k \in \mathbb{T}^*_{n'_k}$. Let $\mu_{A'_k}$ be the only eigenvalue of A'_k, $k = 1, \dots, r'$ and denote by $J_\mathcal{G} = \big\{k \in \{1, \dots, r'\} : \mu_{A'_k} = 1, \forall A \in \mathcal{L}_\mathcal{G}\big\}$. If $J_\mathcal{G} = \varnothing$, we take $E = \{0\}$ and $H = \mathbb{C}^n$. If $J_\mathcal{G} \neq \varnothing$, one can assume that $J_\mathcal{G} = \{1, \dots, s\}$ for some $1 \leq s \leq r'$, by replacing Q by QR, where R is a circular matrix R of $\mathrm{GL}(n, \mathbb{C})$. We let $P_1 = \mathrm{diag}(1, Q) = \Phi(f_1)$, $f_1 = (Q, 0)$. So for every $f = (A, a) \in \mathcal{G}$, we have

$$\Phi(f_1^{-1} \circ f \circ f_1) = P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 \\ Q^{-1}a & Q^{-1}AQ \end{bmatrix} \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big).$$

Proof of (i) If $J_\mathcal{G} = \varnothing$, the assertion is clear. One can assume that $J_\mathcal{G} \neq \varnothing$. We let $p = n'_1 + \cdots + n'_s$, $E = Q(\mathbb{C}^p \times \{0_{\mathbb{C}^{n-p}}\})$ and $H = Q(\{0_{\mathbb{C}^p}\} \times \mathbb{C}^{n-p})$. It is plain that $\mathbb{C}^n = E \oplus H$. Moreover, E and H are $\mathcal{L}_\mathcal{G}$-invariant vector spaces: Indeed, if $A \in \mathcal{L}_\mathcal{G}$ and $x = (x_1, 0) \in \mathbb{C}^p \times \{0_{\mathbb{C}^{n-p}}\}$, one has $AQx = Q(Q^{-1}AQ)x$. Since $Q^{-1}AQ = \mathrm{diag}(A_1, A_2)$ where $A_1 = \mathrm{diag}(A'_1, \dots, A'_s) \in \mathrm{GL}(p, \mathbb{C})$ with $\mu_{A'_k} = 1$, $k = 1, \dots, s$ and $A_2 = \mathrm{diag}(A'_{s+1}, \dots, A'_{r'})$, we have $Q^{-1}AQx = (A_1x_1, 0) \in \mathbb{C}^p \times \{0_{\mathbb{C}^{n-p}}\}$. The same proof holds for H.

Proof of (ii) If $A \in \mathcal{L}_\mathcal{G}$ then $(Q^{-1}AQ)_{/E} = A_1 = \mathrm{diag}(A'_1, \dots, A'_s) \in \mathrm{GL}(p, \mathbb{C})$ with $\mu_{A'_k} = 1$, $k = 1, \dots, s$.

Proof of (iii) Assume that $E \neq \{0\}$ and $H \neq \{0\}$. Then, for every $f = (A, a) \in \mathcal{G}$, we have $Q^{-1}AQ = \mathrm{diag}(A_1, A_2)$ where $A_1 = A_{/E} \in \mathbb{T}_p^*(\mathbb{C})$, $A_2 = A_{/H} \in \mathcal{K}^*_{\eta'',r''}(\mathbb{C})$ with $r'' = r' - s \leq n - p$ and $\eta'' = (n'_{s+1}, \dots, n'_{r'})$. Hence

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 \\ Q^{-1}a & Q^{-1}AQ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix},$$

where $Q^{-1}a = (a_1, a_2) \in \mathbb{C}^p \times \mathbb{C}^{n-p}$. Note that by (ii), 1 is the only eigenvalue of A_1.

Proof of (iv) Assume that $H = \{0\}$. In this case we have $s = r'$ and $J_\mathcal{G} = \{1, \dots, r'\}$. Then for every $f = (A, a) \in \mathcal{G}$, we have $P_1^{-1}\Phi(f)P_1 = \left[\begin{smallmatrix} 1 & 0 \\ a_1 & A_1 \end{smallmatrix}\right]$ with $A = A_1 \in \mathbb{T}_n^*(\mathbb{C})$. So $P_1^{-1}\Phi(f)P_1 \in \mathbb{T}^*_{n+1}(\mathbb{C}) \cap \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$. ∎

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Lemma 2.3 ([2, Lemma 3.1]) *Let $u_1,\dots,u_n\in\mathbb{C}^n$ such that for every $1\le k\le n$, $u_k=(x_{k,1},\dots,x_{k,n})$ with $x_{k,k}\ne 0$. Then $(\mathbb{Z}u_1+\cdots+\mathbb{Z}u_n)\cap(\mathbb{C}^*)^n\ne\varnothing$.*

Lemma 2.4 *Let $\mathcal{G}$ and H are as in Lemma 2.2 If $H\ne\{0\}$ then there exists $B\in\mathcal{L}_{\mathcal{G}}$ such that $B_{/H}-I_{n-p}$ is invertible.*

Proof As $H\ne\{0\}$, then $s<r'$ and for every $1\le k\le r'-s$ there exists $B(k)\in G$ such that $B(k)_{/H}=\operatorname{diag}(B_{k,s+1},\dots,B_{k,r'})$ where

$$B_{k,j}=\begin{bmatrix}\mu_{B_{k,j}} & & & 0\\ b_{2,1}^{(k)} & \ddots & & \\ \vdots & \ddots & \ddots & \\ b_{n'_j,1}^{(k)} & \cdots & b_{n'_j,n'_j-1}^{(k)} & \mu_{B_{k,j}}\end{bmatrix}\in\mathbb{T}^*_{n'_j}(\mathbb{C}),$$

such that $\mu_{B_{k,s+k}}\ne 1$, for every $j=s+1,\dots,r'$.

