Can. J. Math., Vol. XXX, No. 1, 1978, pp. 22–31

GEOMETRY ON THE UNIT BALL OF A COMPLEX HILBERT SPACE

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1. Introduction. Furnishing the open unit ball of a complex Hilbert space with the Carathéodory-differential metric, we construct a model $\mathscr{M}$ which plays a similar role as that of the Poincaré model for the hyperbolic geometry.

In this note we study the question whether or not through a point in the model $\mathscr{M}$ not lying on a given line there exists a unique perpendicular, and give a necessary and sufficient condition for the existence of a unique perpendicular. This enables us to divide a triangle into two right triangles. Many trigonometric identities in a general triangle are easy consequences of various identities which hold on a right triangle.

2. Construction of the model. Let $B = \{x : ||x|| < 1\}$ be the open unit ball in the complex Hilbert space H with the inner product (,) and the norm $|| \ ||$.

Let $\mathscr{H}(B, \Delta)$ be the class of holomorphic maps (in the sense of Fréchet) of B into the open unit disc $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbf{C}$. The Carathéodory-differential metric of B is defined by

$$(1) \qquad \alpha_B(x, \xi) = \sup\{|Df(x)\xi| : f \in \mathscr{H}(B, \Delta)\}, \quad x \in B, \xi \in H,$$

where $Df(x)$ denotes the Fréchet derivative of f at x. See [**1**] or [**8**] for details. This metric has most of the properties that the Poincaré metric does in the unit disc. It has the distance decreasing property, i.e., if $f : B \to B$ is a holomorphic map, then

$$(2) \qquad \alpha_B(f(x), Df(x)\xi) \leqq \alpha_B(x, \xi), \quad x \in B, \xi \in H.$$

In particular, if f is a holomorphic automorphism of B, then

$$(3) \qquad \alpha_B(f(x), Df(x)\xi) = \alpha_B(x, \xi), \quad x \in B, \xi \in H.$$

Therefore, the holomorphic automorphisms of B constitute the group of motions in the model $\mathscr{M} = (B, \alpha_B)$. The holomorphic automorphisms of B are given by the Möbius transformations of the form:

$$(4) \qquad y = T_{-a}(x) = \Gamma(a)\frac{x - a}{1 - (x, a)}, \quad a \in B, x \in B,$$

where

$$(5) \qquad \Gamma(a) = (1 - ||a||^2)^{1/2}(I - aa^*)^{-1/2}$$

Received June 22, 1976 and in revised form, May 31, 1977.

https://doi.org/10.4153/CJM-1978-002-6 Published online by Cambridge University Press

and I the identity operator in H. Furthermore,

(6) $$DT_{-a}(x) = (1 - ||a||^2)^{1/2}(I - aa^*)^{1/2}(I - xa^*)^{-1}/(1 - a^*x).$$

See [**6**]. It is an easy consequence of the Hahn-Banach theorem that for each non-zero vector $\xi \in H$,

(7) $$\alpha_B(0, \xi) = ||\xi||. \quad ([\mathbf{1}])$$

Since the Möbius transformation T_{-x} maps $x \in B$ to $0 \in B$, by (3) and (7),

(8) $$\alpha_B(x, \xi) = \alpha_B(0, DT_{-x}(x)\xi) = ||DT_{-x}(x)\xi||.$$

An explicit computation yields

(8′) $$\alpha_B(x, \xi) = [|(x, \xi)|^2 + (1 - ||x||^2)||\xi||^2]^{1/2}(1 - ||x||^2)^{-1}, \quad x \in B, \xi \in H.$$

The method of [**2**] may be carried over here to prove that for each non-zero $a \in B$ the line segment $\gamma_B(0, a) : \gamma(t) = at$, $t \in [0, 1]$, is the unique shortest geodesic between 0 and a with respect to the metric α_B, and that the length of $\gamma_B(0, a)$ is given by

(9) $$\gamma_B(0, a) = \frac{1}{2}\log\frac{1 + ||a||}{1 - ||a||} = \tan h^{-1}||a||.$$

The shortest geodesic $\gamma_B(a, b)$ between any two points a and b in B is then determined uniquely by the image curve of $\gamma_B(0, T_{-a}(b))$ under

(10) $$T_a : x \to \Gamma(a)\frac{x + a}{1 + (x, a)},$$

the inverse map of T_{-a}.

