ON GENERALISED LAH-NUMBERS

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1. Introduction

In (1) numbers related to the Stirling numbers are defined. Later in (2) these numbers were called Lah-numbers (cf. 2, p. 43, Ex. 16). According to (1) these numbers are of importance in Mathematical Statistics. In this paper we shall generalise the method and apply it to generalised Stirling numbers as defined in (3).

In (3) the polynomials Q and the numbers A_n^s and B_n^s were defined by the relations

$$Q(x;\ M,\ N,\ n) = Q(x,\ n) = \prod_{m=1}^{n} [M(m)+N(m)x] = \sum_{s=0}^{n} A_n^s x^s, \quad n \geqq 1,$$

$$Q(x,\ 0) = M(0), \tag{1}$$

$$x^n = \sum_{m=0}^{n} B_n^m Q(x,\ m),$$

$$1 = B_0^0 Q(x,\ 0) = B_0^0 M(0), \tag{2}$$

where M and N are two functions such that $M(0) \neq 0$, $M(n)$ is defined for all positive integers n, and $N(n)$ is defined for all positive integers n and $N(n) \neq 0$. The numbers A_n^s and B_n^s are called generalised Stirling numbers.

2. Relationship between different Q-polynomials

Let $Q_1(x,\ n) = Q(x;\ M_1,\ N_1,\ n)$ and $Q_2(x,\ n) = Q(x;\ M_2,\ N_2,\ n)$, then

$$Q_1(x,\ n) = \sum_{s=0}^{n} A_{1,n}^s x^s, \quad x^n = \sum_{m=0}^{n} B_{1,n}^m\, Q_1(x,\ m), \tag{3}$$

$$Q_1(x,\ 0) = M_1(0), \quad 1 = B_{1,0}^0 Q_1(x,\ 0) = B_{1,0}^0 M_1(0), \tag{3a}$$

and

$$Q_2(x,\ n) = \sum_{s=0}^{n} A_{2,n}^s x^s, \quad x^n = \sum_{m=0}^{n} B_{2,n}^m Q_2(x,\ m), \tag{4}$$

$$Q_2(x,\ 0) = M_2(0), \quad 1 = B_{2,0}^0 Q_2(x,\ 0) = B_{2,0}^0 M_2(0), \tag{4a}$$

where A_n^m and B_n^m are zero for $m>n$, $m<0$, $n<0$.

Let us express $Q_2(x,\ n)$ in terms of $Q_1(x,\ m)$. We can write

$$Q_2(x,\ n) = \sum_{s=0}^{n} A_{2,n}^s x^s = \sum_{s=0}^{n} A_{2,n}^s \left[\sum_{m=0}^{s} B_{1,s}^m Q_1(x,\ m)\right],$$

$$Q_2(x,\ n) = \sum_{s=0}^{n} \sum_{m=0}^{s} A_{2,n}^s B_{1,s}^m Q_1(x,\ m). \tag{5}$$

Considering the conditions on the numbers A and B we can extend the second summation to n and then change the order of summation, thus,

$$Q_2(x, n) = \sum_{s=0}^{n} \sum_{m=0}^{n} A^s_{2,n} B^m_{1,s} Q_1(x, m) = \sum_{m=0}^{n} Q_1(x, m) \sum_{s=0}^{n} A^s_{2,n} B^m_{1,s}, \tag{6}$$

and, using again the conditions on the A and B numbers, we have,

$$Q_2(x, n) = \sum_{m=0}^{n} Q_1(x, m) \sum_{s=m}^{n} A^s_{2,n} B^m_{1,s}. \tag{7}$$

Let

$$\sum_{s=m}^{n} A^s_{2,n} B^m_{1,s} = L^m_{2,1,n}, \tag{8}$$

where $L^m_{2,1,n} = 0$ for $m<0$, $n<0$, $m>n$.

Similarly, if we express $Q_1(x, n)$ in terms of Q_2-polynomials we obtain

$$Q_1(x, n) = \sum_{m=0}^{n} Q_2(x, m) \sum_{s=m}^{n} A^s_{1,n} B^m_{2,s}, \tag{9}$$

with

$$\sum_{s=m}^{n} A^s_{1,n} B^m_{2,s} = L^m_{1,2,n}, \tag{10}$$

and where $L^m_{1,2,n} = 0$, for $n<0$, $m<0$, $n<m$. It is clear that in the special case where $Q_1 = Q_2$, $L^m_{1,2,n} = L^m_{2,1,n} = \delta^m_n$, where δ^m_n is the Kronecker delta.

