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ON IMPLICATIONAL COMPLETENESS

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A class $\mathscr{K}$ of (universal) algebras [**4**; **5**] of the same type or species τ is called *equationally complete* if the variety [**4**] $\mathrm{Var}(\mathscr{K})$ generated by $\mathscr{K}$ has exactly two subvarieties — namely $\mathrm{Var}(\mathscr{K})$ itself and the class of all one element algebras. It follows that equationally complete varieties are the atoms in the complete lattice of all varieties of a given type of algebras. J. Kalicki, D. Scott [**7**; **8**; **9**; **10**] and others [**2**; **3**; **6**; **12**] have considered several questions about equational completeness. A good many of these results have appeared in books. (See, for instance, [**4**, **5**], to which we also refer the reader for more extensive bibliography on subjects related to equational completeness.)

In the present paper we consider the notion of implicational completeness: A class $\mathscr{K}$ of algebras is called *implicationally complete* if the quasivariety [**1**] $\mathrm{Qua}(\mathscr{K})$ generated by $\mathscr{K}$ has exactly two subquasivarieties—namely $\mathrm{Qua}(\mathscr{K})$ itself and the class of all one element algebras. As in the case of equational completeness to say that $\mathscr{K}$ is an implicationally complete class of algebras of a given type is to say that $\mathrm{Qua}(\mathscr{K})$ is an atom in the complete lattice of quasivarieties of algebras of that type. A class $\mathscr{K}$ can be implicationally complete without being equationally complete and conversely.

We begin in section 1 by defining atomic and semiprojective algebras and proving some lemmas concerning them. In section 2 we give a characterization (Theorem 1) of locally finite implicationally complete quasivarieties in terms of atomic algebras. Theorem 1 and lemmas of section 1 are applied in section 3 to obtain implicational completeness of some special algebras (e.g. two element algebras, primal and point-wise primal algebras, etc.). Theorem 2 gives all implicationally complete quasivarieties of semigroups.

1. Semiprojective and atomic algebras. An algebra on a set A will be usually denoted by $\bar{A}$. We write $Q(\bar{A})$ for the class of subalgebras of isomorphs of cartesian powers of $\bar{A}$.

Let $\mathscr{K}$ be a class of algebras (of the same type). We say that $\bar{A}$ is *semi-projective in* $\mathscr{K}$ if for every epimorphism $\bar{B} \rightarrow \bar{A}$, $\bar{B} \in \mathscr{K}$ there exists a monomorphism $\bar{A} \rightarrow \bar{B}$; i.e., if $\bar{A}$ is a homomorphic image of a $\bar{B}$ in $\mathscr{K}$ then $\bar{A}$ is embeddable in $\bar{B}$. As a justification for the name we note that if $\bar{A}$ is projective in $\mathscr{K}$ then $\bar{A}$ is semiprojective in $\mathscr{K}$. Let $\bar{B} \rightarrow \bar{A}$ be an onto homomorphism, $\bar{B} \in \mathscr{K}$. If $\bar{A}$ is projective then there exists a homomorphism $\bar{A} \rightarrow \bar{B}$ such that

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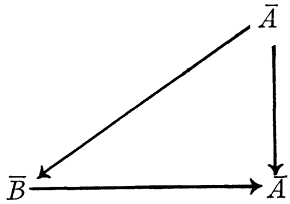
the diagram



commutes, where $\bar{A} \to \bar{A}$ is the identity homomorphism. Since $\bar{A} \to \bar{B} \to \bar{A}$ is mono we conclude that $\bar{A} \to \bar{B}$ is mono. Hence $\bar{A}$ is semiprojective. Note that this observation and the notion of semi-projectivity itself are purely categorical as is the first part of Lemma 1 below. Examples of semiprojective algebras that are not projective exist in abundance.

