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Jackson's Theorem for locally compact abelian groups

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If f is a p-th integrable function on the circle group and $\omega(p; f; \delta)$ is its mean modulus of continuity with exponent p, then an extended version of the classical theorem of Jackson states the for each positive integer n, there exists a trigonometric polynomial t_n of degree at most n for which

$$\|f-t_n\|_p \leq 6\omega(p; f; 1/n) .$$

In this paper it will be shown that for G a Hausdorff locally compact abelian group, the algebra $L^1(G)$ admits a certain bounded positive approximate unit which, in turn, will be used to prove an analogue of the above result for $L^p(G)$.

We shall let λ denote a chosen Haar measure on G. The spectrum (written $\Sigma(f)$) of $f \in L^\infty(G)$ will be defined as in [3], (40.21). For $f \in L^p(G)$ $(p \in [1, \infty))$, we define its spectrum by

$$\Sigma(f) = \bigcup_{\phi \in C_{00}(G)} \Sigma(f*\phi)$$

(where $C_{00}(G)$ denotes the space of continuous functions on G with compact support). Given $K \subset \Gamma$, $V \subset G$ and $f \in L^p(G)$, we put

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$$L_K^p(G) = \{f \in L^p(G) : \Sigma(f) \subset K\} ,$$

$$E_K(p;\ f) = \inf\left\{\|f-g\|_p : g \in L_K^p(G)\right\} ,$$

and

$$\omega(p;\ f;\ V) = \sup\{\|\tau_a f-f\|_p : a \in V\} .$$

We require the following theorem, a corollary of which will serve as the basis of the proof of the main result.

THEOREM 1. *Let V be a neighbourhood of zero in G and $\varepsilon > 0$ be given. Suppose ρ is a locally bounded measurable function on G satisfying:*

(a) $\rho(x) \geq 1$,

(b) $\rho(x+y) \leq \rho(x)\rho(y)$, *and*

(c) $\sum_{n=1}^{\infty} \frac{\log\rho(nx)}{n^2} < \infty$

for all $x, y \in G$. Then there exists a continuous k_V on G such that $k_V \geq 0$, $\int_G k_V d\lambda = 1$, $\operatorname{supp}\hat{k}_V$ is compact, and

$$\int_G \rho k_V d\lambda < \sup_{x\in V} \rho(x) + \varepsilon .$$

Proof. This follows readily from [1], Theorem 2.11 and the proof of [1], Lemma 1.23. //

COROLLARY. *Suppose V is an open neighbourhood of zero generating G , $\varepsilon > 0$ is given, and m_V is the integer-valued function on G defined by*

$$m_V(x) = \min\{m \in \{1, 2, \ldots\} : x \in mV\} .$$

Then there exists a continuous k_V on G such that $k_V \geq 0$, $\int_G k_V d\lambda = 1$, $K_V = \operatorname{supp}\hat{k}_V$ is compact, and

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$$\int_G m_V k_V d\lambda < 1 + \varepsilon .$$

Proof. Put $\rho = 2m_V$ in Theorem 1. //

We are now in a position to prove our promised analogue of the extended version of Jackson's Theorem as stated in the abstract.

THEOREM 2. *Given* $\varepsilon > 0$, *we can find a base* $\{V_i\}_{i\in I}$ *of open neighbourhoods of zero, and a corresponding family* $\{k_i\}_{i\in I}$ *of continuous functions on* G *such that for each* $i \in I$, $k_i \geq 0$, $\int_G k_i d\lambda = 1$, $K_i = \operatorname{supp}\hat{k}_i$ *is compact, and*

(1) $\|k_i * f - f\|_p \leq (1+\varepsilon)\omega(p;\, f;\, V_i)$,

(2) $E_{K_i}(p;\, f) \leq (1+\varepsilon)\omega(p;\, f;\, V_i)$

for every $f \in L^p(G)$ *if* $p \in [1, \infty)$, *or for every bounded uniformly continuous* f *if* $p = \infty$.

Proof. Since, by [2], (24.30), G is topologically isomorphic with $\mathrm{R}^n \times G_0$, where $n \in \{0, 1, \ldots\}$ and G_0 is a Hausdorff locally compact abelian group containing a compact open subgroup H , we need only prove the theorem for $\mathrm{R}^n \times G_0$ (and the result for G will then follow from [2], (24.41) (c)).