We let $u_k=\big(\log(\mu_{B_{k,s+1}}),\dots,\log(\mu_{B_{k,r'}})\big)\in\mathbb{C}^{r'-s}$, $k=1,\dots,r'-s$. For $z=|z|e^{i\arg(z)}\in\mathbb{C}$, $\arg(z)\in[0,2\pi[$, $\log z=|z|+i\arg(z)$. As $\log(\mu_{B_{k,s+k}})\ne 0$ for every $k=1,\dots,r'-s$, by Lemma 2.3, $(\mathbb{Z}u_1+\cdots+\mathbb{Z}u_{r'-s})\cap(\mathbb{C}^*)^{r'-s}\ne\varnothing$. So there exist $m_1,\dots,m_{r'-s}\in\mathbb{Z}$ such that $m_1u_1+\cdots+m_{r'-s}u_{r'-s}\in(\mathbb{C}^*)^{r'-s}$. It follows that for every $j=s+1,\dots,r'$, $\prod_{k=1}^{r'-s}\mu_{B_{k,j}}^{m_k}\ne 1$. If $B=\prod_{k=1}^{r'-s}\big(B(k)\big)^{m_k}$, then $\prod_{k=1}^{r'-s}\mu_{B_{k,j}}^{m_k}$, $j=s+1,\dots,r'$ are the eigenvalues of $B_{/H}$, this implies that $B_{/H}-I_{n-p}$ is invertible. ■

Denote by $\operatorname{Fix}(G)=\{x\in\mathbb{C}^{n+1}: Bx=x, \text{ for every } B\in G\}$.

Lemma 2.5 *Let G and E be as in Lemma 2.2. If $E=\{0\}$ then $\operatorname{Fix}(G)\cap(\{1\}\times\mathbb{C}^n)\ne\varnothing$.*

Proof By hypothesis, $p=0$ and so $H=\mathbb{C}^n$. Then by Lemma 2.4, we have $B\in\mathcal{L}_{\mathcal{G}}$ such that $B-I_n$ is invertible, so 1 is not an eigenvalue of B. We let $f_0=(B,b)\in\mathcal{G}$. As $\Phi(f_0)=\left[\begin{smallmatrix}1&0\\ b&B\end{smallmatrix}\right]$, $F=\operatorname{Fix}\big(\Phi(f_0)\big)=\{x\in\mathbb{C}^{n+1}:\Phi(f_0)x=x\}$ has dimension 1. So $\operatorname{Fix}\big(\Phi(f_0)\big)=\mathbb{C}v$, where $v=(1,v_1)$, $v_1\in\mathbb{C}^n$. Write $P_2=\left[\begin{smallmatrix}1&0\\ v_1&I_n\end{smallmatrix}\right]$. We have $\Phi(f_0)v=v$, so $Bv_1+b=v_1$ and $P_2^{-1}\Phi(f_0)P_2=\left[\begin{smallmatrix}1&0\\ -v_1+b+Bv_1&B\end{smallmatrix}\right]=\left[\begin{smallmatrix}1&0\\ 0&B\end{smallmatrix}\right]$. Similarly, for every $f=(A,a)\in\mathcal{G}$, one has $P_2^{-1}\Phi(f)P_2=\left[\begin{smallmatrix}1&0\\ Av_1+a-v_1&A\end{smallmatrix}\right]$. Write $a'=Av_1+a-v_1$. Since G is abelian, we have $P_2^{-1}\Phi(f_0)\Phi(f)P_2=P_2^{-1}\Phi(f)\Phi(f_0)P_2$, this implies that $Ba'=a'$ and hence $a'=0$. It follows that $P_2^{-1}\Phi(f)P_2e_1=e_1$, hence $P_2e_1\in\operatorname{Fix}(G)$. Since $P_2e_1\in\{1\}\times\mathbb{C}^n$, we conclude that $\operatorname{Fix}(G)\cap(\{1\}\times\mathbb{C}^n)\ne\varnothing$. ■

Proposition 2.6 ([1, Proposition 2.3]) *Let G' be an abelian subgroup of $\operatorname{GL}(m,\mathbb{C})$, $m\ge 1$. Then there exists $P\in\operatorname{GL}(m,\mathbb{C})$ such that $P^{-1}G'P$ is a subgroup of $\mathcal{K}^*_{\eta',r'}(\mathbb{C})$, for some $r'\le m$ and $\eta'=(n'_1,\dots,n'_{r'})\in\mathbb{N}_0^{r'}$.*

Proof of Proposition 2.1 Let $P_1=\operatorname{diag}(1,Q)$, E and H as in Lemma 2.2. We distinguish two cases:

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Case 1: $E \neq \{0\}$ If $H = \{0\}$, then the proposition results from Lemma 2.2 (iv) by taking $P = P_1$.

