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REPRESENTATIONS OF REAL BANACH ALGEBRAS

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Abstract

A commutative complex unital Banach algebra can be represented as a space of continuous complex-valued functions on a compact Hausdorff space via the Gelfand transform. However, in general it is not possible to represent a commutative real unital Banach algebra as a space of continuous real-valued functions on some compact Hausdorff space, and for this to happen some additional conditions are needed. In this note we represent a commutative real Banach algebra on a part of its state space and show connections with representations on the maximal ideal space of the algebra (whose existence one has to prove first).

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1. Representations via the state space of the algebra

In the first two sections of this paper we will look for characterizations of commutative real Banach algebras as spaces of continuous real-valued functions on some compact Hausdorff space, continuing in the spirit of [3] and [2].

Throughout this paper, $\mathcal{A}$ will denote a commutative real Banach algebra with unit e, where $\|e\| = 1$. We use $\mathcal{C}(K)$ to denote the space of continuous real-valued functions on a compact Hausdorff space K equipped with the norm

$$\|f\|_{\mathcal{C}(K)} = \max_{s\in K} |f(s)| = \|f\|_{\infty,K}.$$

The following is our first characterization.

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THEOREM 1.1. *Suppose that whenever $a, b \in \mathcal{A}$,*

$$a^2 + b^2 = e \quad \text{implies } \|a^2\| \leq 1. \tag{1.1}$$

Then $\mathcal{A}$ is isomorphic to a $\mathcal{C}(K)$-space.

In order to prove this theorem we need to recall some concepts and establish a few preliminary lemmas.

The closed unit ball $B_{\mathcal{A}^*}$ of the dual space $\mathcal{A}^*$ is a compact convex set in the weak* topology of $\mathcal{A}^*$. We have a natural representation of the elements of $\mathcal{A}$ as continuous functions on $B_{\mathcal{A}^*}$ via the map that assigns to each $a \in \mathcal{A}$ an element $\hat{a} \in \mathcal{C}(B_{\mathcal{A}^*})$ given by

$$\hat{a}(\varphi) = \varphi(a) \quad \forall \varphi \in \mathcal{A}^*.$$

This mapping is a linear isometry into $\mathcal{C}(B_{\mathcal{A}^*})$. The idea in the proof of Theorem 1.1 as well as in the other proofs is to restrict the functions $\hat{a}$ to a suitable subset of $B_{\mathcal{A}^*}$.

We denote by $\mathcal{A}_+$ the set of squares in $\mathcal{A}$, that is, $\mathcal{A}_+ = \{a^2 : a \in \mathcal{A}\}$. If $\mathcal{A} = \mathcal{C}(K)$, then $\mathcal{A}_+$ is simply the positive cone of the space, that is, $\{f \in \mathcal{C}(K) : f \geq 0\}$. It may also happen that $\mathcal{A}_+$ is all of $\mathcal{A}$; take, for example, $\mathcal{A}$ to be the complex numbers regarded as an algebra over the reals.

Another important ingredient in our arguments is the *state space* of $\mathcal{A}$, that is,

$$\mathcal{S} = \{\varphi \in \mathcal{A}^* : \|\varphi\| = \varphi(e) = 1\}.$$

The Hahn–Banach theorem yields that $\mathcal{S}$ is nonempty, and the example above, where $\mathcal{A}$ stands for the complex numbers, shows that $\mathcal{S}$ may consist of one point only. In general, $\mathcal{S}$ is convex and compact in the weak* topology of $\mathcal{A}^*$.

LEMMA 1.2. *Let $b \in \mathcal{A}$. If $e - b \notin \mathcal{A}_+$, there exists $\varphi \in \mathcal{S}$ such that $\varphi(b) \geq 1$.*

PROOF. The set $e - \mathcal{U}$, where $\mathcal{U}$ denotes the open unit ball of $\mathcal{A}$, is convex and a subset of $\mathcal{A}_+$. This is so because by the square root lemma for Banach algebras (see, for instance, [11, Theorem 3.4.5]) every element of the form $e - a$ with $\|a\| < 1$ is a square. Thus the set