The length of $\gamma_B(a, b)$ is given by

(11) $$\rho_B(a, b) = \rho_B(0, T_{-a}(b)) = \tan h^{-1}||T_{-a}(b)||,$$

where

(12) $$\delta_B(a, b) = ||T_{-a}(b)|| = \frac{[|(a, b)|^2 - ||a||^2||b||^2 + ||a - b||^2]^{1/2}}{|1 - (a, b)|}.$$

The notion of angle can also be introduced in $\mathscr{M}$. Let $\mathscr{T}_x(H)$ denote the set of all (holomorphic) tangent vectors at $x \in H$. As in the finite dimensional case, $\mathscr{T}_x(H)$ may be turned into a complex Hilbert space isomorphic to $H = \mathscr{T}_0(H)$. The inner product of two vectors ξ and η in $\mathscr{T}_x(B)$ is defined by

(13) $$\alpha_B(x; \xi, \eta) = (DT_{-x}(x)\xi, DT_{-x}(x)\eta).$$

The angle between two non-zero vectors ξ and η in $\mathscr{T}_x(H)$ is defined by

(14) $$\cos\Theta = \frac{\operatorname{Re}\alpha_B(x; \xi, \eta)}{\alpha_B(x, \xi)\alpha_B(x, \eta)}, \quad 0 \leqq \Theta \leqq \pi.$$

In particular, if $x = 0$

https://doi.org/10.4153/CJM-1978-002-6 Published online by Cambridge University Press

(15) $$\cos\theta = \frac{\operatorname{Re}(\xi, \eta)}{||\xi||\,||\eta||}, \quad 0 \leqq \theta \leqq \pi,$$

gives the corresponding Hilbert space angle. Therefore, any given three points in $\mathscr{M}$ determine a triangle uniquely. A triangle in $\mathscr{M}$ consists of three sides given by the shortest geodesics and three angles defined by (14).

If two vectors ξ and η in $\mathscr{T}_x(H)$ are independent with respect to the real field $\mathbf{R}$, then

$$S_x(\xi, \eta) = \{\zeta \in \mathscr{T}_x(H) : \zeta = \alpha\xi + \beta\eta\}, \quad \alpha, \beta \in \mathbf{R},$$

determines a two (real) dimensional section in $\mathscr{T}_x(H)$. The section $S_x(\xi, \eta)$ is called *real* if $\operatorname{Im}(\xi, \eta) = 0$, and *holomorphic* if $\eta = \lambda\xi$ for some $\lambda \in \mathbf{C}$. The section $S_x(y - x, z - x)$ formed by three points x, y and z in B is called *tangentially-real* if

(16) $$\operatorname{Im}(T_{-x}(y), T_{-x}(z)) = 0.$$

3. Perpendicularity in $\mathscr{M}$.

THEOREM 1. *Let $\gamma(a, b)$ be the line in $\mathscr{M}$ determined by two distinct points a and b in B which does not pass through* 0. *Then a unique perpendicular line through* 0 *can be drawn to $\gamma(a, b)$ if and only if* $\operatorname{Im}(a, b) = 0$.