If $H \neq \{0\}$, then by Lemma 2.4 there exists $B \in \mathcal{L}_{\mathcal{G}}$ such that $B_{/H} - I_{n-p}$ is invertible. Write $B_1 = B_{/E}$, $B_2 = B_{/H}$ and set $f_0 = (B, b) \in \mathcal{G}$. Since $E \neq \{0\}$, we have by Lemma 2.2 (iii),

$$P_1^{-1}\Phi(f_0)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ b_1 & B_1 & 0 \\ b_2 & 0 & B_2 \end{bmatrix}$$

where $B_1 \in \mathbb{T}_p^*(\mathbb{C})$, $B_2 \in \mathcal{K}^*_{\eta'',r''}(\mathbb{C})$ for some $r'' \leq n-p$, $\eta'' = (n_1'', \dots, n_{r''}'') \in \mathbb{N}_0^{r''}$ and $(b_1, b_2) \in \mathbb{C}^p \times \mathbb{C}^{n-p}$. If

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_p & 0 \\ b_2 & 0 & B_2 - I_{n-p} \end{bmatrix},$$

it is clear that $P_2 \in GL(n+1, \mathbb{C})$. We let $P = P_1 P_2^{-1}$. Then we have $P = \left[\begin{smallmatrix} 1 & 0 \\ d & P_0 \end{smallmatrix}\right]$ where $P_0 = QQ_1^{-1}$, $Q_1 = \left[\begin{smallmatrix} I_p & 0 \\ 0 & B_2 - I_{n-p} \end{smallmatrix}\right]$ and $d = -P_0\left[\left(\begin{smallmatrix} 0 \\ b_2 \end{smallmatrix}\right)\right]$. For $f = (A, a) \in \mathcal{G}$, we have

$$P_1^{-1}\Phi(f)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix}$$

where $A_1 \in \mathbb{T}_p^*(\mathbb{C})$ and $A_2 \in \mathcal{K}^*_{\eta'',r''}(\mathbb{C})$. Since G is abelian, $P_1^{-1}\Phi(f)\Phi(f_0)P_1 = P_1^{-1}\Phi(f_0)\Phi(f)P_1$, and therefore $A_2B_2 = B_2A_2$ and $-(A_2 - I_{n-p})b_2 + (B_2 - I_{n-p})a_2 = 0$. It follows that

$$\begin{aligned} P^{-1}\Phi(f)P &= P_2P_1^{-1}\Phi(f)P_1P_2^{-1} \\ &= P_2 \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ a_2 & 0 & A_2 \end{bmatrix} P_2^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ -(A_2 - I_{n-p})b_2 + (B_2 - I_{n-p})a_2 & 0 & A_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ a_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}. \end{aligned}$$

Therefore, $P^{-1}\Phi(f)P = \operatorname{diag}(A_1', A_2) \in \mathcal{K}^*_{\eta',r''+1}(\mathbb{C})$ where $A_1' = \left[\begin{smallmatrix} 1 & 0 \\ a_1 & A_1 \end{smallmatrix}\right] \in \mathbb{T}^*_{p+1}(\mathbb{C})$ $A_2 \in \mathcal{K}^*_{\eta'',r''}(\mathbb{C})$ and $\eta' = (p+1, n_1'', \dots, n_{r''}'')$. This completes the proof in this case.

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Case 2: $E = \{0\}$ Let $B \in \mathcal{L}_{\mathcal{G}}$ such that $(B - I_n)$ is invertible (Lemma 2.4). We let $f_0 = (B, b) \in \mathcal{G}$. By Proposition 2.6, there exists $Q \in \mathrm{GL}(n, \mathbb{C})$ such that $Q^{-1}\mathcal{L}_{\mathcal{G}}Q$ is a subgroup of $\mathcal{K}^*_{\eta',r'}(\mathbb{C})$ for some $r' \leq n$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$ where $n'_1 + \cdots + n'_{r'} = n$. By Lemma 2.5, there exists $w = (1, w_1) \in \mathrm{Fix}(G) \cap (\{1\} \times \mathbb{C}^n)$. Set $P = \left[\begin{smallmatrix} 1 & 0 \\ w_1 & Q \end{smallmatrix}\right]$. For every $f = (A, a) \in \mathcal{G}$, $\Phi(f)w = w$, so $Aw_1 + a = w_1$. Therefore

$$
\begin{aligned}
P^{-1}\Phi(f)P &= \begin{bmatrix} 1 & 0 \\ -Q^{-1}w_1 & Q^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ w_1 & Q \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ Q^{-1}(Aw_1 + a - w_1) & Q^{-1}AQ \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1}AQ \end{bmatrix}.
\end{aligned}
$$

Hence $P^{-1}\Phi(f)P \in \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$, where $r = r' + 1$ and $\eta = (1, n'_1, \dots, n'_{r'})$. This completes the proof. ∎

Lemma 2.7 ([1, Proposition 3.2]) $\exp\big(\mathcal{K}_{\eta,r}(\mathbb{C})\big) = \mathcal{K}^*_{\eta,r}(\mathbb{C})$.

Lemma 2.8 $\exp(\Psi(\mathrm{MA}(n, \mathbb{C}))) = \Phi(\mathrm{GA}(n, \mathbb{C}))$.