$$\mathcal{C} = \bigcup_{\lambda > 0} \lambda(e - \mathcal{U})$$

is contained in $\mathcal{A}_+$. Moreover, $\mathcal{C}$ is clearly open and is easily seen to be convex. Since $e - b \notin \mathcal{C}$, by the Hahn–Banach theorem there exists $\psi \in \mathcal{A}^*$ with $\|\psi\| = 1$ such that $\psi(e - b) > \psi(c)$ for all $c \in \mathcal{C}$. Since $\mathcal{C}$ is closed under multiplication by positive numbers, $\psi(c) \leq 0$ for all $c \in \mathcal{C}$. Let $\varphi = -\psi$. Then $\varphi(e - u) \geq 0$ for $u \in \mathcal{U}$, so that $\varphi(e) \geq \|\varphi\| = 1$, and hence $\varphi \in \mathcal{S}$. Also $\varphi(b) - \varphi(e) \geq 0$, so that $\varphi(b) \geq 1$. □

The next lemma is a characterization of the elements in $\mathcal{S}$.

LEMMA 1.3. *Suppose that $\mathcal{A}$ satisfies the condition of Theorem 1.1. Let $\varphi \in \mathcal{A}^*$ with $\|\varphi\| = 1$. Then $\varphi \in \mathcal{S}$ if and only if $\varphi \geq 0$ on $\mathcal{A}_+$.*

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PROOF. If $\varphi \geq 0$ on $\mathcal{A}_+$ then $\varphi(e-u) \geq 0$ for all $u \in \mathcal{U}$. It follows immediately that $\|\varphi\| = \varphi(e)$. Conversely, suppose that $\varphi \in \mathcal{S}$ and that $\|a^2\| < 1$. Then, by assumption, $\|e - a^2\| \leq 1$, so that $|\varphi(e-a^2)| \leq 1$. And thus, since $\varphi(e) = 1$, we have $\varphi(a^2) \geq 0$. Since this inequality holds if $\|a^2\| < 1$, it holds on $\mathcal{A}_+$. □

LEMMA 1.4. *Suppose that $\mathcal{A}$ satisfies the condition of Theorem 1.1. Let $a \in \mathcal{A}_+$ with $\|a\| > 1$. Then $\|a\| = \|\hat{a}\|_{\infty,\mathcal{S}}$.*

PROOF. If $e - a \in \mathcal{A}_+$ then $\|a\| \leq 1$. Thus $e - a \notin \mathcal{A}_+$ and so by Lemma 1.2, $\varphi(a) \geq 1$ for some $\varphi \in \mathcal{S}$. We deduce that $\|\hat{a}\|_{\infty,\mathcal{S}} \geq 1$. Since $\|a\| \leq \|\hat{a}\|_{\infty,\mathcal{S}}$, we are done. □

The set $\partial_e\mathcal{S}$ of extreme points of $\mathcal{S}$ is nonempty by the Krein–Milman theorem. Let X be the weak* closure of $\partial_e\mathcal{S}$. We shall represent $\mathcal{A}$ as a space of real-valued continuous functions on the compact set X by restricting the functions $\hat{a}$ to X. Since the functions $\hat{a}$ are affine on $\mathcal{S}$, we have $\|\hat{a}\|_{\infty,X} = \|\hat{a}\|_{\infty,\mathcal{S}}$, where the norms denote the sup norms on X and $\mathcal{S}$ respectively, and where $\hat{a}$ denotes as well its restriction to X or $\mathcal{S}$.

The next lemma will yield that the image of $\mathcal{A}$ in $\mathcal{C}(X)$ is closed.

LEMMA 1.5. *Suppose that $\mathcal{A}$ satisfies the condition of Theorem 1.1. Then we have*

$$\|\hat{a}\|_{\infty,X} \leq \|a\| \leq 2\|\hat{a}\|_{\infty,X} \quad \forall a \in \mathcal{A}.$$

PROOF. From Lemma 1.4, if $a \in \mathcal{A}_+$ then $\|a\| = \|\hat{a}\|_{\infty,X}$. Suppose that $b \in \mathcal{A}$ with $\|\hat{b}\|_{\infty,X} < 1$. Then, by Lemma 1.2, $e \pm b \in \mathcal{A}_+$, so that

$$\begin{aligned}\|b\| &= \tfrac{1}{2}\|(e+b) - (e-b)\| \leq \tfrac{1}{2}(\|e+b\| + \|e-b\|)\\ &\leq \tfrac{1}{2}(1 + \|\hat{b}\|_{\infty,X} + 1 + \|\hat{b}\|_{\infty,X}) < 2.\end{aligned}$$ □