Let $\{V_i\}_{i\in I}$ be a base of open neighbourhoods of zero in $\mathrm{R}^n \times G_0$ such that for each $i \in I$,

$$V_i = U_i \times W_i ,$$

where U_i (respectively W_i) is open in R^n (respectively H). Let $\mathcal{W}_i$ be the subgroup of G_0 generated by W_i . Clearly $\mathcal{W}_i \subset H$ is open and compact. Let λ_{R^n}, λ_{G_0} and $\lambda_{\mathcal{W}_i}$ denote the Haar measures on R^n, G_0 and $\mathcal{W}_i$ respectively, where $\lambda_{\mathcal{W}_i}$ is chosen such that $\lambda_{G_0}(\mathcal{W}_i) = \lambda_{\mathcal{W}_i}(\mathcal{W}_i)$.

By the corollary to Theorem 1, we can find continuous k_{U_i}, k_{W_i} on $\mathbb{R}^n$, W_i respectively such that $k_{U_i}, k_{W_i} \geq 0$,

$$\int_{\mathbb{R}^n} k_{U_i} d\lambda_{\mathbb{R}^n} = \int_{W_i} k_{W_i} d\lambda_{W_i}$$
$$= 1 ,$$

$\text{supp}\hat{k}_{U_i}$ and $\text{supp}\hat{k}_{W_i}$ are compact, and

$$\max\left\{\int_{\mathbb{R}^n} m_{U_i} k_{U_i} d\lambda_{\mathbb{R}^n}, \int_{W_i} m_{W_i} k_{W_i} d\lambda_{W_i}\right\} < (1+\varepsilon)^{1/2} .$$

Define k_i on $\mathbb{R}^n \times G_0$ by

$$k_i[(x, y)] = k_{U_i}(x) k'_{W_i}(y) ,$$

where

$$k'_{W_i}(y) = \begin{cases} k_{W_i}(y) , & y \in W_i , \\ 0 , & y \in G_0 \backslash W_i . \end{cases}$$

We shall show that $\{k_i\}_{i \in I}$ has the desired properties.

Clearly each k_i is continuous (k'_{W_i} is continuous on G_0 since W_i is both open and closed, and k_{W_i} is continuous on W_i) and non-negative. An application of [2], (13.4) gives

$$\int_{\mathbb{R}^n \times G_0} k_i d\lambda_{\mathbb{R}^n} \times \lambda_{G_0} = 1 .$$

The fact that $\text{supp}\hat{k}'_{W_i}$ is compact follows from the compactness of $\text{supp}\hat{k}_{W_i}$, [2], (24.5) and [2], (5.24) (a). Appealing to [3], (31.7) (b), we see that $K_i = \text{supp}\hat{k}_i$ is compact.

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Now let $f \in L^p\left(\mathbb{R}^n \times G_0\right)$ $\left(p \in [1, \infty)\right)$ or, if $p = \infty$, take f to be uniformly continuous. Then we have

$$k_i * f - f = \int_{\mathbb{R}^n \times G_0} \left(\tau_{(x,y)} f - f\right) k_i(x, y) d\lambda_{\mathbb{R}^n \times G_0}(x, y)$$

(interpreting the right-hand side as a vector-valued integral), and

$$\begin{aligned} \|k_i * f - f\|_p &\le \int_{\mathbb{R}^n \times W_i} \|\tau_{(x,y)} f - f\|_p k_i(x, y) d\lambda_{\mathbb{R}^n} \times \lambda_{G_0}(x, y) \\ &\le \int_{\mathbb{R}^n \times W_i} \omega\left(p;\ f;\ m_{U_i}(x) U_i \times m_{W_i}(y) W_i\right) k_i(x, y) d\lambda_{\mathbb{R}^n} \times \lambda_{G_0}(x, y) \\ &\le \omega\left(p;\ f;\ U_i \times W_i\right) \int_{\mathbb{R}^n \times W_i} m_{U_i}(x) m_{W_i}(y) k_i(x, y) d\lambda_{\mathbb{R}^n} \times \lambda_{G_0}(x, y) . \end{aligned}$$

It follows from [2], (13.12) that

$$\begin{aligned} \|k_i * f - f\|_p &\le \omega\left(p;\ f;\ V_i\right) \int_{\mathbb{R}^n} m_{U_i}(x) k_{U_i}(x) d\lambda_{\mathbb{R}^n}(x) \int_{W_i} m_{W_i}(y) k_{W_i}(y) d\lambda_{G_0}(y) \\ &\le (1+\varepsilon)\omega\left(p;\ f;\ V_i\right) , \end{aligned}$$

proving (1).

The proof of (2) is immediate since

$$K_i = \operatorname{supp} \hat{k}_i = \Sigma\left(k_i\right)$$

and hence $k_i * f \in L^p_{K_i}(G)$. //

If we partially order I so that

$$i \ge j \text{ if and only if } V_i \subset V_j ,$$

then the case $p = 1$ of Theorem 2 shows that the k_i form a bounded positive approximate unit in $L^1(G)$ $\left(cf.\ [3], (28.51)\right)$.