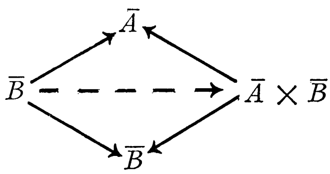
We call $\bar{A}$ *semiprojective* if $\bar{A}$ is semiprojective in $Q(\bar{A})$.

An algebra will be called *singleton* if it has only one element. A nonsingleton semiprojective algebra will be called *atomic* if it has no proper nonsingleton subalgebras.

LEMMA 1. (i) *Let* $\mathscr{K}$ *be closed under subdirect products. Then* $\bar{A} \in \mathscr{K}$ *is semiprojective in* $\mathscr{K}$ *if and only if* $\bar{A}$ *is embeddable in every subdirect product of* $\bar{A}$ *and* $\bar{B}$, *for all* $\bar{B} \in \mathscr{K}$.

(ii) *A nonsingleton finite algebra* $\bar{A}$ *is atomic if and only if* $\bar{A}$ *is embeddable in every nonsingleton subalgebra of* $\bar{A}^2$.

Proof. (i) Let $\bar{A}$ be semiprojective in $\mathscr{K}$ and let $\bar{C}$ be a subdirect product of $\bar{A}$ with another algebra $\bar{B} \in \mathscr{K}$. Then $\bar{C} \in \mathscr{K}$ and the projection map from C to A is an onto homomorphism and hence there exists an embedding $\bar{A} \to \bar{C}$. Conversely, let such an embedding exist for every such subdirect product, $\bar{C}$, and let $\phi : \bar{B} \to \bar{A}$ be an epimorphism. Then $\bar{B}$ is isomorphic to the subdirect product $\bar{C}$ of $\bar{A}$ and $\bar{B}$, where $C = \{\langle \phi(b), b \rangle, b \in B\}$. Hence $\bar{A}$ is embeddable in $\bar{B}$. (We note in passing that in arbitrary categories the embedding $\bar{A} \to \bar{B}$ will be obtained by considering the diagram



where $\bar{B} \to \bar{A}$ is the given epimorphism, $\bar{B} \to \bar{B}$ is identity, $\bar{A} \times \bar{B} \to \bar{A}$, $\bar{A} \times \bar{B} \to \bar{B}$ are projections. The map $\bar{B} \to \bar{A} \times \bar{B}$ exists by definition of $\bar{A} \times \bar{B}$ and makes the diagram commute and hence is mono. Under such circumstances $\bar{B}$ may be called a subdirect product of $\bar{A}$ and $\bar{B}$ and then the existence of a monomorphism $\bar{A} \to \bar{B}$ follows.)

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(ii) Let $\bar{A}$ be atomic and finite and let $\bar{B}$ be a subalgebra of $\bar{A}^2$. If $\pi_1 : \bar{B} \to \bar{A}$, $\pi_2 : \bar{B} \to \bar{A}$ denote the restrictions to $\bar{B}$ of the projections from $\bar{A}^2$ to $\bar{A}$, then $\pi_1(B)$, $\pi_2(\bar{B})$ are either singleton algebras or equal $\bar{A}$. If $\bar{B}$ is nonsingleton then at least one of $\pi_1(\bar{B})$, $\pi_2(\bar{B})$ is $\bar{A}$, say $\pi_1(\bar{B})$. Then $\bar{B}$ is a subdirect product of $\bar{A}$ and $\pi_2(\bar{B})$, $\pi_2(\bar{B}) \in Q(\bar{A})$. Hence $\bar{B}$ contains a subalgebra isomorphic to $\bar{A}$.