LEMMA 1.6. *Suppose that $\mathcal{A}$ satisfies the condition of Theorem 1.1. Let $\varphi \in \partial_e\mathcal{S}$. Then $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$.*

PROOF. Let $a \in \mathcal{A}_+$ with $\|a\| < 1$. Define φ_a on $\mathcal{A}$ by $\varphi_a(c) = \varphi(ac)$. Clearly, $\varphi_a \geq 0$ on $\mathcal{A}_+$ because $\mathcal{A}_+$ is closed under multiplication by elements of itself. Lemma 1.3 yields that $\|\varphi_a\| = \varphi_a(e) = \varphi(a)$. Since $e - a \in \mathcal{A}_+$ we can apply the same argument to $\varphi_{(e-a)}$. Since φ is an extreme point we deduce that either $\varphi_a = \varphi$ or $\varphi_{(e-a)} = \varphi$. It follows that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $b \in \mathcal{A}$. Since, for $\|a\| < 1$, we can write $a = \frac{1}{2}(e+a) - \frac{1}{2}(e-a) \in \mathcal{A}_+ - \mathcal{A}_+$, the proof is finished. □

We are now in a position to conclude the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Suppose that $\mathcal{A}$ satisfies condition (1.1). Define the map $J : \mathcal{A} \to \mathcal{C}(X)$ by $a \to J(a) = \hat{a}$. By Lemma 1.6, J is an algebra homomorphism so that its image is an algebra, and by Lemma 1.5 this image is closed. Finally, by the Stone–Weierstrass theorem we infer that $J(\mathcal{A})$ is all of $\mathcal{C}(X)$. □

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REMARK. We cannot hope for an isometry in Theorem 1.1. The example in [2, p. 741] with the algebra $\mathcal{C}[0, 1]$ equipped with the norm

$$|||f||| = \|f_+\|_{\mathcal{C}[0,1]} + \|f_-\|_{\mathcal{C}[0,1]},$$

where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$, is an example of an algebra satisfying the condition in Theorem 1.1 which is only 2-isomorphic to $\mathcal{C}[0, 1]$.

2. Purely algebraic characterizations of real $\mathcal{C}(K)$-spaces

In 1947, Arens [3] gave the earliest known criterion for checking whether a commutative real Banach algebra with unit *is* a $\mathcal{C}(K)$-space.

THEOREM 2.1 (Arens). *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity e such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space K if and only if*

$$\|a\|^2 \le \|a^2 + b^2\| \quad \forall a, b \in \mathcal{A}. \tag{2.1}$$

Fifty years later, unaware of the existence of Arens's theorem, the authors established in [2] a second characterization of real $\mathcal{C}(K)$-spaces amongst real commutative unital Banach algebras (see also [1, Section 4.2]).

THEOREM 2.2. *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity e such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space K if and only if*

$$\|a^2 - b^2\| \le \|a^2 + b^2\| \quad \forall a, b \in \mathcal{A}. \tag{2.2}$$

Unlike the proof of Theorem 2.1, which uses the complexification technique, Theorem 2.2 relies exclusively on *intrinsic* methods, that is, it does not require any complex function theory and, instead, the proof remains inside the structure of real Banach algebras.

One may wonder whether there is a condition of this type that characterizes real $\mathcal{C}(K)$-spaces involving just the two operations of the algebra and its norm. In the Banach algebra $\mathcal{C}(K)$ the addition and multiplication are related by the inequality

$$2|f(s)g(s)| \le f^2(s) + g^2(s) \quad \forall s \in K,$$

whence

$$\|2fg\|_{\mathcal{C}(K)} \le \|f^2 + g^2\|_{\mathcal{C}(K)} \quad \forall f, g \in \mathcal{C}(K).$$

As it turns out, this easily checked condition happens to characterize the commutative real Banach algebras with unit which are (isometrically isomorphic to) $\mathcal{C}(K)$-spaces.

