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A nonlinear complementarity problem in mathematical programming in Hilbert space

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In this paper we prove the following existence and uniqueness theorem for the nonlinear complementarity problem by using the Banach contraction principle. If $T : K \to H$ is strongly monotone and lipschitzian with $k^2 < 2c < k^2+1$, then there is a unique $y \in K$, such that $Ty \in K^*$ and $(Ty, y) = 0$ where H is a Hilbert space, K is a closed convex cone in H, and K^* the polar cone.

1. Introduction and statement of the theorem

Let H be a real Hilbert space and let K be a closed convex cone in H with the vertex at 0. The polar of K is the cone K^*, defined by

$$K^* = \{y \in H : (x, y) \geq 0 \text{ for every } x \in K\} .$$

A mapping $T : H \to H$ is said to be monotone on K if $(Tx-Ty, x-y) \geq 0$ for all $x, y \in K$ and strictly monotone if strict inequality holds whenever $x \neq y$. T is called strongly monotone if there is a constant $c > 0$ such that $(Tx-Ty, x-y) \geq c\|x-y\|^2$. T is said to be lipschitzian if there is a constant $k > 0$ such that $\|Tx-Ty\| \leq k\|x-y\|$ for all $x, y \in H$ whenever $x \neq y$, and a contraction if $0 < k < 1$.

The purpose of this note is to prove the following existence and uniqueness theorem for the nonlinear complementarity problem.

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THEOREM. *Let* $T : K \to H$ *be strongly monotone and lipschitzian with* $k^2 < 2c < k^2+1$. *Then there is a unique* y_0 *such that*

(1.1) $$y_0 \in K\ ,\quad Ty_0 \in K^*\ ,\ \text{and}\quad (Ty_0, y_0) = 0\ .$$

2. Proof of the theorem

Since K is a nonempty closed convex set in H , for every $y \in K$ there is a unique $x \in K$ closest to $y - Ty$; that is,

$$\|x-y+Ty\| \le \|z-y+Ty\|$$

for every $z \in K$. Let the correspondence $y \mapsto x$ be denoted by θ . Let z be any element of K and let $0 \le \lambda \le 1$. Since K is convex, $(1-\lambda)x + \lambda z \in K$. Define a function $h : [0, 1] \to R^+$ by the rule

$$h(\lambda) = \|y-Ty-(1-\lambda)x-\lambda z\|^2 .$$

Then h is a twice continuously differentiable function of λ and

$$h'(\lambda) = 2(y-Ty-\lambda z-(1-\lambda)x,\ x-z) .$$

Since x is the unique element closest to $y - Ty$, we must have $h'(0) \ge 0$, and therefore

(2.1) $$(y-Ty-x,\ x-z) \ge 0$$

for every $z \in K$. Let y_1 and y_2 be two elements of K and $y_1 \ne y_2$. Let $\theta(y_1) = x_1$ and $\theta(y_2) = x_2$. Putting $y = y_1$ and $z = \theta(y_2)$ in (2.1) we get

(2.2) $$\left(y_1-Ty_1-\theta(y_1),\ \theta(y_1)-\theta(y_2)\right) \ge 0 .$$

Again, putting $y = y_2$ and $z = \theta(y_1)$ in (2.1), we get

(2.3) $$\left(y_2-Ty_2-\theta(y_2),\ \theta(y_2)-\theta(y_1)\right) \ge 0 .$$

From (2.2) and (2.3) we have

$$\left(y_1-Ty_1-\theta(y_1)-y_2+Ty_2+\theta(y_2),\ \theta(y_1)-\theta(y_2)\right) \ge 0 .$$

Hence

$$\left(y_1-Ty_1-y_2+Ty_2,\ \theta(y_1)-\theta(y_2)\right) \geq \left(\theta(y_1)-\theta(y_2),\ \theta(y_1)-\theta(y_2)\right)$$
$$= \|\theta(y_1)-\theta(y_2)\|^2 .$$

Therefore,

$$\|\theta(y_1)-\theta(y_2)\|^2 \leq \left|\left(y_1-Ty_1-y_2+Ty_2,\ \theta(y_1)-\theta(y_2)\right)\right|$$
$$\leq \|y_1-Ty_1-y_2+Ty_2\|\|\theta(y_1)-\theta(y_2)\| .$$

Thus

(2.4) $$\|\theta(y_1)-\theta(y_2)\| \leq \|Ty_1-Ty_2-y_1+y_2\| .$$

Since T is strongly monotone and lipschitzian, it follows from the inequality (2.4) that

$$\|\theta(y_1)-\theta(y_2)\|^2 \leq \|Ty_1-Ty_2-y_1+y_2\|^2$$
$$= \left(Ty_1-Ty_2-y_1+y_2,\ Ty_1-Ty_2-y_1+y_2\right)$$
$$= \|Ty_1-Ty_2\|^2 + \|y_1-y_2\|^2 - 2\left(Ty_1-Ty_2,\ y_1-y_2\right)$$
$$\leq \left(k^2+1-2c\right)\|y_1-y_2\|^2 .$$

Since $k^2 < 2c < k^2+1$, we have $0 < k^2+1-2c < 1$. Putting $a^2 = k^2 + 1 - 2c$ in the above inequality we obtain

$$\|\theta(y_1)-\theta(y_2)\| \leq a\|y_1-y_2\|$$

where $0 < a < 1$. Thus θ is a contraction. Now applying the Banach contraction principle (see, for example, [1]) we conclude that θ has a unique fixed point, say y_0. Now putting $y = y_0$ in (2.1) we get

(2.5) $$\left(Ty_0,\ z-y_0\right) \geq 0$$

for every $z \in K$. Since $0 \in K$ we have from (2.5) that $\left(Ty_0,\ y_0\right) \leq 0$. Again since K is a convex cone, $2y_0 \in K$ and therefore putting $z = 2y_0$ in (2.5) we get $\left(Ty_0,\ y_0\right) \geq 0$. Thus $\left(Ty_0,\ y_0\right) = 0$ and $\left(Ty_0,\ z\right) \geq 0$ for every $z \in K$, showing that $Ty_0 \in K^*$. Therefore y_0 is the unique solution to the complementarity problem (1.1) and this completes the proof.

Reference

[1] Casper Goffman, George Pedrick, *First course in functional analysis* (Prentice/Hall of India, New Delhi, 1974).

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