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THEORETICAL PEARL

Yet yet a counterexample for λ+SP

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In 1979, Klop (1980), answering a question raised by Mann in 1972, showed that the extension of λ-calculus with subjective pairing is not confluent. We refer to Klop (1980) and Barendregt (1981, revised 1984) for a perspective. The term presented by Klop to provide a counterexample is fairly simple, but the proof of non-confluence, although intuitively quite simple, involves some technical properties. Among others, a suitable standardization result on derivations in the extended system is needed in the proof. Klop's proof was revisited by Bunder (1985), who seemingly uses less technical apparatus than Klop, starting with the same term as Klop. Although Bunder's proof does not explicitly use a standardization result, his proof proceeds internally with some rearrangements of derivations, so that it is fair to say that some standardization technique is present in Bunder (1985).

In 1986, Hardin (1989) proposed a different example (clearly inspired by Klop's original example though) leading to a proof of non-confluence with less technical apparatus (in particular, it does not need any standardization). But her counterexample, unlike Klop's counterexample, cannot be typed in a simply typed λ-calculus extended with a recursion operator (and surjective pairing). We present here yet another proof, based on a third term, which is also close to Klop's original term, and which like Klop's counterexample can be typed. The proof of non-confluence does not involve any special technical apparatus, and is simple and short.

We recall that (one of the versions of) λ-calculus with subjective pairing is the following extension of λ-calculus with β:

1 There are three constants: D, F and S.
2 The following rules are added:

(Fst)	$F(Dxy)$	$\rightarrow$	x
(Snd)	$S(Dxy)$	$\rightarrow$	y
(SP)	$D(Fx)(Sx)$	$\rightarrow$	x

The rules (Fst) and (Snd) are optional, i.e., non-confluence holds with or without them. The non-confluence of the larger system implies the non-confluence of the smaller, because the two reductions from B to A and from B to CA, considered below, live in the smaller system. Thus we keep going on with the system consisting of β, (Fst), (Snd) and (SP). The last rule is called the subjective pairing rule.

Recall Turing's fixpoint combinator:

$$Y =_{def} \Theta\Theta \quad \text{where} \quad \Theta =_{def} \lambda xy.y((xx)y)$$

The term providing the counterexample of this note is:

$$B =_{def} YC \text{ where } C =_{def} YV \text{ where } V =_{def} \lambda xy.D(F(Ey))(S(E(xy)))$$

where E is some free variable. For comparison, the term proposed by Klop (1980) was:

$$B =_{def} YC \text{ where } C =_{def} YV \text{ where } V =_{def} \lambda xy.E(D(Fy)(S(xy)))$$

One first observes, for any M:

$$CM \to^{\star} VCM \to^{\star} D((F(EM))(S(E(CM))) \tag{1}$$

One has thus, writing $E(CB) = A$:

$$\begin{aligned} B &\to^{\star} CB \\ &\to^{\star} D((F(EB))(S(E(CB))) \\ &\to^{\star} D((F(E(CB)))(S(E(CB))) \\ &\to A \end{aligned}$$

and

$$\begin{aligned} B &\to^{\star} CB \\ &\to^{\star} CA \quad (\text{since } B \to^{\star} A) \end{aligned}$$

We show that A and CA cannot have a common reduct, thus that the reduction is not confluent. We prove this by contradiction. Any term L can be written uniquely as $L = E(E(\ldots(EL')\ldots))$, where L' is not of the form EL''. We say that those E's are L's head E's. Take a common reduct K of A and CA which has a minimum number of head E's. We analyse the reduction of CA to K.

First C and A are reduced independently:

$$\begin{aligned} C &\to (\lambda z.z(Yz))V \quad (\text{no choice}) \\ &\to^{\star} (\lambda z.zZ)V \quad (\text{where } Yz \to^{\star} Z) \\ &\to V(Z[V/z]) \\ &\to^{\star} VC' \quad (\text{where } Z[V/z] \to^{\star} C') \\ &\to \lambda y.D(F(Ey))(S(E(C'y))) \\ &\to^{\star} \lambda y.D(F(Ey))(S(EC'')) \quad (\text{where } C'y \to^{\star} C'') \end{aligned}$$

We note that $C = YV \to^{\star} Z[V/z] \to^{\star} C'$, since $Yz \to^{\star} Z$ implies $YV \to^{\star} Z[V/z]$. Surjective pairing can be applied to a reduct of $\lambda y.D(F(Ey))(S(EC''))$ only if $C'y \to^{\star} y$, which implies $Cy \to^{\star} y$.

But one has, by (1):

$$Cy \to^{\star} D((F(Ey))(S(E(Cy)))$$

If $Cy \to^{\star} y$, then $D((F(Ey))(S(E(Cy))) \to^{\star} Ey$, hence $Cy \to^{\star} Ey$, and this provides a counterexample to confluence, since y and Ey are two distinct normal forms. Thus we may assume that the independent reduction of C stops as above. Since the only way to reach K (which has the form EL) is to apply surjective pairing at some stage, we have:

$$\begin{array}{lll} CA & \to^\star & D(F(EA')(S(E(C''[A'/y])))) \quad (\text{where } A \to^\star A') \\ & \to^\star & EQ \quad (\text{where } A' \to^\star Q \text{ and } C''[A'/y] \to^\star Q) \\ & \to^\star & K \end{array}$$

Now we observe:

1 Q has less head E's than K, since all the head E's of EQ are left untouched by any reduction from EQ.
2 $A \to^\star A' \to^\star Q$ and $CA \to^\star C'A' \to^\star C''[A'/y] \to^\star Q$.

Thus Q is a common reduct of A and CA with less head E's than K: contradiction.

We believe that what makes the proof of non-confluence easier with this term than with Klop's original term is that the creation of new head E's in our example is conditioned by the application of a surjective pairing.

An adaptation of the present counterexample serves also to prove the non-confluence of a locally confluent rewriting system in the $\lambda\sigma$-calculus, an extension of λ-calculus with explicit substitution. The non-confluence, conjectured in Abadi *et al.* (1991), is proved in Curien *et al.* (1993, submitted).

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