THEOREM 2.3. *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity e such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space K if and only if*

$$\|2xy\| \le \|x^2 + y^2\| \quad \forall x, y \in \mathcal{A}. \tag{2.3}$$

PROOF. We have already pointed out that every $\mathcal{C}(K)$-space satisfies (2.3). If we conversely assume (2.3) and if a and b belong to $\mathcal{A}$, then

$$\|a^2 - b^2\| = \tfrac{1}{2}\|2(a-b)(a+b)\| \leq \tfrac{1}{2}\|(a-b)^2 + (a+b)^2\| = \|a^2 + b^2\|,$$

and the result follows from Theorem 2.2. □

REMARK. Of course, all three conditions (2.1), (2.2), and (2.3) must be equivalent since they characterize the same space. In [2] it was shown that (2.2) is an immediate consequence of (2.1). That (2.2) implies (2.3) follows very easily as well: given x and y in $\mathcal{A}$, we pick a, b in $\mathcal{A}$ such that $x = a - b$ and $y = a + b$ (that is, $a = (x+y)/2$, $b = (y-x)/2$). Then,

$$\begin{aligned}\|2xy\| &= 2\|(a-b)(a+b)\| \\ &\leq 2\|a^2 + b^2\| \\ &= 2\left\|\left(\frac{x+y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2\right\| \\ &= \|x^2 + y^2\|.\end{aligned}$$

To close the circle of these equivalences let us show that (2.3) implies (2.1). If we put $y = e$ in (2.3), we get

$$2\|x\| \leq \|x^2 + e\| \leq \|x^2\| + 1.$$

Thus for all x in $\mathcal{A}$ of norm $\|x\| = 1$, we have $2 \leq \|x^2\| + 1$, that is, $\|x^2\| \geq 1$. Therefore,

$$1 \leq \left\|\frac{x^2}{\|x\|^2}\right\| \quad \forall x \in \mathcal{A},\, x \neq 0,$$

which yields

$$\|x\|^2 \leq \|x^2\| \quad \forall x \in \mathcal{A}.$$

Hence, given any $a, b \in \mathcal{A}$,

$$\begin{aligned}\|a\|^2 = \|a^2\| &= \tfrac{1}{2}\|(a^2 - b^2) + (b^2 + a^2)\| \\ &\leq \tfrac{1}{2}(\|a^2 - b^2\| + \|a^2 + b^2\|) \\ &\leq \|a^2 + b^2\|.\end{aligned}$$

REMARK. Notice that we can obtain Theorem 2.1 using Theorem 1.1. Indeed, the condition in Theorem 1.1 is implied by Arens's condition (2.1), and so the map $a \to \hat{a}$ is an isomorphism. Equation (2.1) gives $\|a^2\| \leq \|a\|^2 = \|a^2\|$ so that, by Lemma 1.4,

$$\|a\|^2 = \|a^2\| = \|\widehat{a^2}\|_{\infty,X} = \|\hat{a}\|^2_{\infty,X},$$

showing that the map is an isometry.

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There is another condition, weaker than (2.1), which is also mentioned in [2], namely

$$\|a^2\| \le \|a^2 + b^2\| \quad \forall a, b \in \mathcal{A}. \tag{2.4}$$

This condition clearly implies (1.1), but since it is satisfied by the aforementioned example from [2], it is strictly weaker than (2.1). However, (2.4) implies an Arens condition with a constant. To be specific, we state the following proposition.

PROPOSITION 2.4. *Suppose that $\mathcal{A}$ satisfies condition (2.4). Then*

$$\|a\|^2 \le 4\|a^2 + b^2\| \quad \forall a, b \in \mathcal{A}. \qquad \square$$

PROOF. We have

$$\begin{aligned}\|a\| &= \tfrac{1}{4}\|(e+a)^2 - (e-a)^2\| \le \tfrac{1}{4}\|(e+a)^2\| + \tfrac{1}{4}\|(e-a)^2\| \\ &\le \tfrac{1}{2}\|(e+a)^2 + (e-a)^2\| = \tfrac{1}{2}\|2 + 2a^2\| \\ &= \|e + a^2\| \le 1 + \|a^2\|,\end{aligned}$$

where in order to obtain the second inequality we used condition (2.4) twice. It follows that $\|a\|^2 \le 4$ if $\|a^2\| = 1$, so that $\|a\|^2 \le 4\|a^2\|$, and the proof is complete. $\square$

REMARK. Proposition 2.4 yields that $\|a\|^k \le 4^k\|a^k\|$ if k is a power of 2. Hence $\|a\| \le 4r(a)$, where $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ is the *spectral radius* of a. Since we have $r(a) \le \|a\|$ for every $a \in \mathcal{A}$, we conclude that the spectral radius is a norm on $\mathcal{A}$ equivalent to the given one.

3. Representations via the maximal ideal space

Here, influenced by [9], we obtain results similar to those in Section 1 via a different path, through the maximal ideal space.