Proof. We follow the method of [**3**, Theorem 4.2]. The Möbius transformation

$$T_{-a} : x \to \Gamma(a)\frac{x - a}{1 - (x, a)}$$

maps a to 0 and b to $\beta = T_{-a}(b)$. Since the line $\gamma(0, \beta)$ joining 0 and β is given by $\gamma(t) = \beta t$, $t \in \mathbf{R}$, $\gamma(a, b)$ is the image of $\gamma(0, \beta)$ under the map

$$T_a : x \to \Gamma(a)\frac{x + a}{1 + (x, a)}$$

and it is given by

(1) $$\gamma : t \to \Gamma(a)\frac{\beta t + a}{1 + (\beta, a)t} = \frac{a + \Gamma(a)\beta t}{1 + (\beta, a)t}, \quad t \in \mathbf{R}.$$

A formal computation yields

(2) $$D\gamma(t) = \gamma'(t) = \frac{\Gamma(a)\beta - (\beta, a)a}{[1 + (\beta, a)t]^2}.$$

A perpendicular line from 0 to $\gamma(a, b)$ exists if and only if

(3) $$\alpha_B(\gamma(t); \gamma(t), \gamma'(t)) = 0$$

for some $t \in \mathbf{R}$. From (6) and (13),

(4) $$\alpha_B(\gamma(t); \gamma(t), \gamma'(t)) = \frac{(\gamma(t), \gamma'(t))}{1 - ||\gamma(t)||^2}.$$

Hence, a perpendicular line from 0 to $\gamma(a, b)$ exists if and only if

(5) $(\gamma(t), \gamma'(t)) = 0,$

or

(6) $[(\Gamma(a)\beta, a)(a, \beta) - ||\Gamma(a)\beta||^2]t = (a, \beta)(1 - ||a||^2)$

for some $t \in \mathbf{R}$. It follows from (4) and (5) of §2 through some calculations that

(7) $\Gamma(a)\beta = -a + \dfrac{1 - ||a||^2}{1 - (b, a)}\, b,$

(8) $(a, \beta) = \dfrac{(a, b) - ||a||^2}{1 - (a, b)},$

(9) $(\Gamma(a)\beta, a) = (\beta, a),$

(10) $(\Gamma(a)\beta, \Gamma(a)\beta) = (1 - ||a||^2)||\beta||^2 + (\beta, a)(a, \beta),$

and

(11) $(\beta, \beta) = \dfrac{|1 - (a, b)|^2 - (1 - ||a||^2)(1 - ||b||^2)}{|1 - (a, b)|^2}.$

From (6) together with subsequent identities, we have

(12) $t = -\dfrac{(a, \beta)}{(\beta, \beta)} = \dfrac{[||a||^2 - (a, b)][1 - (b, a)]}{|1 - (b, a)|^2 - (1 - ||a||^2)(1 - ||b||^2)}, \quad a \neq b,$

which implies that the equation (6) has exactly one real solution if and only if $\operatorname{Im}(a, b) = 0$.

COROLLARY 1. *Through a point c in $\mathscr{M}$ not lying on a given line γ in $\mathscr{M}$ there exists a unique perpendicular to γ if and only if there are two distinct points a and b on γ such that a, b and c form a tangentially-real section.*

Proof. By the Möbius transformation

$$T_{-c} : x \rightarrow \Gamma(c)\, \frac{x - c}{1 - (x, c)},$$

the points a, b and c are mapped to $T_{-c}(a)$, $T_{-c}(b)$ and 0, respectively. By Theorem 1, a unique perpendicular line from 0 to the line joining $T_{-c}(a)$ and $T_{-c}(b)$ exists if and only if $\operatorname{Im}(T_{-c}(a), T_{-c}(b)) = 0$, from Corollary 1 follows.