When G is connected, Theorem 2 will hold for any base of open neighbourhoods of zero since, by [2], (7.9), every neighbourhood of zero generates the group; in this case the proof is greatly simplified, needing

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only the corollary to Theorem 1 and the final two paragraphs of the proof of Theorem 2.

When G is totally disconnected, then (see [2], (7.7)) taking each V_i to be a compact open subgroup of G, Theorem 2 holds with $\varepsilon = 0$,

$$k_i = \lambda(V_i)^{-1}\xi_{V_i} ,$$

and

$$K_i = A(\Gamma, V_i)$$

(the annihilator of V_i in Γ); see [3], (31.7) (a).

In the classical situation when G is taken to be the circle group $\mathbf{T}$, it is easily shown that the so-called kernel $\{k_n\}_{n=1}^{\infty}$ of Fejér-Korovkin (see [4], p. 75 with $k_n = 2u_n$) satisfies the conditions of the corollary to Theorem 1 with $\varepsilon = 5$,

$$V_n = \{e^{i\theta} : \theta \in \mathbf{R} \text{ and } |\theta| < 1/n\}$$

and

$$\begin{aligned} K_n &= \operatorname{supp}\hat{k}_n \\ &= \{-n, -n+1, \ldots, n-1, n\} . \end{aligned}$$

This simple dependence of K_n on V_n also appears when $G = \mathbf{R}$.

THEOREM 3. *There exists a number $C > 0$ and $\{k_n\}_{n=1}^{\infty}$ with the following properties: for each $n \in \{1, 2, \ldots\}$, k_n is continuous and non-negative, $\int_{\mathbf{R}} k_n(x)dx = 1$, $\operatorname{supp}\hat{k}_n \subset [-n, n]$, and*

(1) $\|k_n*f-f\|_p \leq C\omega(p; f; (-1/n, 1/n))$,

(2) $E_{[-n,n]}(p; f) \leq C\omega(p; f; (-1/n, 1/n))$

for every $f \in L^p(\mathbf{R})$ if $p \in [1, \infty)$, or for every bounded uniformly continuous f if $p = \infty$.

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Proof. Choose g to be a non-negative function on $\mathbb{R}$ with two continuous derivatives such that $\operatorname{supp} g \subset [-1/2, 1/2]$ and $g(0) > 0$. Put

$$h = g*g_{\vee}/g*g_{\vee}(0) ,$$

where $g_{\vee} : x \to g(-x)$. Then $h, \hat{h} \geq 0$, $h(0) = 1$, $\operatorname{supp} h \subset [-1, 1]$, h has four continuous derivatives, and hence

$$\widehat{h^{(iv)}}(x) = x^4\hat{h}(x)$$

for all $x \in \mathbb{R}$. It follows that

$$\hat{h}(x) \leq B\left(1+x^4\right)^{-1} , \quad (-\infty < x < \infty) ,$$

where $B = \|h\|_1 + \|h^{(iv)}\|_1$.

Now define the continuous non-negative function $k \in L^1(\mathbb{R})$ by

$$\hat{k} = h ,$$

and for each $n \in \{1, 2, \ldots\}$, k_n by

$$k_n(x) = nk(nx) , \quad x \in \mathbb{R} .$$

Then k_n is non-negative, continuous and integrable, and for all $x \in \mathbb{R}$,

$$\hat{k}_n(x) = \hat{k}(x/n) .$$

Hence $\hat{k}_n(0) = 1$, $\operatorname{supp}\hat{k}_n \subset [-n, n]$, and

$$\begin{aligned}
\|k_n*f-f\|_p &\leq \int_{\mathbb{R}} \|\tau_x f-f\|_p k_n(x)dx \\
&\leq \omega\left(p;\ f;\ (-1/n,\ 1/n)\right) \int_{\mathbb{R}} m_{(-1/n,\ 1/n)}(x)k_n(x)dx \\
&\leq \omega\left(p;\ f;\ (-1/n,\ 1/n)\right) \int_{\mathbb{R}} \frac{B(1+|x|)}{1+x^4}\, dx \\
&\leq C\omega\left(p;\ f;\ (-1/n,\ 1/n)\right) ,
\end{aligned}$$

proving (1).

Once again (2) follows immediately from the fact that $\Sigma\left(k_n\right) \subset [-n, n]$. //

REMARK. It is easily shown that an analogue of Theorem 2, exhibiting

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a simple dependence of K_i on V_i, can be obtained for all groups of the form $\mathbb{R}^m \times \mathbb{T}^n \times G_0$, where m, n are non-negative integers and G_0 is totally disconnected.

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