Proof It is clear that $\exp(\Psi(\mathrm{MA}(n, \mathbb{C}))) \subset \Phi(\mathrm{GA}(n, \mathbb{C}))$. Conversely, let $M \in \Phi(\mathrm{GA}(n, \mathbb{C}))$. By Proposition 2.1, there exists $P \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$ such that $M' = P^{-1}MP \in \mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$. By Lemma 2.7, $\exp\big(\mathcal{K}_{\eta,r}(\mathbb{C})\big) = \mathcal{K}^*_{\eta,r}(\mathbb{C})$, then $M' = e^{N'}$ for some $N' \in \mathcal{K}_{\eta,r}(\mathbb{C})$. So $N'' = PN'P^{-1} \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ and $e^{N''} = PM'P^{-1} = M \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$. By Lemma 2.9, $N = N'' - 2ik\pi I_{n+1} \in \Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$ for some $k \in \mathbb{Z}$ and N satisfies $e^N = e^{2ik\pi}e^{N''} = M$. It follows that $M \in \exp(\Psi(\mathrm{MA}(n, \mathbb{C})))$. ∎

Lemma 2.9 *If $N \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ such that $e^N \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$, then there exists $k \in \mathbb{Z}$ such that $N - 2ik\pi I_{n+1} \in \Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$.*

Proof Let $N' = P^{-1}NP \in \mathcal{K}_{\eta,r}(\mathbb{C})$, $M = e^N$ and $M' = P^{-1}MP$. We have $e^{N'} = M'$ and by Lemma 2.7, $M' \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Write $M' = \mathrm{diag}(M'_1, \dots, M'_r)$ and $N' = \mathrm{diag}(N'_1, \dots, N'_r)$, $M'_k, N'_k \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \dots, r$. Then $e^{N'} = \mathrm{diag}(e^{N'_1}, \dots, e^{N'_r})$, so $e^{N'_1} = M'_1$. As 1 is the only eigenvalue of M'_1, N'_1 has an eigenvalue $\mu \in \mathbb{C}$ such that $e^\mu = 1$. Thus $\mu = 2ik\pi$ for some $k \in \mathbb{Z}$. Therefore, $N'' = N' - 2ik\pi I_{n+1} \in \Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$ and $e^{N''} = e^{-2ik\pi}e^{N'} = M'$. It follows that $N - 2ik\pi I_{n+1} = PN''P^{-1} \in P\Psi\big(\mathrm{MA}(n, \mathbb{C})\big)P^{-1} = \Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$, since $P \in \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$. ∎

Lemma 2.10 ([1, Lemma 4.2]) *One has* $\exp(\mathrm{g}) = G$.

Corollary 2.11 *Let $G = \Phi(\mathcal{G})$. We have* $\mathrm{g} = \mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1}$.

Proof Let $N \in \mathrm{g}$. By Lemma 2.10, $exp(N) \in G \subset \Phi\big(\mathrm{GA}(n, \mathbb{C})\big)$. Then by Lemma 2.9, there exists $k \in \mathbb{Z}$ such that $N' = N - 2ik\pi I_{n+1} \in \Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$.

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As $e^{N'} = e^N \in G$ and $N' \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ then $N' \in \mathrm{g} \cap \Psi\big(\mathrm{MA}(n,\mathbb{C})\big) = \mathrm{g}^1$. Hence $\mathrm{g} \subset \mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1}$. Conversely, as $\mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1} \subset P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ and $\exp(\mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1}) = \exp(\mathrm{g}^1) \subset G$, hence $\mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1} \subset \mathrm{g}$. ■

Corollary 2.12 *We have* $\exp\big(\Psi(\mathfrak{q})\big) = \Phi(\mathcal{G})$.

Proof By Lemmas 2.10 and 2.11, we have $G = \exp(\mathrm{g}) = \exp(\mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1}) = \exp(\mathrm{g}^1)$. Since $\mathrm{g}^1 = \Psi(\mathfrak{q})$, we get $\exp\big(\Psi(\mathfrak{q})\big) = \Phi(\mathcal{G})$. ■

3 Proof of Theorem 1.1

Let $\widetilde{G}$ be the group generated by G and $\mathbb{C}^*I_{n+1} = \{\lambda I_{n+1} : \lambda \in \mathbb{C}^*\}$. Then $\widetilde{G}$ is an abelian subgroup of $\mathrm{GL}(n+1,\mathbb{C})$. By Proposition 2.1, there exists $P \in \Phi\big(\mathrm{GA}(n,\mathbb{C})\big)$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ for some $r \leq n+1$ and $\eta = (n_1,\dots,n_r) \in \mathbb{N}_0^r$, and this also implies that $P^{-1}\widetilde{G}P$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. Set $\widetilde{\mathrm{g}} = \exp^{-1}(\widetilde{G}) \cap \big(P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}\big)$ and $\widetilde{\mathrm{g}}_{v_0} = \{Bv_0 : B \in \widetilde{\mathrm{g}}\}$. Then we have the following theorem, applied to $\widetilde{G}$.

Theorem 3.1 ([1, Theorem 1.1]) *Under the notations above, the following properties are equivalent:*

(i) *$\widetilde{G}$ has a dense orbit in $\mathbb{C}^{n+1}$;*
(ii) *the orbit $\widetilde{G}(v_0)$ is dense in $\mathbb{C}^{n+1}$;*
(iii) *$\widetilde{\mathrm{g}}_{v_0}$ is an additive subgroup dense in $\mathbb{C}^{n+1}$.*

Lemma 3.2 ([1, Lemma 4.1]) *The sets g and $\widetilde{\mathrm{g}}$ are additive subgroups of $M_{n+1}(\mathbb{C})$. In particular, g_{v_0} and $\widetilde{\mathrm{g}}_{v_0}$ are additive subgroups of $\mathbb{C}^{n+1}$.*

Recall that $\mathrm{g}^1 = \mathrm{g} \cap \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$ and $\mathfrak{q} = \Psi^{-1}(\mathrm{g}^1) \subset \mathrm{MA}(n,\mathbb{C})$.