Conversely, let $\bar{A}$ be embeddable in every nonsingleton subalgebra of $\bar{A}^2$. $\bar{A}$ and hence every subalgebra of $\bar{A}$ is embeddable in $\bar{A}^2$. Thus $\bar{A}$ is embeddable in every nonsingleton subalgebra of $\bar{A}$, which means that $\bar{A}$ has no proper nonsingleton subalgebras. It remains to be shown that $\bar{A}$ is semiprojective (in $Q(\bar{A})$). For this it is enough to show that $\bar{A}$ is embeddable in $\bar{B}$ for all $\bar{B} \in Q(\bar{A})$. If $\bar{A}$ is finite then every finitely generated algebra in $Q(\bar{A})$ is finite and hence embeddable in $\bar{A}^n$ for some positive integer n. Thus we need only show that $\bar{A}$ is embeddable in every nonsingleton subalgebra of $\bar{A}^n$ for all positive integers n. Since by assumption this last assertion is true for $n = 2$ and since $\bar{A}^n$ is isomorphic to $\bar{A}^{n-1} \times \bar{A}$ the proof can be concluded by induction without difficulty.

A somewhat restricted but well-studied class of atomic algebras is the class of functionally complete or primal algebras. (See, for example, [**2**] for definition, results and references to original papers concerning primal algebras.) A wider class is that of what we call *point-wise primal algebras*: $\bar{A}$ is point-wise primal if $\bar{A}$ is finite, nonsingleton and for every $a \in A$ there exists a unary polynomial [**2**] $u(x)$ such that $u(b) = a$, for all b in A. Every primal algebra is point-wise primal but not conversely. Atomicity of point-wise primal algebras $\bar{A}$ can be easily seen as follows: For every $a \in A$ let $u_a(x)$ be a polynomial such that $u_a(b) = a$ for all $b \in A$. It follows that $\bar{A}$ is generated by every element of $\bar{A}$. Now if $\bar{B}$ is a subalgebra of $\bar{A}^2$ and $\langle a, b\rangle \in B$ then $u_a(\langle a, b\rangle) = \langle a, a\rangle \in B$ and the subalgebra of $\bar{B}$ generated by $\langle a, a\rangle$ is isomorphic to $\bar{A}$.

A class of atomic algebras (not contained in the class of point-wise primal algebras) is given by the following lemma.

LEMMA 2. *Every two element algebra is atomic.*

Proof. Let $\bar{A}$ be any algebra on $A = \{a, b\}$ and let $\bar{B}$ be a nonsingleton subalgebra of $\bar{A}^2$. If $C = \{\langle a, a\rangle, \langle b, b\rangle\} \subseteq B$ then the subalgebra of $\bar{B}$ on C is isomorphic to $\bar{A}$. Let us assume $C \not\subseteq B$, so that $|B| = 2$ or 3. If $|B| = 3$ then we can take $B = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle\}$. We show that $\{a\}$ forms a subalgebra of $\bar{A}$ and hence $\{\langle a, a\rangle, \langle a, b\rangle\}$ forms a subalgebra of $\bar{B}$ isomorphic to $\bar{A}$. If $\{a\}$ is not a subalgebra of $\bar{A}$ then there exists a polynomial $w(x)$ such that $w(a) = b$. Then $w(\langle a, a\rangle) = \langle w(a), w(a)\rangle = \langle b, b\rangle$. Hence $\langle b, b\rangle \in B$, a contradiction. The case $|B| = 2$ is easy too. Let π_1, π_2 be the projections from $\bar{A}^2$ to $\bar{A}$. Then one of $\pi_1(\bar{B})$, $\pi_2(\bar{B})$ must have two elements, say $\pi_1(\bar{B})$. This means that the restriction of π_1 to $\bar{B}$ is an isomorphism onto $\bar{A}$.

2. Implicational completeness of locally finite classes.

From the point of view of implicational completeness, as will become apparent in the sequel, there are two sharply distinguished types of quasivarieties: Those that have a non-singleton finite algebra and those that do not. In the first case we have the following result.

THEOREM 1. *Let* $\mathscr{K}$ *be a quasivariety containing a nonsingleton finite algebra. Then* $\mathscr{K}$ *is implicationally complete if and only if* $\mathscr{K} = Q(\bar{A})$ *for some finite atomic algebra* $\bar{A}$.

