A LIMIT THEOREM FOR MARKOV CHAINS WITH CONTINUOUS STATE SPACE

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1. Introduction

Let R denote the set of real numbers, B the σ-field of all Borel subsets of R. A homogeneous Markov Chain with state space a Borel subset Ω of R is a sequence $\{a_n\}$, $n \geqq 0$, of random variables, taking values in Ω, with one-step transition probabilities $P^{(1)}(\xi, A)$ defined by

$$(1.1) \quad P^1(\xi, A) = \Pr\{a_{n+1} \in A | a_n = \xi, a_{n-1} = \xi_{n-1}, \cdots a_0 = \xi_0\} \quad n \geqq 0$$

for each choice of $\xi, \xi_0, \cdots, \xi_{n-1}$ in Ω and all Borel subsets A of Ω. The fact that the right-hand side of (1.1) does not depend on the ξ_i, $0 \leqq i < n$, is of course the Markovian property, the non-dependence on n is the homogeneity of the chain.

The one-step transition probabilities are assumed to be such that for fixed ξ, $P^{(1)}(\xi, A)$ is a probability measure over the σ-field of Borel subsets of Ω and for fixed A, a Borel measurable function of ξ. The n-step transition probabilities are defined by

$$P^{(n)}(\xi, A) = \Pr\{a_{n+j} \in A | a_j = \xi\}$$

and satisfy

$$(1.2) \qquad P^{(i+j)}(\xi, A) = \int_\Omega P^{(j)}(\eta, A) P^{(i)}(\xi, \mathrm{d}\eta)$$

We suppose also that there is given an initial probability measure $P^{(0)}(A) = \Pr\{a_0 \in A\}$. If $P^{(n)}(A) = \Pr\{a_n \in A\}$ then

$$(1.3) \qquad P^{(n)}(A) = \int_\Omega P^{(n)}(\xi, A) P^{(0)}(d\xi)$$

Throughout this paper we write "set" for "Borel subset of Ω". The Borel σ-field over Ω will be denoted by $B\Omega$.

Theorem 1 to be proved in the next section remains valid when the state space Ω is the set R but the interesting applications seem to be when Ω is a compact set in which there is a particular point w, usually a boundary point, and an integer $r \geqq 1$, such that G.L.B. $P^{(r)}(\xi, \{w\}) > 0$. Here $\{w\}$ denotes the set containing the single point w.

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The proof of Theorem 1 is based on ideas which go back to Markov's method of proving ergodicity for a Markov Cahin with finite number of states, Doob [1]. A similar argument is used by Doob [1] to establish a limit theorem based on a condition on the one-step transition probability density.

2. The limit theorem

THEOREM 1. *If $\{a_n\}$ is a homogeneous Markov Chain with state space Ω then either* (1), *for each $r \geqq 1$ and arbitrarily small $\varepsilon > 0$ there is at least one pair of points $\xi, \eta \in \Omega$ and a set S, depending on ξ and η, such that*

$$P^{(r)}(\xi, S) > P^{(r)}(\eta, S)+1-\varepsilon. \tag{2.1}$$

The set S is such that

$$\Phi^{(r)}(\xi, \eta, A) = P^{(r)}(\xi, A)-P^{(r)}(\eta, A) \tag{2.2}$$

is non-negative on every subset of S and negative on every subset of $\bar{S}$ the complement of S in Ω.

Or (2), *there is an integer $r \geqq 1$, a constant δ, and a probability measure $P(A)$ defined over $B\Omega$ such that*

$$|P^{(n)}(\xi, A)-P(A)| \leqq \delta^{(n/r)-1} < 1, \text{ for all } \xi \in \Omega, \tag{2.3}$$

all $A \in B\Omega$.
and

$$P(A) = \int_\Omega P^{(n)}(\xi, A)P(d\xi), \qquad n \geqq 1. \tag{2.4}$$

REMARK. The disjunction of the theorem is meant in the sense that if the first alternative is false then the second alternative is true. It is possible, however, that both alternatives are true.