In contrast with the complex case, an element in a real Banach algebra may have empty spectrum; take, for example, the algebra of operators in the plane generated by a rotation and the identity. Recall that the *spectrum* of an element a in $\mathcal{A}$ is the (possibly empty) set

$$\sigma(a) = \{s \in \mathbb{R} : (a - s) \text{ not invertible in } \mathcal{A}\}.$$

In turn, the *complex spectrum* of a is the set

$$\sigma_{\mathbb{C}}(a) = \{(s, t) \in \mathbb{R}^2 : (a - s)^2 + t^2 \text{ not invertible in } \mathcal{A}\}.$$

It is shown in [9] that the complex spectrum of every a in $\mathcal{A}$ is nonempty. We give a similar but slightly shorter proof here.

PROPOSITION 3.1. *For $a \in \mathcal{A}$, the complex spectrum $\sigma_{\mathbb{C}}(a)$ is nonempty and*

$$\sup_{(s,t)\in\sigma_{\mathbb{C}}(a)} (s^2 + t^2)^{1/2} = r(a).$$

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PROOF. Let $\varphi \in \mathcal{A}^*$ and put

$$u(s, t) = \varphi((a - s)((a - s)^2 + t^2)^{-1})$$

and

$$v(s, t) = \varphi(t((a - s)^2 + t^2)^{-1}).$$

Straightforward calculations show that these functions are harmonic and harmonic conjugates of each other. We have

$$\lim_{(s^2+t^2)\to\infty} (s^2 + t^2)^{-1}(a - s) = 0$$

and

$$\lim_{(s^2+t^2)\to\infty} (s^2 + t^2)^{-1}(a^2 - 2sa) = 0.$$

Thus, for $\|a^2 - 2sa\| < (s^2 + t^2)$,

$$\begin{aligned} \|(a - s)((a - s)^2 + t^2)^{-1}\| &= \|(a - s)(a^2 - 2sa + s^2 + t^2)^{-1}\| \\ &\le \frac{\|a - s\|}{s^2 + t^2} \cdot \sum_{n=0}^{\infty} \left\| \frac{a^2 - 2sa}{s^2 + t^2} \right\|^n \\ &= \frac{\|a - s\|}{s^2 + t^2} \cdot \frac{1}{1 - (\|a^2 - 2sa\|/(s^2 + t^2))}, \end{aligned}$$

which tends to 0 as $(s^2 + t^2) \to \infty$. If u and v were everywhere defined, the maximum principle for harmonic functions shows that they are identically 0. But clearly we can find $\varphi \in \mathcal{A}^*$ and real numbers s, t such that $u(s, t) \neq 0$. This contradiction shows that the complex spectrum is nonempty. It is easily seen to be closed.

For a complex number $z = s + it$, put $w(z) = u(s, t) + iv(s, t)$. Then w is an analytic function on the complement of the complex spectrum and thus analytic outside the smallest disk with center 0 containing the complex spectrum. Let R denote the radius of this disk, $R = \sup(s^2 + t^2)^{1/2}$, the supremum taken over all (s, t) in the complex spectrum of a. Since $\lim_{z\to\infty} w(z) = 0$, we can write

$$w(z) = \sum_{n=0}^{\infty} a_n\, 1/z^n,$$

converging for $|z| > R$.

Now, for $z = s$ we have $v(s, 0) = 0$ so that $w(z) = w(s) = \varphi((a - s)^{-1})$ and thus

$$w(s) = \sum_{n=0}^{\infty} a_n\, 1/s^n = \varphi((a - s)^{-1}) = -\sum_{n=0}^{\infty} \varphi(a^n)\, 1/s^n.$$

It follows that $a_n = \varphi(a^n)$ for all n, and so the first and the last series have the same radius of convergence, R. For $|s| > R$, the uniform boundedness principle shows

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that there exists an number M such that $\|a^n/s^n\| < M$ for all n. It follows that $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \leq R$. On the other hand, the series $\sum_{n=0}^{\infty} a^n/z^n$ converges for $|z| > \lim_{n\to\infty} \|a^n\|^{1/n}$. We conclude that

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \sup_{(s,t)\in\sigma_{\mathbb{C}}(a)} (s^2+t^2)^{1/2}. \qquad \square$$