Lemma 3.3 *Under the notations above, one has*

(i) $\widetilde{\mathrm{g}} = \mathrm{g}^1 + \mathbb{C}I_{n+1}$,
(ii) $\{0\} \times \mathfrak{q}_{w_0} = \mathrm{g}^1_{v_0}$.

Proof (i) Let $B \in \widetilde{\mathrm{g}}$, then $e^B \in \widetilde{G}$. One can write $e^B = \lambda A$ for some $\lambda \in \mathbb{C}^*$ and $A \in G$. Let $\mu \in \mathbb{C}$ such that $e^\mu = \lambda$, then $e^{B-\mu I_{n+1}} = A$. Since $B - \mu I_{n+1} \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$, so $B - \mu I_{n+1} \in \exp^{-1}(G) \cap P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1} = \mathrm{g}$. By Corollary 2.11, there exists $k \in \mathbb{Z}$ such that $B' := B - \mu I_{n+1} + 2ik\pi I_{n+1} \in \mathrm{g}^1$. Then $B \in \mathrm{g}^1 + \mathbb{C}I_{n+1}$ and hence $\widetilde{\mathrm{g}} \subset \mathrm{g}^1 + \mathbb{C}I_{n+1}$. Since $\mathrm{g}^1 \subset \widetilde{\mathrm{g}}$ and $\mathbb{C}I_{n+1} \subset \widetilde{\mathrm{g}}$, it follows that $\mathrm{g}^1 + \mathbb{C}I_{n+1} \subset \widetilde{\mathrm{g}}$ (since $\widetilde{\mathrm{g}}$ is an additive group, by Lemma 3.2). This proves (i).

(ii) Since $\Psi(\mathfrak{q}) = \mathrm{g}^1$ and $v_0 = (1, w_0)$, we obtain for every $f = (B, b) \in \mathfrak{q}$,

$$\Psi(f)v_0 = \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ b + Bw_0 \end{bmatrix} = \begin{bmatrix} 0 \\ f(w_0) \end{bmatrix}.$$

Hence $\mathrm{g}^1_{v_0} = \{0\} \times \mathfrak{q}_{w_0}$. ■

Lemma 3.4 *The following assertions are equivalent:*

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(i) $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$;
(ii) $\overline{\mathfrak{g}^1_{v_0}} = \{0\} \times \mathbb{C}^n$;
(iii) $\overline{\widetilde{\mathfrak{g}}_{v_0}} = \mathbb{C}^{n+1}$.

Proof (i) $\Leftrightarrow$ (ii) follows from the fact that $\{0\} \times \mathfrak{q}_{w_0} = \mathfrak{g}^1_{v_0}$ (Lemma 3.3 (ii)).

(ii) $\Rightarrow$ (iii) By Lemma 3.3 (ii), $\widetilde{\mathfrak{g}}_{v_0} = \mathfrak{g}^1_{v_0} + \mathbb{C}v_0$. Since $v_0 = (1, w_0) \notin \{0\} \times \mathbb{C}^n$ and $\mathbb{C}I_{n+1} \subset \widetilde{\mathfrak{g}}$, we obtain $\mathbb{C}v_0 \subset \widetilde{\mathfrak{g}}_{v_0}$ and so $\mathbb{C}v_0 \subset \overline{\widetilde{\mathfrak{g}}_{v_0}}$. Therefore $\mathbb{C}^{n+1} = \{0\} \times \mathbb{C}^n \oplus \mathbb{C}v_0 = \overline{\mathfrak{g}^1_{v_0}} \oplus \mathbb{C}v_0 \subset \overline{\widetilde{\mathfrak{g}}_{v_0}}$ (since, by Lemma 3.2, $\widetilde{\mathfrak{g}}_{v_0}$ is an additive subgroup of $\mathbb{C}^{n+1}$). Thus $\overline{\widetilde{\mathfrak{g}}_{v_0}} = \mathbb{C}^{n+1}$.

(iii) $\Rightarrow$ (ii) Let $x \in \mathbb{C}^n$, then $(0, x) \in \overline{\widetilde{\mathfrak{g}}_{v_0}}$ and there exists a sequence $(A_m)_{m\in\mathbb{N}} \subset \widetilde{\mathfrak{g}}$ such that $\lim_{m\to+\infty} A_m v_0 = (0, x)$. By Lemma 3.3, we can write $A_m v_0 = \lambda_m v_0 + B_m v_0$ with $\lambda_m \in \mathbb{C}$ and $B_m = \left[\begin{smallmatrix} 0 & 0 \\ b_m & B^1_m \end{smallmatrix}\right] \in \mathfrak{g}^1$ for every $m \in \mathbb{N}$. Since $B_m v_0 \in \{0\} \times \mathbb{C}^n$ for every $m \in \mathbb{N}$, we have $A_m v_0 = (\lambda_m, b_m + B^1_m w_0 + \lambda_m w_0)$. It follows that $\lim_{m\to+\infty} \lambda_m = 0$ and $\lim_{m\to+\infty} A_m v_0 = \lim_{m\to+\infty} B_m v_0 = (0, x)$, thus $(0, x) \in \overline{\mathfrak{g}^1_{v_0}}$. Hence $\{0\} \times \mathbb{C}^n \subset \overline{\mathfrak{g}^1_{v_0}}$. Since $\mathfrak{g}^1 \subset \Psi\big(\mathrm{MA}(n, \mathbb{C})\big)$, $\mathfrak{g}^1_{v_0} \subset \{0\} \times \mathbb{C}^n$, and we conclude that $\overline{\mathfrak{g}^1_{v_0}} = \{0\} \times \mathbb{C}^n$. ∎

Lemma 3.5 *Let $x \in \mathbb{C}^n$ and $G = \Phi(\mathcal{G})$. The following are equivalent:*

(i) $\overline{\mathcal{G}(x)} = \mathbb{C}^n$;
(ii) $\overline{G(1, x)} = \{1\} \times \mathbb{C}^n$;
(iii) $\overline{\widetilde{G}(1, x)} = \mathbb{C}^{n+1}$.