THEOREM 2. *Let $\gamma = \gamma(a, b)$ be the line determined by two distinct points a and b in $\mathscr{M}$ such that $\operatorname{Im}(a, b) = 0$. If the point $0 \in \mathscr{M}$ is not on γ, then the distance from 0 to γ is given by*

(13) $\rho_B(0, \gamma) = \tan h^{-1} \dfrac{[||a||^2||b||^2 - |(a, b)|^2]^{1/2}}{||a - b||}.$

Proof. By Theorem 1, the distance $\rho_B(0, \gamma)$ is given by the distance from 0 to $m = \gamma(t_0)$ with t_0 given in (12). From (1),

$$m = \gamma(t_0) = \frac{a||\beta||^2 - \Gamma(a)\beta(a, \beta)}{||\beta||^2 - |(a, \beta)|^2}, \tag{14}$$

$$\text{where } \beta = T_{-a}(b) = \Gamma(a)\frac{b-a}{1-(b, a)}.$$

A formal computation yields

$$||m||^2 = \frac{||a||^2||\beta||^2 - |(a, \beta)|^2}{||\beta||^2 - |(a, \beta)|^2} \tag{15}$$

when the formulas (9) and (10) are taken into account. From (8) and (11),

$$||\beta||^2 - |(a, \beta)|^2 = \frac{(1 - ||a||^2)||a - b||^2}{|1 - (a, b)|^2} \tag{16}$$

and

$$||a||^2||\beta||^2 - |(a, \beta)|^2 = \frac{(1 - ||a||^2)(||a||^2||b||^2 - |(a, b)|^2)}{|1 - (a, b)|^2}. \tag{17}$$

Therefore, Theorem 2 follows from (15), (16) and (17).

With the help of the Möbius transformation $T_{-c} : B \to B$ as in the proof of Corollary 1, we have

COROLLARY 2. *Let c be a point in $\mathcal{M}$ not lying on the line $\gamma = \gamma(a, b)$ determined by two points a and b in $\mathcal{M}$ such that a, b and c form a tangentially-real section. Then the distance from c to γ is given by*

$$\rho_B(c, \gamma) = \tan h^{-1}\left[\frac{||\alpha||^2||\beta||^2 - |(\alpha, \beta)|^2}{||\alpha||^2 + ||\beta||^2 - 2|(\alpha, \beta)|}\right]^{1/2}, \tag{18}$$

where $\alpha = T_{-c}(a)$ and $\beta = T_{-c}(b)$.

Remark. Any three distinct points a, b and c in the complex Hilbert space H which are not on the same line determines a triangle, $\boldsymbol{t}(abc)$, in H. The lengths of the sides of $\boldsymbol{t}(abc)$ are given by the Hilbert space norm, while the angles are measured by the usual Hilbert space angle, that is, the angle defined in terms of the inner product in H. It is easy to see that the length of the perpendicular from c to the opposite side is given by

$$\frac{[||a - c||^2||b - c||^2 - |(a - c, b - c)|^2]^{1/2}}{||a - b||} \tag{19}$$

and that

$$||a - b||^2||c - b||^2 - |(a - b, c - b)|^2 = ||b - c||^2||a - c||^2 - |(b - c, a - c)|^2 = ||c - a||^2||b - a||^2 - |(c - a, b - a)|^2. \tag{20}$$

Therefore, the area of $\boldsymbol{t}(abc)$ is well-defined and given by

$$\tfrac{1}{2}[||a - c||^2||b - c||^2 - |(a - c, b - c)|^2]^{1/2}. \tag{21}$$

If the section $S_c(a, b)$ is real, i.e., $\text{Im}(a - c, b - c) = 0$, then the area becomes

(22) $\frac{1}{2}[||a - c||^2||b - c||^2 - [\text{Re}(a - c, b - c)]^2]^{1/2}$.

THEOREM 3. *Let $\gamma(a, b)$ be the line determined by two distinct points a and b in $\mathcal{M}$ such that $\text{Im}(a, b) = 0$, and let $0 \notin \gamma(a, b)$. Suppose that the perpendicular from 0 to $\gamma(a, b)$ meets $\gamma(a, b)$ at m_0. Then $\rho_B(0, a) = \rho_B(0, b)$ if and only if $\rho_B(a, m_0) = \rho_B(b, m_0)$.*

Proof. The perpendicular line $\gamma(0, m_0)$ divides the triangle $\tau(0ab)$ into two right triangles: $\tau(0am_0)$ and $\tau(0bm_0)$. Applying the Pythagorean theorem (see Theorem 5 of §4) to each of these right triangles, we have

(23) $\cosh \rho_B(0, a) = \cosh \rho_B(0, m_0) \cosh \rho_B(a, m_0)$

and

(24) $\cosh \rho_B(0, b) = \cosh \rho_B(0, m_0) \cosh \rho_B(b, m_0)$.