We are concerned with homomorphisms on $\mathcal{A}$, especially those that are real-valued. Suppose φ is a nonzero real- or complex-valued homomorphism. Then its nullspace, $\mathcal{M}$, is a maximal ideal in $\mathcal{A}$. Conversely, if $\mathcal{M}$ is a maximal ideal then the quotient algebra $\mathcal{A}/\mathcal{M}$ is a division algebra (that is, every nonzero element has an inverse) and thus isomorphic to $\mathbb{R}$ or $\mathbb{C}$ by [4, Theorem 14.7] or [10, Theorem 1.1.23]. The space of homomorhisms, X, is given the usual Gelfand topology and $\mathcal{A}$ is represented as a space of continuous functions on X via $a \to \hat{a}$ where $\hat{a}(x) = x(a)$. We are interested in finding conditions on $\mathcal{A}$ where this representation gives real-valued functions.

LEMMA 3.2. *Every homomorphism on $\mathcal{A}$ is real-valued if and only if we have $\sigma_{\mathbb{C}}(a) = \sigma(a)$ for every a in $\mathcal{A}$. Also, $(s, t) \in \sigma_{\mathbb{C}}(a)$ if and only if $\varphi(a) = s + it$ for some homomorphism φ, and*

$$\sup_{(s,t)\in\sigma_{\mathbb{C}}(a)} (s^2+t^2)^{1/2} = \|\hat{a}\|_{\infty,X} = r(a),$$

where $\|\hat{a}\|_{\infty,X} = \sup_{x\in X} |\hat{a}(x)|$.

PROOF. Suppose that $\varphi(a) = s + it$, where $t \neq 0$ for some homomorphism φ and some a in $\mathcal{A}$. Then

$$\varphi((a-s)^2+t^2) = (\varphi(a)-s)^2 + t^2 = 0,$$

so that $(a-s)^2+t^2$ is not invertible in $\mathcal{A}$. Conversely, suppose that $(a-s)^2+t^2$, where $t \neq 0$, is not invertible. The ideal $((a-s)^2+t^2)\mathcal{A}$ is contained in a maximal ideal and thus there is a homomorphism φ on $\mathcal{A}$ vanishing on this maximal ideal. In particular,

$$\varphi((a-s)^2) + t^2 = \varphi((a-s)^2+t^2) = 0,$$

so that $\varphi(a-s)$ is not a real number.

If $(a-s)^2+t^2$ is not invertible then $((a-s)^2+t^2)\mathcal{A}$ is an ideal in $\mathcal{A}$ and thus $\varphi((a-s)^2+t^2) = 0$ for some nonzero homomorphism φ. It follows immediately that $\varphi(a-s) = it$ so that $\varphi(a) = s + it$. (If $\varphi(a-s) = -it$ we use $\overline{\varphi}$.) The last claim follows from the first part and Proposition 3.1. $\square$

Let us have a look at the conditions

$$\|a\|^2 \leq k\|a^2+b^2\| \quad \forall a, b \in \mathcal{A}, \tag{3.1}$$

$$\|a^2\| \leq k\|a^2+b^2\| \quad \forall a, b \in \mathcal{A}, \tag{3.2}$$

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and

$$r(a^2) \le kr(a^2 + b^2) \quad \forall a, b \in \mathcal{A}, \tag{3.3}$$

where k is a positive constant that may differ from inequality to inequality. It is easy to see that (3.1) implies (3.2), which in turn implies (3.3).

EXAMPLE 1. The algebra $\mathcal{A} = \mathcal{C}^{(1)}[0, 1]$, consisting of all continuously differentiable real-valued functions on the interval [0, 1], equipped with the norm given by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

is a commutative Banach algebra with unit. Clearly, $1 + f^2$ is invertible so that (s, t) is not in the complex spectrum of f for $t \ne 0$, and $\sigma(f) = f([0, 1])$ for every f in $\mathcal{A}$. It follows that the Gelfand transform in this case is just the inclusion map from $\mathcal{A}$ into $\mathcal{C}[0, 1]$. Thus $\mathcal{A}$ satisfies (3.3) with $k = 1$.

The first two conditions above are actually equivalent, as we now show.