Proof. Let $\mathscr{K}$ be implicationally complete and let $\bar{F} \in \mathscr{K}$, where $|F|$ is finite and greater than 1. Then there exists a minimal subalgebra $\bar{A}$ of $\bar{F}$. We call an algebra minimal if it is nonsingleton and has no proper nonsingleton subalgebras. By [**8**], $Q(\bar{A})$ is a locally finite quasivariety and by implicational completeness and $Q(\bar{A}) \subseteq \mathscr{K}$ we deduce $\mathscr{K} = Q(\bar{A})$. It remains to be shown that $\bar{A}$ is semiprojective in $Q(\bar{A})$. By local finiteness of $Q(\bar{A})$ it is enough to show that $\bar{A}$ is embeddable in every nonsingleton finite algebra $\bar{B}$ of $Q(\bar{A})$. Since every such $\bar{B}$ contains a minimal algebra it is enough to show that every two minimal algebras in $Q(\bar{A})$ are isomorphic. Let then $\bar{A}'$ be a minimal algebra of $Q(\bar{A})$. Then by local finiteness of $Q(\bar{A})$, $\bar{A}'$ must be finite and as in the case of $\bar{A}$ we deduce $\mathscr{K} = Q(\bar{A}')$. By $\bar{A}' \in Q(\bar{A})$ and minimality of $\bar{A}$ it follows that there exists an onto homomorphism $\phi : \bar{A}' \rightarrow \bar{A}$. Hence $|A'| \geqq |A|$. Similarly $|A| \geqq |A'|$. Hence $|A| = |A'|$ and ϕ must be an isomorphism.

Conversely, let $\mathscr{K} = Q(\bar{A})$, where $\bar{A}$ is finite atomic. Let $\mathscr{K}'$ be a subquasivariety of $\mathscr{K}$ with $\bar{B} \in \mathscr{K}'$, $|B| > 1$. We show that $\mathscr{K}' = \mathscr{K}$. Since $\bar{B} \in Q(\bar{A})$ and $\bar{A}$ is minimal we see that there exists an epimorphism $\bar{B} \rightarrow \bar{A}$. By semiprojectivity of $\bar{A}$ there exists an embedding $\bar{A} \rightarrow \bar{B}$. Hence $\bar{A} \in \mathscr{K}'$ and $Q(\bar{A}) \subseteq \mathscr{K}' \subseteq \mathscr{K} = Q(\bar{A})$. This completes the proof of the theorem.

The above proof shows that minimal algebras of a locally finite, implicationally complete quasivariety are isomorphic. The converse of this is false, as is proved by the following example: Let $\bar{A}$ be any groupoid on $\{a, b\}$ such that $a^2 = b^2 = a$. Let $\bar{B}$ be a groupoid obtained from $\bar{A}$ by adding a new element c satisfying $c^2 = b$. Then $\bar{B} \notin Q(\bar{A})$, since $\bar{A}$ satisfies the identity $x^2 = y^2$ while $\bar{B}$ does not. Hence $Q(\bar{A})$ is a proper subquasivariety of $Q(\bar{B})$ and the latter is therefore not implicationally complete. However, it is easily verified that every nonsingleton minimal subalgebra in $Q(\bar{B})$ is isomorphic to $\bar{A}$.

The following corollary to Theorem 1 needs no proof.

COROLLARY 1. *Implicationally complete subquasivarieties of a locally finite quasivariety* $\mathscr{K}$ *are in one-to-one correspondence with the isomorphism classes of atomic algebras in* $\mathscr{K}$.