Proof (i) $\Leftrightarrow$ (ii) is obvious, since $\{1\} \times \mathcal{G}(x) = G(1, x)$ by construction.

(iii) $\Rightarrow$ (ii) Let $y \in \mathbb{C}^n$ and $(B_m)_m$ a sequence in $\widetilde{G}$ with $\lim_{m\to+\infty} B_m(1, x) = (1, y)$. One can write $B_m = \lambda_m \Phi(f_m)$ with $f_m \in \mathcal{G}$ and $\lambda_m \in \mathbb{C}^*$, thus $B_m(1, x) = \big(\lambda_m, \lambda_m f_m(x)\big)$, so $\lim m \to +\infty \lambda_m = 1$. Therefore,

$$\lim_{m\to+\infty} \Phi(f_m)(1, x) = \lim_{m\to+\infty} \frac{1}{\lambda_m} B_m(1, x) = (1, y).$$

Hence, $(1, y) \in \overline{G(1, x)}$.

(ii) $\Rightarrow$ (iii) Since $\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n) = \bigcup_{\lambda\in\mathbb{C}^*} \lambda(\{1\} \times \mathbb{C}^n)$ and for every $\lambda \in \mathbb{C}^*$, $\lambda G(1, x) \subset \widetilde{G}(1, x)$, we get

$$\mathbb{C}^{n+1} = \overline{\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n)} = \overline{\bigcup_{\lambda\in\mathbb{C}^*} \lambda(\{1\} \times \mathbb{C}^n)} = \overline{\bigcup_{\lambda\in\mathbb{C}^*} \lambda \overline{G(1, x)}} \subset \overline{\widetilde{G}(1, x)}.$$

Hence $\mathbb{C}^{n+1} = \overline{\widetilde{G}(1, x)}$. ∎

Proof of Theorem 1.1 (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{G}$ is hypercyclic, so $\overline{\mathcal{G}(x)} = \mathbb{C}^n$ for some $x \in \mathbb{C}^n$. By Lemma 3.5 (iii), $\overline{\widetilde{G}(1, x)} = \mathbb{C}^{n+1}$, and by Theorem 3.1, $\overline{\widetilde{G}(v_0)} = \mathbb{C}^{n+1}$. Then by Lemma 3.5, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$, since $v_0 = (1, w_0)$.

(ii) $\Rightarrow$ (iii) Suppose that $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. By Lemma 3.5, $\overline{\widetilde{G}(v_0)} = \mathbb{C}^{n+1}$, and by Theorem 3.1, $\overline{\widetilde{\mathfrak{g}}_{v_0}} = \mathbb{C}^{n+1}$. Then by Lemma 3.4, $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$.

(iii) $\Rightarrow$ (ii) Suppose that $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$. By Lemma 3.4, $\overline{\widetilde{\mathfrak{g}}_{v_0}} = \mathbb{C}^{n+1}$, and by Theorem 3.1, $\overline{\widetilde{G}(v_0)} = \mathbb{C}^{n+1}$. Then by Lemma 3.5, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. ■

Proof of Corollary 1.2 Assume that $\mathcal{G} \subset \mathrm{GL}(n,\mathbb{C})$. Then take $P = \mathrm{diag}(1,Q)$ and $G = \Phi(\mathcal{G})$, so $P^{-1}GP \subset \mathcal{K}_{\eta,r'+1}(\mathbb{C})$ where $\eta = (1, n'_1, \dots, n'_{r'})$. Hence $u_0 = (1, u'_0)$, $v_0 = Pu_0 = (1, Qu'_0)$ and thus $w_0 = Qu'_0 = v'_0$. Every $f = (A,0) \in \mathcal{G}$ is simply noted A. Then for every $A \in \mathcal{G}$, $\Phi(A) = \mathrm{diag}(1,A)$. We can verify that $\mathfrak{g}^1 = \{\mathrm{diag}(0,B) : B \in \mathfrak{g}'\}$ where $\mathfrak{g}' = \exp^{-1}(\mathcal{G}) \cap Q\big(\mathcal{K}_{\eta',r'}(\mathbb{C})\big)Q^{-1}$, and so $\mathfrak{q} = \Psi^{-1}(\mathfrak{g}^1) = \mathfrak{g}'$. Hence the proof of Corollary 1.2 follows directly from Theorem 1.1. ■

4 Finitely Generated Subgroups

Recall the following result, proved in [1], which, applied to G, can be stated as follows.