PROOF. Write

$$M^{(n)}(A) = \underset{\xi\in\Omega}{\text{L.U.B.}}\ P^{(n)}(\xi, A),\ m^{(n)}(A) = \underset{\xi\in\Omega}{\text{G.L.B.}}\ P^{(n)}(\xi, A)$$

for $A \in B\Omega$. It follows from (1.2) that for fixed A, $M^{(n)}(A)$ is a monotonic non-increasing sequence. Similarly, for fixed A, $m^{(n)}(A)$ is a monotonic non-decreasing sequence. Thus for any integers, n, $r \geqq 1$,

$$\begin{aligned} 0 &\leqq M^{(n+r)}(A)-m^{(n+r)}(A) \\ &\leqq \underset{\xi,\eta}{\text{L.U.B.}} \int_\Omega P^{(n)}(\zeta, A)\{P^{(r)}(\xi, d\zeta)-P^{(r)}(\eta, d\zeta)\} \end{aligned} \tag{2.5}$$

$\Phi^{(r)}(\xi, \eta, A)$ defined by (2.2) is a completely additive set function over

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$B\Omega$ and there is a set S, depending on ξ and η on which $\Phi^{(r)}(\xi, \eta, A)$ is a maximum. This set has the properties stated in the theorem. It follows from (2.5) that

$$(2.6) \qquad 0 \leqq M^{(n+r)}(A) - m^{(n+r)}(A) \leqq \{M^{(n)}(A) - m^{n}(A)\} \operatorname*{L.U.B.}_{\xi, \eta} \Phi^{(r)}(\xi, \eta, S)$$

Either

$$(2.7) \qquad \operatorname*{L.U.B.}_{\xi, \eta} \Phi^{(r)}(\xi, \eta, S) = 1$$

for all $r \geqq 1$ in which case the first alternative of the theorem is true, or there is some $r \geqq 1$ such that

$$(2.8) \qquad 0 < \operatorname*{L.U.B.}_{\xi, \eta} \Phi^{(r)}(\xi, \eta, S) = \delta < 1.$$

When (2.8) is true it follows from (2.6) that

$$|M^{(nr)}(A) - m^{(nr)}(A)| \leqq \delta^{n}$$

Thus $P(A) = \lim_{n\to\infty} M^{(n)}(A) = \lim_{n\to\infty} m^{(n)}(A)$ exists and (2.3) follows from

$$|P^{(n)}(\xi, A) - P(A)| \leqq |M^{(n)}(A) - m^{(n)}(A)| \leqq \delta^{(n/r)-1}$$

$P(\Omega) = 1$, $P(A) \geqq 0$ and $P(A)$ is the uniform limit of completely additive set functions, thus $P(A)$ is a probability measure. Equation (2.4) follows from (1.2).

We mention the following corollary which is easily proved.

COROLLARY. *When* (2.3) *is true* $\lim_{n\to\infty} P^{(n)}(A) = P(A)$, *further*

$$|P^{(n)}(A) - P(A)| \leqq \delta^{(n/r)-1}$$

$P(A)$ *does not depend on the initial distribution* $P^{0}(A)$ *and is the unique probability measure solution to the integral equation*

$$P(A) = \int_{\Omega} P^{(1)}(\xi, A) P(d\xi).$$

We deduce from theorem 1 the following theorem which is useful in applications.

THEOREM 2. *Let* a_n *be a homogeneous Markov Chain with state space* $\Omega \in B$. *If there is an integer* $r \geqq 1$ *and a point* $\{w\}$ *such that*

$$(2.9) \qquad \operatorname*{G.L.B.}_{\xi \in \Omega} P^{(r)}(\xi, \{w\}) > 0$$

then there exists a constant δ, *and a probability measure* $P(A)$ *over* $B\Omega$ *such that*

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(2.10) $\quad |P^{(n)}(\xi, A)-P(A)| \leqq \delta^{(n/r)-1}$ *for all* $\xi \in \Omega$, *all* $A \in B\Omega$.

(2.11) $$P(A) = \int_\Omega P^{(n)}(\xi, A)P(d\xi), \qquad n \geqq 1$$

$P(A)$ *is the unique probability measure solution of* (2.11) *with* $n = 1$.