COROLLARY 2. *If* Ω *has operators of positive rank then there are infinitely many implicationally complete, locally finite quasivarieties of* Ω*-algebras.*

Proof. Let $C_p = \{1, \ldots, p\}$, where p is any positive integer. We define an

Ω-algebra $\bar{C}_p$ on C_p as follows: For $a_1, \ldots, a_n, \ldots \in C_p$, $\omega \in \Omega$ define

$\omega = p$, if ω is nullary,

$(a_1, \ldots, a_n)\omega = a_1 + 1(\bmod p)$, if ω is not nullary.

We leave the easy details of the verification of the fact that all algebras so defined are atomic by Lemma 1. The proof then is completed by Corollary 1.

3. Some implicationally complete algebras. For implicationally complete quasivarieties of semigroups, as in the case of their equationally complete varieties [**5**], it is possible to give a reasonably complete characterization.

THEOREM 2. *A quasivariety $\mathscr{K}$ of semigroups is implicationally complete if and only if $\mathscr{K}$ is generated by one of the following types of semigroups:*

(1) *Two element semigroups and groups of prime order* (*as semigroups*).

(2) *The free monogenic semigroup.*

Proof. Let $\mathscr{K}$ be an implicationally complete quasivariety of semigroups. First note that if the idempotence law $x^2 = x$ holds in a nonsingleton semigroup $\bar{S}$ of $\mathscr{K}$ then for $s, t \in S$, $s \neq t$, at least one of the sets $\{s, t\}$, $\{s, st\}$, $\{s, ts\}$, $\{s, sts\}$ forms a two element subsemigroup of $\bar{S}$; this two element semigroup must generate $\mathscr{K}$, by the implicational completeness of $\mathscr{K}$. We therefore assume that $x^2 = x$ does not hold in any nonsingleton semigroup of $\mathscr{K}$. Then the free monogenic semigroup $F_1(\mathscr{K})$ of $\mathscr{K}$ consists of more than one element and hence generates $\mathscr{K}$. If $F_1(\mathscr{K})$ is finite and has order greater than 2 then it follows from elementary semigroup theory that $F_1(\mathscr{K})$ contains a group of prime order or a two element semigroup; so that $\mathscr{K}$ must be generated by a semigroup of type (1). If $F_1(\mathscr{K})$ is infinite then it is isomorphic to the monogenic free semigroup.

It remains to be seen that every semigroup of types (1) and (2) generates an implicationally complete quasivariety. Every semigroup of type (1) is atomic, by Lemma 2 and elementary group theory. Hence in view of Theorem 1 we need only consider the case of the quasivariety $\mathscr{K}^*$ generated by the free monogenic semigroup. Implicational completeness of $\mathscr{K}^*$, in this case, follows by noting that every semigroup in $\mathscr{K}^*$ is either infinite or singleton and every infinite monogenic semigroup is free.

The implicationally complete quasivarieties generated by semigroups of type (1) are all varieties and precisely those that are shown by J. Kalicki and D. Scott [**5**] to be the only equationally complete varieties of semigroups. The quasivariety $\mathscr{K}^*$ generated by the free monogenic semigroup F_1 is proper, since the finite nonsingleton homomorphic images of F_1 do not belong to $\mathscr{K}^*$, as has been noted above. Thus locally finite implicationally complete quasivarieties of semigroups are the same as the equationally complete varieties and there is exactly one locally nonfinite implicationally complete quasivariety of semigroups.

It follows immediately from Lemma 2 and Theorem 1 that every two element algebra is implicationally complete. As noted above every two element semigroup is equationally complete too. In the rest of this section we shall be looking at two element groupoids for equational completeness.