Proposition 4.1 ([1, Proposition 8.1]) *Suppose that G is generated by $A_1, \dots, A_p$ and let $B_1, \dots, B_p \in \mathfrak{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$, and $P \in \mathrm{GL}(n+1,\mathbb{C})$ satisfying $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Then*

$$\mathfrak{g} = \sum_{k=1}^{p} \mathbb{Z}B_k + 2i\pi \sum_{k=1}^{r} \mathbb{Z}PJ_kP^{-1} \quad \text{and} \quad \mathfrak{g}_{v_0} = \sum_{k=1}^{p} \mathbb{Z}B_k v_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z}Pe^{(k)},$$

where $J_k = \mathrm{diag}(J_{k,1}, \dots, J_{k,r})$ with $J_{k,i} = 0 \in \mathbb{T}_{n_i}(\mathbb{C})$ if $i \neq k$ and $J_{k,k} = I_{n_k}$.

Proposition 4.2 *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GA}(n,\mathbb{C})$ generated by $f_1, \dots, f_p$ and let $f'_1, \dots, f'_p \in \mathfrak{q}$ such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1, \dots, p$. Let P be as in Proposition 2.1. Then*

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{Z}f'_k(w_0) + \sum_{k=2}^{r} 2i\pi\mathbb{Z}p_2(Pe^{(k)}) & \text{if } r \geq 2, \\ \sum_{k=1}^{p} \mathbb{Z}f'_k(w_0) & \text{if } r = 1. \end{cases}$$

Proof Let $G = \Phi(\mathcal{G})$. Then G is generated by $\Phi(f_1), \dots, \Phi(f_p)$. Apply Proposition 4.1 to G, $A_k = \Phi(f_k)$, $B_k = \Psi(f'_k) \in \mathfrak{g}^1$, then we have

$$\mathfrak{g} = \sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k) + 2i\pi\mathbb{Z}\sum_{k=1}^{r} PJ_kP^{-1}.$$

We have $\sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k) \subset \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$. Moreover, for every $k = 2, \dots, r$, $J_k \in \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$, hence $PJ_kP^{-1} \in \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$, since $P \in \Phi\big(\mathrm{GA}(n,\mathbb{C})\big)$. However, $mPJ_1P^{-1} \notin \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$ for every $m \in \mathbb{Z}\backslash\{0\}$, since J_1 has the form $J_1 = \mathrm{diag}(1, J')$ where $J' \in M_n(\mathbb{C})$. As $\mathfrak{g}^1 = \mathfrak{g} \cap \Psi\big(\mathrm{MA}(n,\mathbb{C})\big)$, then $mPJ_1P^{-1} \notin \mathfrak{g}^1$ for every $m \in \mathbb{Z}\backslash\{0\}$. Hence we obtain

$$\mathfrak{g}^1 = \begin{cases} \sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k) + \sum_{k=2}^{r} 2i\pi\mathbb{Z}PJ_kP^{-1} & \text{if } r \geq 2, \\ \sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k) & \text{if } r = 1. \end{cases}$$

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Since $J_k u_0 = e^{(k)}$, we get

$$\mathfrak{g}^1_{v_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k)v_0 + \sum_{k=2}^{r} 2i\pi\mathbb{Z}Pe^{(k)} & \text{if } r \geq 2, \\ \sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k)v_0 & \text{if } r = 1. \end{cases}$$

By Lemma 3.3 (iii), one has $\{0\} \times \mathfrak{q}_{w_0} = \mathfrak{g}^1_{v_0}$ and $\Psi(f'_k)v_0 = \big(0, f'_k(w_0)\big)$, so $\mathfrak{q}_{w_0} = p_2(\mathfrak{g}^1_{v_0})$. It follows that

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{Z}f'_k(w_0) + \sum_{k=2}^{r} 2i\pi\mathbb{Z}p_2(Pe^{(k)}) & \text{if } r \geq 2, \\ \sum_{k=1}^{p} \mathbb{Z}f'_k(w_0) & \text{if } r = 1. \end{cases}$$

The proof is complete. ■

Recall the following proposition, which was proved in [7].

Proposition 4.3 (*cf.* [7, p. 35]) *Let* $F = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p$ *with* $u_k = \mathrm{Re}(u_k) + i\,\mathrm{Im}(u_k)$, *where* $\mathrm{Re}(u_k), \mathrm{Im}(u_k) \in \mathbb{R}^n$, $k = 1, \ldots, p$. *Then* F *is dense in* $\mathbb{C}^n$ *if and only if for every* $(s_1, \ldots, s_p) \in \mathbb{Z}^p \backslash \{0\}$:

$$\operatorname{rank} \begin{bmatrix} \mathrm{Re}(u_1) & \cdots & \mathrm{Re}(u_p) \\ \mathrm{Im}(u_1) & \cdots & \mathrm{Im}(u_p) \\ s_1 & \cdots & s_p \end{bmatrix} = 2n + 1.$$

Proof of Theorem 1.3 This follows directly from Theorem 1.1, Propositions 4.2 and 4.3.

Proof of Corollary 1.4 First, by Proposition 4.3, if $F = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then F cannot be dense in $\mathbb{C}^n$. Now, by the form of $\mathfrak{q}_{w_0}$ in Proposition 4.2, $\mathfrak{q}_{w_0}$ cannot be dense in $\mathbb{C}^n$, and so Corollary 1.4 follows by Theorem 1.3. ■

Proof of Corollary 1.5 Since $n \leq 2n - r + 1$ (because $r \leq n + 1$), Corollary 1.5 follows from Corollary 1.4. ■

5 Example

Example 5.1 Let $\mathcal{G}$ the subgroup of $\mathrm{GA}(2, \mathbb{C})$ generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$, $f_3 = (A_3, a_3)$ and $f_4 = (A_4, a_4)$, where

$$a_1 = I_2, \qquad a_1 = (1 + i, 0),$$

$$A_2 = \mathrm{diag}(1, e^{-2+i}), \qquad a_2 = (0, 0),$$

$$A_3 = \mathrm{diag}(1, e^{\frac{-\sqrt{2}}{\pi} + i(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2})}), \qquad a_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\Big(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\Big), 0\right),$$

$$A_4 = I_2, \qquad a_4 = (2i\pi, 0).$$

Then $\mathcal{G}$ is hypercyclic.