PROPOSITION 3.3. *Suppose that $\mathcal{A}$ satisfies condition (3.2). Then*

$$\|a\|^2 \le 4k^2\|a^2\| \quad \forall a \in \mathcal{A}. \qquad \square$$

PROOF. For $a \in \mathcal{A}$,

$$\begin{aligned}
\|a\| &= \frac{1}{4}\|(e+a)^2 - (e-a)^2\| \\
&\le \frac{1}{4}(\|(e+a)^2\| + \|(e-a)^2\|) \\
&\le \frac{k}{4}(\|(e+a)^2 + (e-a)^2\| + \|(e-a)^2 + (e+a)^2\|) \\
&= k\|e + a^2\| \\
&\le k(1 + \|a^2\|).
\end{aligned}$$

Replacing a by ta, where t is any positive number, we see that $a^2 = 0$ implies that $a = 0$. Thus $\|a\|^2 \le 4k^2$ if $\|a^2\| \le 1$, and the inequality follows. $\square$

The next lemma is crucial in obtaining representations of $\mathcal{A}$ as a space of real-valued functions.

LEMMA 3.4. *Suppose that there is an element $a \in \mathcal{A}$ whose complex spectrum is not a subset of the real line. Then there exist a subset F of X and elements b and u in $\mathcal{A}$ such that $\hat{b} = 1$ on F, $|\hat{b}| < 1$ on $X \backslash F$, and $1 + \hat{u}^2 = 0$ on F.*

PROOF. Let X be the set of homomorphisms on $\mathcal{A}$. Choose (s_0, t_0) in $\sigma_{\mathbb{C}}(a)$ and x_0 in X such that $t_0 \ge t$ if $(s, t) \in \sigma_{\mathbb{C}}(a)$ and $\hat{a}(x_0) = s_0 + it_0$. Since $(s, -t) \in \sigma_{\mathbb{C}}(a)$ if $(s, t) \in \sigma_{\mathbb{C}}(a)$, we have $t_0 > 0$. For $t_1 > t_0$ the element $(a - s_0)^2 + t_1^2$ is invertible in $\mathcal{A}$. Denote its inverse by b. For x in X and $\hat{a}(x) = s + it$,

$$|\hat{b}(x)| = |((s + it - s_0)^2 + t_1^2)|^{-1} \le (t_1^2 - t^2)^{-1} \le \hat{b}(x_0),$$

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with equality only if $\hat{a}(x) = s_0 \pm it_0$. Let F be the subset of X where $\hat{b}$ takes the value $\hat{b}(x_0)$ and put $u = t_0^{-1}(a - s_0)$. Replacing b by $\|\hat{b}\|_{\infty,X}^{-1} b$, the proof is finished. □

PROPOSITION 3.5. *Suppose that $\mathcal{A}$ satisfies condition (3.3). Then the complex spectrum of a is a nonempty subset of the interval $(-r(a), r(a))$.*

PROOF. If not, let F, b and u be as in Lemma 3.4. From (3.3) we get

$$r(b^{2n}) \leq kr(b^{2n} + u^2 b^{2n}) = kr(b^{2n}(1 + u^2)).$$

Since $1 + \hat{u}^2 = 0$ on F, and $|\hat{b}| < 1$ on $X \backslash F$, we deduce that the right-hand side tends to zero as n increases. However, the left-hand side is equal to 1 since $r(b) = 1$, a contradiction. □

We thus have a representation of $\mathcal{A}$ as a space of continuous real-valued functions on the space X of homomorphisms on $\mathcal{A}$ if (3.3) is satisfied. If (3.2) is satisfied, more can be said and we can state the following theorem.

THEOREM 3.6. *Suppose that $\mathcal{A}$ satisfies*

$$\|a^2\| \leq k\|a^2 + b^2\| \quad \forall a, b \in \mathcal{A}.$$

Then $\mathcal{A}$ is isomorphic to a $\mathcal{C}(K)$-space.

PROOF. We must show that the Gelfand transform is an isomorphism. By the Stone–Weierstrass theorem, since $r(a) = \sup_{x \in X} |\hat{a}(x)|$, it suffices to prove that the spectral radius semi-norm is equivalent to the given norm. Pick a in $\mathcal{A}$. By Proposition 3.3,

$$\|a\|^{2^n} \leq (4k^2)^{2^n} \|a^{2^n}\|.$$

Taking roots and letting n tend to infinity, we obtain

$$\|a\| \leq 4k^2 r(a).$$

Since $r(a) \leq \|a\|$, we are done. □

A similar theorem with (3.1) instead of (3.2) is to be found in [9].