PROOF. We have only to show that the first alternative of theorem 1 cannot occur. To do so choose $\varepsilon <$ G.L.B.$_{\xi\in\Omega} P^{(r)}(\xi, \{w\})$. Equation (2.1) implies that $w \in S$. Thus $P^{(r)}(\eta, S) \geqq P^{(r)}(\eta, \{w\}) > \varepsilon$. But (2.1) implies that $P^{(r)}(\eta, S) \leqq \varepsilon$. This contradiction establishes that the second alternative of Theorem 1 is true and this proves Theorem 2.

3. Applications

We prove

THEOREM 3. *Define a sequence* $\{w_n\}$ *of random variables by*

(3.1) $$w_{n+1} = \begin{cases} 0 & \text{if } \; w_n+u_n \leqq 0 \\ w_n + u_n & \text{if } \; 0 < w_n+u_n \leqq W \\ W & \text{if } \; W < w_n+u_n \end{cases}$$

where $\{u_n\}$ *is a sequence of independently and identically distributed random variables with common distribution function (d.f.)*$G(x)$. *Write* $F_n(x) = \Pr\{w_n \leqq x\}$ *then* $F(x) = \lim_{n\to\infty} F_n(x)$ *exists and is a d.f., it is independent of the initial distribution* $F_0(x)$ *and is the unique (d.f.) solution of the following integral equation*

(3.2) $$F(x) = \begin{cases} 0 & x < 0 \\ \int_{-\infty}^{x} F(x-y)dG(y) & 0 \leqq x < W \\ 1 & W \leqq x. \end{cases}$$

This theorem is proved in Finch [2] by a generalisation of an argument due to Lindley [5]. Equations (3.1) occur in certain storage problems, Finch [2], and in certain queueing systems with customer impatience, Finch [3], [4]. The equations (3.1) can be regarded as a random walk with independent steps on the closed interval $(0, W)$.

PROOF. The sequence of random variables $\{w_n\}$ is a homogeneous Markov Chain with state space the closed interval $(0, W)$ In order to prove Theorem 3 it is sufficient to verify that (2.9) is true. If the random variables u_n take on only nonpositive values the theorem is trivial, we can assume therefore that there is an integer $r \geqq 1$ such that $\Pr\{u \geqq r^{-1}W\} > 0$. It follows from (3.1) that

$$\Pr\{w_{n+r} = 1 | w_n = \xi\} \geqq \Pr\{u_{n+j} \geqq r^{-1}W, j = 1, 2, \cdots, r\} > 0.$$

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Thus G.L.B.$_\xi P^{(r)}(\xi, \{1\}) > 0$ and this proves the theorem.

This proof of Theorem 3 gives more than the proof in Finch (2) since by means of Theorem 2. We have established not only ergodicity but geometric ergodicity. Thus there is a constant δ such that

$$|F_n(x)-F(x)| \leqq \delta^{(n/r)-1} < 1.$$

In many cases of practical interest it is possible to obtain an upper bound for δ means of Theorem 1, namely

$$\delta \leqq \underset{\xi,\eta}{\text{L.U.B.}}\ \Phi^{(r)}(\xi, \eta, (S)). \tag{3.3}$$

For example, if the d.f. $G(x)$ is given by

$$G(x) = \begin{cases} \mu(\lambda+\mu)^{-1}e^{\lambda x}, & x \leqq 0 \\ 1-\lambda(\lambda+\mu)^{-1}e^{-\mu x}, & x > 0 \end{cases} \tag{3.4}$$

the random variables u_n are of the form (s_n-t_n) where $\{s_n\}$, $\{t_n\}$ are mutually independent sequences of independently and identically distributed non-negative random variables with

$$\Pr\{s_n \leqq x\} = 1-e^{-\mu x}, x \geqq 0 \quad \text{and} \quad \Pr\{t_n \leqq x\} = 1-e^{-\lambda x}, x \geqq 0.$$

This case occurs in the single server queueing system with Poisson arrivals at a rate λ, exponential service times and customers departing impatiently if they wait as long as the fixed time W. It is shown in Finch (3) that

$$F(x) = \begin{cases} [1-\lambda\mu^{-1}e^{-(\mu-\lambda)x}][1-\lambda^2\mu^{-2}e^{-(\mu-\lambda)W}]^{-1}, & \lambda \neq \mu, \quad 0 \leqq x < W \\ (1+\mu x)(2+\mu W)^{-1}, & \lambda = \mu, \quad 0 \leqq x < W. \end{cases}$$

We shall now prove that

$$|F_n(x)-F(x)| \leqq [1-\exp\{-K\rho(1+\rho)^{-1}\}]^{n-1} \tag{3.5}$$

where $\rho = \lambda/\mu$ is the traffic intensity and $K = \mu W$ is the maximum waiting time expressed in units of mean service time.