3. Quasi-orthogonality of the L-numbers

For obvious reasons we shall call the L-numbers generalised Lah-numbers (cf. Ex. 2 hereafter). Using (7) and (8) we can write

$$Q_2(x, n) = \sum_{m=0}^{n} Q_1(x, m) L^m_{2,1,n}, \tag{11}$$

and using (9) and (10)

$$Q_1(x, n) = \sum_{m=0}^{n} Q_2(x, m) L^m_{1,2,n}. \tag{12}$$

Substituting (12) into (11) we obtain

$$Q_2(x, n) = \sum_{m=0}^{n} L^m_{2,1,n} \sum_{s=0}^{n} L^s_{1,2,m} Q_2(x, s),$$

and using the conditions on the L-numbers we obtain

$$Q_2(x, n) = \sum_{s=0}^{n} Q_2(x, s) \sum_{m=s}^{n} L^m_{2,1,n} L^s_{1,2,m},$$

thus

$$\sum_{m=s}^{n} L^m_{2,1,n} L^s_{1,2,m} = \delta^s_n. \tag{13}$$

(13) expresses the quasi-orthogonality of L-numbers for reversed indices.

4. Recurrence relations

According to (1) we can write

$$Q_1(x, n+1) = \sum_{m=0}^{n+1} A^m_{1,n+1}x^m = [M_1(n+1)+N_1(n+1)x] \sum_{m=0}^{n} A^m_{1,n}x^m,$$

and,

$$Q_2(x, n+1) = \sum_{m=0}^{n+1} A^m_{2,n+1}x^m = [M_2(n+1)+N_2(n+1)x] \sum_{m=0}^{n} A^m_{2,n}x^m,$$

which according to (11) yields

$$Q_2(x, n+1) = \sum_{m=0}^{n+1} Q_1(x, m)L^m_{2,1,n+1} = [M_2(n+1)+N_2(n+1)x] \sum_{m=0}^{n} Q_1(x, m)L^m_{2,1,n}. \tag{14}$$

But

$$Q_1(x, m+1) = [M_1(m+1)+N_1(m+1)x]Q_1(x, m),$$

so that

$$xQ_1(x, m) = [Q_1(x, m+1)-M_1(m+1)Q_1(x, m)]/N_1(m+1). \tag{15}$$

Substituting (15) into (14) we obtain

$$\sum_{m=0}^{n+1} Q_1(x, m)L^m_{2,1,n+1} = M_2(n+1) \sum_{m=0}^{n} Q_1(x, m)L^m_{2,1,n}$$
$$+N_2(n+1) \sum_{m=0}^{n} [Q_1(x, m+1)-M_1(m+1)Q_1(x, m)]L^m_{2,1,n}/N_1(m+1).$$

By equating the coefficients of $Q_1(x, m)$ we obtain

$$L^m_{2,1,n+1} = M_2(n+1)L^m_{2,1,n} + \frac{N_2(n+1)}{N_1(m)} L^{m-1}_{2,1,n} - \frac{N_2(n+1)M_1(m+1)}{N_1(m+1)} L^m_{2,1,n},$$

or,

$$L^m_{2,1,n} = \left[M_2(n) - \frac{N_2(n)M_1(m+1)}{N_1(m+1)}\right] L^m_{2,1,n-1} + \frac{N_2(n)}{N_1(m)} L^{m-1}_{2,1,n-1} \tag{16}$$

and by inverting the indices 2 and 1,

$$L^m_{1,2,n} = \left[M_1(n) - \frac{N_1(n)M_2(m+1)}{N_2(m+1)}\right] L^m_{1,2,n-1} + \frac{N_1(n)}{N_2(m)} L^{m-1}_{1,2,n-1}. \tag{17}$$

5. Examples

1. Let $M_1(\alpha) = \alpha$, $N_1(\alpha) = \alpha-1$, $M_2(\alpha) = \alpha+1$, $N_2(\alpha) = 1/\alpha$. We obtain, taking $L^1_{2,1,1} = L^1_{1,2,1} = 1$, and $\alpha>0$,

$$L^m_{2,1,n} = \left[(n+1) - \frac{m+1}{nm}\right] L^m_{2,1,n-1} + \frac{1}{n(m-1)} L^{m-1}_{2,1,n-1},$$

$$L^m_{1,2,n} = [n-(n-1)(m+1)(m+2)]L^m_{1,2,n-1} + m(n-1)L^{m-1}_{1,2,n-1},$$

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and the following numerical values:

$L^m_{2,1,n}$: $m =$	1	2	3	4
n:				
1	1			
2	2	1/2		
3	20/3	29/12	1/12	
4	30	1233/96	199/288	1/144

$L^m_{1,2,n}$: $m =$	1	2	3	4
n:				
1	1			
2	-4	2		
3	36	-58	12	
4	-504	2072	-1194	144

2. Let $M_1(\alpha) = 1-\alpha$, $N_1(\alpha) = 1$, $M_2(\alpha) = \alpha-1$, $N_2(\alpha) = 1$, then the numbers obtained are the Lah-numbers as studied in (**1**).

REFERENCES

(1) I. Lah, Eine neue Art von Zahlen, ihre Eigenschaften und Anwendungen in der mathematischen Statistik, *Mitteilungsblatt fuer Mathematische Statistik*, **7** (1955), 203-212.

(2) J. Riordan, *An Introduction to Combinatorial Analysis* (New York, 1958).

(3) S. Tauber, On quasi-orthogonal numbers, *Amer. Math. Monthly*, **69** (1962), 365-372.

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