EXAMPLE 2. Another example of an algebra satisfying (3.3) is the algebra $\mathcal{A}$ of all continuous real-valued functions on the unit circle with absolutely convergent Fourier series, that is, the subalgebra of $\ell_1(\mathbb{Z})$ consisting of those sequences $(a_k)_{k \in \mathbb{Z}}$ in $\mathbb{C}$ for which $\overline{a_k} = a_{-k}$ for every natural number k. With convolution as multiplication and the norm given by $\|(a_k)_{k \in \mathbb{Z}}\| = \sum_{\mathbb{Z}} |a_k|$, $\mathcal{A}$ is a real commutative Banach algebra with unit. The space of homomorphisms can be identified with the unit circle. Both this example and Example 1 are examples of real Banach algebras which consist of Hermitian elements in *-algebras. In this situation every a in $\mathcal{A}$ satisfying $\hat{a} > 0$ is a square because we have complex functional calculus at our disposal. However, there are elements a in $\mathcal{A}$ satisfying $\hat{a} \geq 0$ which are not squares. This is a consequence of Katznelson's square root theorem [8] which says that if that were the case, then $\mathcal{A} = \mathcal{C}(X)$.

4. Two comments on representations of C^*-algebras

(a) In [2, Theorem 4.1] it was shown that the Gelfand–Naimark [6] representation of *complex* commutative C^*-algebras can be obtained from Theorem 2.2 without using general methods of Banach algebras that depend heavily on the use of complex scalars. Let us state this theorem for reference.

THEOREM 4.1 (Gelfand–Naimark). *If $\mathcal{A}$ is a commutative complex Banach algebra with an identity e such that $\|e\| = 1$ and an involution * such that*

$$\|a^*a\| = \|a\|^2 \quad \forall a \in \mathcal{A}, \tag{4.1}$$

then $\mathcal{A}$ is isometrically $*$-algebra isomorphic to $\mathcal{C}_{\mathbb{C}}(K)$ for some compact Hausdorff space K.

We note that the proof of Theorem 4.1 given in [2] also works if (4.1) is relaxed by just requiring the norm to be multiplicative for products a^*a, that is,

$$\|a^*a\| = \|a\|\|a^*\| \quad \forall a \in \mathcal{A}. \tag{4.2}$$

Obviously, (4.1) implies (4.2) since for $a \in \mathcal{A}$,

$$\|a\|^2\|a^*\|^2 = \|a^*a\|\|(a^*)^*a^*\| = \|aa^*\|^2.$$

Then one has to work a bit to deduce from (4.2) that the involution is isometric on $\mathcal{A}$, which in turn will yield the equivalence of both C^*-conditions. With hindsight, the advantage of (4.1) versus the more natural 'weak' C^*-condition (4.2) is that the former immediately yields that $\|a\| = \|a^*\|$ for every $a \in \mathcal{A}$, and from here the proof of Theorem 4.1 can be completed, avoiding a detour into complex analysis.

As Doran and Belfi point out in [5, p. 5] and show in [5, Ch. III], it turns out that in a (complex) Banach $*$-algebra, (4.2) also implies (4.1) without the assumption that the involution is isometric, but they warn the reader that this is highly nontrivial.

(b) Note that the C^*-condition (4.1) is not sufficient to guarantee the representation of a Banach real *-algebra $\mathcal{A}$ as $\mathcal{C}(K)$-space. Recall that a *real $*$-algebra* is a real algebra $\mathcal{A}$ equipped with an involution on $\mathcal{A}$, that is, a map $^* : \mathcal{A} \to \mathcal{A}$ that is real linear (that is, $(x + y)^* = x^* + y^*$ and $(\lambda x)^* = \lambda x^*$ for all $x, y \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$), reverse-multiplicative (that is, $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{A}$), and has period at most 2 (that is, $x^{**} = x$ for all $x \in A$). For example (see [7, p. 83]), if $\mathbb{C}$ is viewed as a real Banach *-algebra with the absolute value as norm and the identity as involution (that is, $x^* = x$ for all $x \in \mathbb{C}$), it satisfies all the axioms of Banach real *-algebra but cannot be isometric to a $\mathcal{C}(K)$-space since $1 + i^*i = 0$. The problem is that, unlike the situation for complex C^*-algebras, in the real case the invertibility of elements of the form $1 + x^*x$ cannot be deduced from the other axioms and hence must be included in the definition of real C^*-algebra [5, p. 274].

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