Theorem 3 follows from (23) and (24).

COROLLARY 3. *Let $\tau(abc)$ be a triangle in $\mathcal{M}$ such that the vertices form a tangentially-real section. Let m_c be the point of intersection of $\gamma(a, b)$ and the perpendicular line from c to $\gamma(a, b)$. Then $\rho_B(a, c) = \rho_B(b, c)$ if and only if $\rho_B(a, m_c) = \rho_B(b, m_c)$.*

COROLLARY 4. *Let $\tau(abc)$ be a triangle in $\mathcal{M}$ whose vertices form a tangentially-real section. If $\rho_B(a, c) = \rho_B(b, c)$, then the perpendicular line through c to $\gamma(a, b)$ divides the triangle into two congruent right triangles.*

In the following we give a slight refinement of Theorem 3.

THEOREM 4. *Let $\gamma(a, b)$ and m_0 be given as in Theorem 3. The following hold:*
(1°) *$||a|| = ||b||$ if and only if m_0 lies at the midpoint of $\gamma(a, b)$.*
(2°) *Let $||a|| > ||b||$. If $||a||^2 > ||b||^2 > (a, b)$, then m_0 lies on the line segment $\gamma(a, b)$. If $||a||^2 > (a, b) > ||b||^2$, then m_0 lies on the extension of $\gamma(a, b)$ across b.*
(3°) *Let $||b|| > ||a||$. If $||b||^2 > ||a||^2 > (a, b)$, then m_0 lies on the line segment $\gamma(a, b)$. If $||b||^2 > (a, b) > ||a||^2$, then m_0 lies on the extension of $\gamma(a, b)$ across a.*

Proof. (1°) was proved in Theorem 3. Since $||\beta||^2 > 0$ from (11) we have

$$|1 - (b, a)|^2 - (1 - ||a||^2)(1 - ||b||^2) > 0.$$

Therefore, the denominator of t_0 in (12) is positive. If $||a|| > ||b||$, $t_0 > 0$ by the Schwarz inequality. Let

$$D \equiv (||a||^2 - (a, b))(1 - (a, b)) - |1 - (a, b)|^2 + (1 - ||a||^2)(1 - ||b||^2).$$

By simple calculation, $D = (1 - ||a||^2)((a, b) - ||b||^2)$. If $||b||^2 > (a, b)$, then $D < 0$, and hence, $t_0 < 1$. Thus, $||a||^2 > ||b||^2 > (a, b)$ implies $0 < t_0 < 1$. This means that $m_0 = \gamma(t_0)$ lies on $\gamma(a, b)$, since $\gamma(0) = a$ and $\gamma(1) = b$. If $(a, b) > ||b||^2$, then $D > 0$ and $t_0 > 1$. This implies that m_0 lies on the extension of $\gamma(a, b)$ across b. (3°) may be proved in the same manner.

4. Trigonometry in $\mathscr{M}$. As in the case of plane hyperbolic geometry, various trigonometric identities hold in $\mathscr{M}$. In this section we establish some of the basic trigonometric identities which hold in $\mathscr{M}$.

Let $\boldsymbol{\tau}(abc)$ be a triangle in $\mathscr{M}$ determined by the vertices a, b and c. We denote by Θ_a, Θ_b and Θ_c the angles of $\boldsymbol{\tau}(abc)$ at a, b and c, and by ρ_a, ρ_b and ρ_c the lengths of the sides opposite to a, b and c, respectively. First, we prove the Pythagorean theorem.