First we list all two element groupoids on $\{a, b\}$ (within isomorphism of course). There are ten of them:

S^0, the two element semilattice
S^+, the trival semigroup satisfying $xy = x$
S^-, the trivial semigroup satisfying $xy = y$
A_1^0 the "fixated" semigroup satisfying $xy = x'y'$
A_2^0, the abelian group of order 2
A^+, the groupoid defined by: $a^2 = b^2 = a, ab = a, ba = b.$
A^-, the groupoid defined by: $a^2 = b^2 = a, ab = b, ba = a$
G^0, the groupoid defined by: $a^2 = b, b^2 = a, ab = ba = a$
G^+, the groupoid defined by: $a^2 = b, b^2 = a, ab = a, ba = b$
G^-, the groupoid defined by: $a^2 = b, b^2 = a, ab = b, ba = a.$

In S^0, S^+, S^- both a and b are idempotent. A_1^0, A_2^0, A^+, A^- have exactly one idempotent element and one element which generates them. In G^0, G^+, G^- no element is idempotent and both elements are generators. S^0, S^+, S^-, A_1^0, A_2^0 are semigroups while the other five groupoids are not. S^0, A_1^0, A_2^0, G^0 are commutative. In S^+, A^+, G^+ $xy = x$ if $x \neq y$ and in S^-, A^-, G^- $xy = y$ if $x \neq y$. The varieties generated by the five semigroups are well-known. A^-, G^- behave like A^+, G^+ respectively. We therefore consider A^+, G^+, G^0.

A^+ is equationally complete.

Proof. The identities $x^2 = y^2$, $x^2x = x^2$, $xx^2 = x$ can be easily checked to hold in A^+. These identities show that the free monogenic groupoid in $\mathrm{Var}(A^+)$ has at most two elements. A^+, being monogenic, is a homomorphic image of this free groupoid. Hence A^+ is free in $\mathrm{Var}(A^+)$. If V' is a subvariety of $\mathrm{Var}(A^+)$ then the free monogenic groupoid, being a homomorphic image of A^+ is either A^+ or the singleton groupoid. In the first case $A^+ \in V'$ and $V' = \mathrm{Var}(A^+)$ and in the second case $x^2 = x$ is an identity of V', which by $x^2 = y^2$, implies $x = y$.

G^0 is also equationally complete, since G^0 is nothing but the two element Boolean algebra with the Sheffer stroke.

But G^+ is *not* equationally complete. *There is exactly one subvariety of* $\mathrm{Var}(G^+)$ *and that is defined by* $xy = y$.

Proof. G^+ satisfies the identities: $xy = y^2$, $(x^2)^2 = x$. As in the case of A^+ we can easily see that a proper subvariety of $\mathrm{Var}(G^+)$ must have $x^2 = x$, as one of its identities. By $xy = y^2$ we deduce $xy = y$. Since this latter identity defines equationally complete variety we see that there is at most one proper subvariety of $\mathrm{Var}(G^+)$. To conclude the proof, therefore, it is enough to show that $xy = y$ indeed defines a proper subvariety of $\mathrm{Var}(G^+)$, i.e., there is a non-singleton groupoid in $\mathrm{Var}(G^+)$ which satisfies $xy = y$. Consider the equivalence relation θ over $G^+ \times G^+$, with partition classes $\{\langle a, a\rangle, \langle b, b\rangle\}$, $\{\langle a, b\rangle, \langle b, a\rangle\}$. Then it is easy to verify that θ is a congruence and $G^+ \times G^+/\theta$ satisfies $xy = y$.

The example of G^+ shows, in particular, that there are implicationally complete finite algebras that are not equationally complete.

We conclude this paper by the following

Remark. By a well-known result of Lindenbaum, first proved for the sentential calculus but true for a much wider class of systems (cf. [**11**, Theorem 56], it follows that every quasivariety with nonsingleton algebras contains an implicationally complete quasivariety. This together with the main result of [**7**] then shows that there are uncountably many implicationally complete quasivarieties of groupoids..

More generally, there are at least as many implicationally complete subquasivarieties of a variety as there are equationally complete subvarieties. This observation has little connection with the fact that every variety is a quasivariety. More precisely we can have a complete sublattice L' of a complete lattice L such that L' has more atoms than L. (Example: Take L to be the set of non-negative real numbers plus ∞ and L' to be the subset consisting of ∞ and the non-negative integers.)

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