To prove (3.5) we note first that from (3.4) $\Pr\{u_n > W\} > 0$ so that we can take $r = 1$ in (3.3). We have to find the set S on which $P^{(1)}(\xi, A)-P^{(1)}(\eta, A)$ is a maximum and to take the least upper bound over these maxima for ξ, η varying in $(0, W)$.

It is easily verified that

$$P^{(1)}(\xi, x) = \Pr\{w_{n+1} \leqq x | w_n = \xi\} = G(x-\xi), \quad 0 \leqq \xi \leqq w, \quad 0 \leqq x < W \tag{3.6}$$

and that

$$\int_{0-}^{0+} [dP^{(1)}(\xi, x)-dP^{(1)}(\eta, x)] = G(-\xi)-G(-\eta)$$

$$\int_{W-}^{W+} [dP^{(1)}(\xi, x)-dP^{(1)}(\eta, x)] = G(W-\xi)-G(W-\eta)$$

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Thus the set S must contain the point 0 if $\xi < \eta$ and the point W if $\xi > \eta$. For $0 < x < W$, the set S consists of these points x for which $g(x-\xi) \geqq g(x-\eta)$, where $g(x) = G'(x)$ is obtained from (3.4). It will be found that

$$S = \begin{cases} [0, (\mu\xi+\lambda\eta)(\lambda+\mu)^{-1}] & \text{if } \xi < \eta \\ [(\lambda\xi+\mu\eta)(\lambda+\mu)^{-1}, W] & \text{if } \xi > \eta \end{cases}$$

and that

$$P^{(1)}(\xi, S)-P^{(1)}(\eta, S) = 1-\exp-\{\lambda\mu(\lambda+\mu)^{-1}|\xi-\eta|\} \tag{3.7}$$

The least upper bound of the expression (3.7) is obtained by taking $|\xi-\eta| = W$, thus giving (3.5).

As a second example of the use of theorems 1 and 2 we consider the problem of the finite dam in discrete time. At time n let the content of a dam of capacity K be Z_n. In the time interval $(n, n+1)$ an amount $\text{Min}(X_n, K-Z_n)$ is put into the dam where (X_n) is a sequence of independently and identically distributed non-negative random variables. Just before the instant $(n+1)$ an amount Y_n or $\text{Min}(K, X_n+Z_n)$, whichever is the smaller, is released from the dam, where (Y_n) is a sequence of independently and identically distributed non-negative random variables distributed independently of the sequence (X_n). Then

$$Z_{n+1} = \text{Max}[0, \text{Min}(K, Z_n+X_n)-Y_n]. \tag{3.8}$$

The sequence (Z_n) is a homogeneous Markov Chain with state space the interval $(0, K-M)$ where M is the greatest lower bound with probability one of the sequence (Y_n). In order to prove the ergodicity of this chain it is sufficient, by Theorem 1, to prove that there is a positive probability of emptiness in exactly r steps, for some integer $r \geqq 1$, independently of the initial level. Alternatively it is sufficient to prove that there is a positive probability of overflow in a finite number of steps independently of the initial level. We shall not consider the general case here but will confine ourselves to a model due to Moran [6] in which the release quantity $Y_n = M$ a constant. There is a probability concentration at the point $(K-M)$ and if we assume that $\Pr(X_n \geqq K) > 0$ then $\text{G.L.B.}_\xi\, P^{(1)}(\xi, \{K-M\}) > 0$ since

$$P^{(1)}(\xi, \{K-M\}) = \Pr\{Z_{n+1} = K-M | Z_n = \xi\} = \Pr\{X_n \geqq K-\xi\}.$$

The geometric ergodicity of the cain follows from Theorem 2.