THEOREM 5. *Let $\boldsymbol{\tau}(abc)$ be a triangle in $\mathscr{M}$ whose vertices form a tangentially-real section. Then*

(1) $\quad \cosh \rho_c = \cosh \rho_a \cosh \rho_b$

holds if and only if $\Theta_c = \pi/2$.

Proof. Since the Möbius transformation

$$T_{-c}(x) = \Gamma(c) \frac{x - c}{1 - (x, c)}$$

maps a, b and c to $\alpha = T_{-c}(a)$, $\beta = T_{-c}(b)$ and $0 = T_{-c}(c)$, respectively, it is enough to prove that

(2) $\quad \cosh \rho_0 = \cosh \rho_\alpha \cosh \rho_\beta$

holds if and only if $(\alpha, \beta) = 0$. Clearly,

(3) $\quad ||\alpha - \beta||^2 = ||\alpha||^2 + ||\beta||^2$

holds if and only if $(\alpha, \beta) = 0$. From (11) and (12) of §2,

(4) $\quad ||\alpha - \beta||^2 = |1 - (\alpha, \beta)|^2 \tanh^2 \rho_0 + ||\alpha||^2||\beta||^2 - |(\alpha, \beta)|^2.$

Combining (3) and (4), we have that

(5) $\quad \tanh^2 \rho_0 = ||\alpha||^2 + ||\beta||^2 - ||\alpha||^2||\beta||^2$

if and only if $(\alpha, \beta) = 0$. (2) now follows from (5) when we observe: $||\alpha|| = \tanh \rho_\beta = \tanh \rho_b$, $||\beta|| = \tanh \rho_\alpha = \tanh \rho_a$ and the identity $\tanh^2 x = 1 - \operatorname{sech}^2 x$.

THEOREM 6. *Let $\boldsymbol{\tau}(abc)$ be a right triangle in $\mathscr{M}$ with right angle at b*, i.e., *$\Theta_b = \pi/2$. If the vertices of $\boldsymbol{\tau}(abc)$ form a tangentially-real section, then the following identities hold:*

(6a) $\quad \tanh \rho_a = \tanh \rho_b \cos \Theta_c$

(6b) $\quad \tanh \rho_c = \tanh \rho_b \cos \Theta_a$

(7a) $\quad \sinh \rho_c = \sinh \rho_b \sin \Theta_c$

(7b) $\sinh \rho_a = \sinh \rho_b \sin \Theta_a$

(8a) $\tanh \rho_c = \sinh \rho_a \tan \Theta_c$

(8b) $\tanh \rho_a = \sinh \rho_c \tan \Theta_a$

(9a) $\cos \Theta_c = \cosh \rho_c \sin \Theta_a$

(9b) $\cos \Theta_a = \cosh \rho_a \sin \Theta_c$

(10) $\cosh \rho_b = \cot \Theta_a \cot \Theta_c.$

Proof. Let α, β, and 0 be the image of a, b and c under $T_{-c} : B \rightarrow B$. Since $\text{Im}(\alpha, \beta) = 0$, $\text{Im}(\alpha - \beta, \beta) = 0$. A formal computation leads to

$$\cos^2 \theta_\beta = \cos^2 \Theta_\beta[1 - ||\beta||^2 + |(\alpha - \beta, \beta)|^2/||\alpha - \beta||^2].$$

This, together with $\cos \Theta_\beta = \cos \Theta_b = \cos \pi/2 = 0$ and

$$\cos \theta_\beta = \frac{\text{Re } (\alpha - \beta, \beta)}{||\alpha - \beta|| \, ||\beta||},$$

implies $\text{Re } (\alpha - \beta, \beta) = 0$, and hence $(\alpha - \beta, \beta) = 0$ or $(\alpha, \beta) = ||\beta||^2$. From (4),

(11) $||\alpha - \beta||^2 = (1 - ||\beta||^2) \tanh^2 \rho_0 + ||\beta||^2(||\alpha||^2 - ||\beta||^2).$