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Proof First one can check that $\mathcal{G}$ is abelian: $f_i \circ f_j = f_j \circ f_i$ for every $i, j = 1, 2, 3, 4$. Let by $G = \Phi(\mathcal{G})$. Then G is generated by

$$\Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2+i} \end{bmatrix},$$

$$\Phi(f_3) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right) & 1 & 0 \\ 0 & 0 & e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)} \end{bmatrix}, \quad \Phi(f_4) = \begin{bmatrix} 1 & 0 & 0 \\ 2i\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $f_i' = (B_i, b_i)$, $i = 1, 2, 3, 4$ where

$$B_1 = \operatorname{diag}(0, 0) = 0, \qquad b_1 = (1 + i, 0),$$

$$B_2 = \operatorname{diag}(0, -2 + i), \qquad b_2 = (0, 0),$$

$$B_3 = \operatorname{diag}\left(0, \frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)\right), \qquad b_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right), 0\right),$$

$$B_4 = \operatorname{diag}(0, 0) = 0, \qquad b_4 = (2i\pi, 0).$$

Then we have $e^{\Psi(f_i')} = \Phi(f_i)$, $i = 1, 2, 3, 4$.

Here $r = 2$, $\eta = (2, 1)$, G is an abelian subgroup of $\mathcal{K}^*_{(2,1),2}(\mathbb{C})$. We have $P = I_3$, $\varphi = (I_2, 0)$, $u_0 = v_0 = (1, 0, 1)$, $e^{(2)} = (0, 0, 1)$ and $w_0 = (0, 1)$. By Proposition 4.2, $\mathfrak{q}_{w_0} = \sum_{k=1}^{4} \mathbb{Z} f_k'(w_0) + 2i\pi\mathbb{Z} p_2(e^{(2)})$. On the other hand, for every $(s_1, s_2, s_3, s_4, t_2) \in \mathbb{Z}^5 \backslash \{0\}$, write

$$M_{(s_1,s_2,s_3,s_4,t_2)} = \begin{bmatrix} \operatorname{Re}(B_1 w_0 + b_1) & \operatorname{Re}(B_2 w_0 + b_2) & \operatorname{Re}(B_3 w_0 + b_3) & \operatorname{Re}(B_4 w_0 + b_4) & 0 \\ \operatorname{Im}(B_1 w_0 + b_1) & \operatorname{Im}(B_2 w_0 + b_2) & \operatorname{Im}(B_3 w_0 + b_3) & \operatorname{Im}(B_4 w_0 + b_4) & 2\pi e^{(2)} \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{bmatrix}.$$

Then the determinant:

$$\Delta = \det(M_{(s_1,s_2,s_3,s_4,t_2)}) = \begin{vmatrix} 1 & 0 & -\frac{\sqrt{3}}{2\pi} & 0 & 0 \\ 0 & -2 & -\frac{\sqrt{2}}{\pi} & 0 & 0 \\ 1 & 0 & \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 2\pi & 0 \\ 0 & 1 & \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 2\pi \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{vmatrix}$$
$$= 2\pi(-s_1\sqrt{3} + 2s_2\sqrt{2} - 4s_3\pi + s_4\sqrt{5} - t_2\sqrt{7}).$$

Since π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are rationally independent, $\Delta \neq 0$ for every $(s_1, s_2, s_3, s_4, t_2) \in \mathbb{Z}^5 \backslash \{0\}$. It follows that $\operatorname{rank}(M_{(s_1,s_2,s_3,s_4,t_2)}) = 5$. Hence $f_1, \dots, f_4$ satisfy the property $\mathcal{D}$. By Theorem 1.3, $\mathcal{G}$ is hypercyclic. ■

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References

[1] A. Ayadi and H. Marzougui, *Dense orbits for abelian subgroups of* $\mathrm{GL}(n, C)$. In: Foliations 2005, World Sci. Publ., Hackensack, NJ, 2006, 47–69.
[2] A. Ayadi, H. Marzougui and Y. N'dao, *On the dynamic of abelian groups of affine maps on* $\mathbb{C}^n$ *and* $\mathbb{R}^n$. Preprint, ICTP, IC /2009/062, 2009.
[3] F. Bayart and E. Matheron, *Dynamics of Linear Operators.* Cambridge Tracts in Math. **179**, Cambridge University Press, 2009.
[4] M. Javaheri, *A generalization of Dirichlet approximation theorem for the affine actions on real line.* J. Number Theory **128**(2008), 1146–1156. http://dx.doi.org/10.1016/j.jnt.2007.08.008
[5] R. S. Kulkarni, *Dynamics of linear and affine maps.* Asian J. Math. **12**(2008), 321–344.
[6] V. Bergelson, M. Misiurewicz and S. Senti, *Affine actions of a free semigroup on the real line*. Ergodic Theory Dynam. Systems **26**(2006), 1285–1305. http://dx.doi.org/10.1017/S014338570600037X
[7] M. Waldschmidt, *Topologie des points rationnels*. Cours de Troisième Cycle, Université P. et M. Curie (Paris VI), 1994/95.

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