We examine now the particular case in which the random variables X_n have an exponential distribution $\Pr(X_n \leqq x) = 1-e^{-\lambda x}$, $x \geqq 0$. The limiting distribution of dam content for this case has been obtained by Moran [7]. Write $P_n(x) = \Pr\{Z_n \leqq x]$, $P(x) = \lim_{n\to\infty} P_n(x)$, we prove

$$|P_n(x)-P(x)| \leqq [1-\exp\{-\lambda(K-M)\}]^{n-1}. \tag{2.9}$$

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The state space is the closed interval $(0, K-M)$, the one-step transition probability has a probability concentration at the point $(K-M)$, namely

$$P^{(1)}(\xi, \{K-M\}) = e^{-\lambda(K-\xi)}, \quad 0 \leqq \xi \leqq K-M \tag{3.10}$$

For $0 \leqq \xi < M$ there is also a probability concentration at the point $\{0\}$, namely

$$P^{(1)}(\xi, \{0\}) = 1-e^{-\lambda(M-\xi)}, \quad 0 \leqq \xi \leqq M \tag{3.11}$$

For $0 < x < (K-M)$ the one-step transition probabilities have a density given by

$$p^{(1)}(\xi, x) = \begin{cases} \lambda e^{-\lambda(x+M-\xi)} & \text{if} \quad (\xi-M)^+ < x < K-M \\ 0 & \qquad\qquad 0 < x < \xi-M \end{cases}$$

where $a^+ = \operatorname{Max}(0, a)$.

The set S on which $P^{(1)}(\xi, A)-P^{(1)}(\eta, A)$ is a maximum is given by $[(\xi-M)^+, (K-M)]$ if $\xi > \eta$ and by $[(\xi-M)^+, (\eta-M)^+]$ if $\xi < \eta$. Thus

$$\underset{\xi,\eta}{\text{L.U.B.}}\ P^{(1)}(\xi, S)-P^{(1)}(\eta, S) = \underset{\xi,\eta}{\text{L.U.B.}}\ \{1-e^{-\lambda|\xi-\eta|}\}. \tag{3.12}$$

The least upper bound is attained when $|\xi-\eta| = K-M$ and this proves (3.9).

4. Concluding remarks

The applications of theorem 2 have been to Markov Chains with an interval state space with a probability concentration at some point of the space in the sense of (2.9). It is worthwhile noting that we can ensure that the first alternative of Theorem 1 cannot occur if (1) the space Ω is compact, (2) the bivariate function $\Phi^r(\xi, \eta, S)$, given by (2.2) where S is the set occurring in Theorem 1, is a continuous function of ξ, η in the product space $\Omega \times \Omega$ and (3) the chain is such that if there is an integer $r \geqq 1$, a point $w_1 \in \Omega$ and a set $A \in B\Omega$ such that $P^{(r)}(w_1, A) = 1$ then $P^{(r)}(w, A) > 0$ for all $\omega \in \Omega$.

To prove that the first alternative of Theorem 1 cannot occur under these conditions we note that if it does occur then (2.7) holds. By conditions (1) and (2) above the least upper bound must be attained for points $\xi, \eta \in \Omega$. This implies the existence of a set S and points ξ, η for which $P^{(r)}(\xi, S) = 1$ and $P^{(r)}(\eta, S) = 0$ thus contradicting condition (3). In fact the ergodicity of the examples considered in section 3 could have been proved in this way, for from (3.7) and (3.12) we see that condition (2) is satisfied and it is not difficult to establish condition (3). However the approach through Theorem 2 seems more natural and intuitive for these examples.

Finally we remark that although we have formulated Theorem 1 in terms of a particular state space, namely a Borel subset of the set of real numbers, it is possible to formulate the theorem in terms of an abstract state space.

In particular, the theorem can be applied to an m-dependent Markov Process, in an obvious way, by regarding the random variable a_n as an m-tuple $(\alpha_n, \alpha_{n+1}, \cdots, \alpha_{n+m-1})$ in R^m.

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