But $(\alpha - \beta, \beta) = 0$ if and only if $||\alpha||^2 - ||\beta||^2 = ||\alpha - \beta||^2$. Therefore, (11) becomes

(12) $||\alpha - \beta||^2 = \text{sech}^2 \rho_a \tanh^2 \rho_c$

when we observe: $\rho_0 = \rho_c$ and $\rho_\alpha = \rho_a$. Let $\boldsymbol{t}(0\alpha\beta)$ be the Hilbert space triangle with 0, α and β as its vertices. It is a right triangle with the right angle at β. Therefore, the following identities hold:

(13) $||\alpha|| \cos \Theta_0 = ||\beta||$

(14) $||\alpha|| \sin \Theta_0 = ||\alpha - \beta||.$

(15) $||\beta|| \tan \Theta_0 = ||\alpha - \beta||.$

Identity (6a) follows from (13), since $||\alpha|| = \tanh \rho_b$, $||\beta|| = \tanh \rho_a$ and $\Theta_0 = \Theta_c$, while (7a) follows from (14) together with (12) and the Pythagorean theorem. Identity (8a) is an immediate consequence of (15) and (12). Identities (6b), (7b) and (8b) may be obtained in the same way if the Möbius transformation

$$T_{-a}(x) = \Gamma(a) \frac{x - a}{1 - (x, a)}$$

is used, instead.

From (6a) and (6b), we get

(16) $\tanh \rho_a \cos \Theta_a = \tanh \rho_c \cos \Theta_c.$

(9a) and (9b) follow from (16) together with (8b) and (8a), while (10) is a consequence of (8a), (8b) and the Pythagorean theorem.

The following corollaries are simple conseqnences of the above theorems:

COROLLARY 5. *Let $\tau(abc)$ be a triangle in $\mathcal{M}$ whose vertices form a tangentially-real section. Then the sum of the angles of $\tau(abc)$ is less than π.*

Proof. Assume first that $\tau(abc)$ is a right triangle with right angle at b. By (10),

$$\cot \Theta_a \cot \Theta_c > 1$$

or

$$\tan(\pi/2 - \Theta_c) > \tan \Theta_a.$$

Since the tangent function is monotone increasing,

$$\pi/2 - \Theta_c > \Theta_a$$

or

(17) $\Theta_a + \Theta_c < \pi/2.$

Corollary 5 follows when we divide $\tau(abc)$ into two right triangles and use (17).

COROLLARY 6. (Law of Sines) *Let $\tau(abc)$ be a triangle in $\mathcal{M}$ whose vertices for a tangentially-real section. Then*

(18) $$\frac{\sin \Theta_a}{\sinh \rho_a} = \frac{\sin \Theta_b}{\sinh \rho_b} = \frac{\sin \Theta_c}{\sinh \rho_c}.$$

Proof. Draw a perpendicular line $\gamma(c, m_c)$ to $\gamma(a, b)$ through c. Applying (7) to the right triangles $\tau(acm_c)$ and $\tau(bcm_c)$ we have

(19) $\sinh \rho_a \sin \Theta_b = \sinh \rho_b \sin \Theta_a.$

Similarly, we have

(20) $\sinh \rho_b \sin \Theta_c = \sinh \rho_c \sin \Theta_b.$

Corollary 6 follows from (19) and (20).

COROLLARY 7. (Law of Cosines) *Let $\tau(abc)$ be a triangle in $\mathcal{M}$ whose vertices for a tangentially-real section. Then*

$$\cosh \rho_c = \cosh \rho_a \cosh \rho_b - \sinh \rho_a \sinh \rho_b \cos \Theta_c.$$

Proof. As in the proof of Corollary 6, we draw a perpendicular line $\gamma(a, m_a)$ to $\gamma(b, c)$ through a. Corollary 7 follows when we apply the Pythagorean theorem to the right triangles $\tau(abm_a)$ and $\tau(acm_a)$ and use the identity (6a).

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