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BANACH SPACES THAT ARE UNIFORMLY ROTUND IN WEAKLY COMPACT SETS OF DIRECTIONS

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Introduction. In a Banach space, the *directional modulus of rotundity*, $\delta(\epsilon, z)$, measures the minimum depth at which the midpoints of all chords of the unit ball which are parallel to z and of length at least ϵ are buried beneath the surface. A Banach space is *uniformly rotund in every direction* (*URED*) if $\delta(\epsilon, z)$ is positive for every positive ϵ and every nonzero element z. This concept of directionalized uniform rotundity was introduced by Garkavi [**6**] to characterize those Banach spaces in which every bounded subset has at most one Čebyšev center. More interest in directionalized uniform rotundity was aroused by Zizler [**10**] who showed that a Banach space that is *URED* possesses normal structure, a property that quarantees the existence of fixed points for certain nonexpansive mappings (see Day [**2**, p. 106]).

In this paper we express several well known rotundity conditions in terms of the directional modulus of rotundity. The various conditions are organized by assuming a positive minimum of $\delta(\epsilon, z)$ as z varies over sets in prescribed families of subsets. This systematization lists the familiar conditions in order of strength and isolates a new condition, *uniformly rotund in weakly compact sets of directions*. This new notion is investigated and its relationships to the known rotundity notions are established.

1. Definitions and preliminaries. For a Banach space B, the symbols Σ and U denote the unit sphere and closed unit ball of B respectively. The symbols Σ' and U' denote the analogous sets in the conjugate space B^*. A subspace of B always means a nonempty closed linear manifold of B with the norm induced from B.

For an index set S, the Banach spaces $m(S)$, $c_0(S)$, and $l^p(S)$, where $1 \leqq p < \infty$, are defined as in Day [**2**, p. 32]. Unless otherwise stated, $\|\cdot\|_p$ denotes the usual norm on $l^p(S)$. For a full function space X on S and a collection $\{B_s: s \in S\}$ of Banach spaces, the product space $P_X B_s$ is defined as in Day [**2**, p. 35]. The letter N denotes the natural numbers. In particular, c_0 and l^p denote the spaces $c_0(N)$ and $l^p(N)$; in these spaces let $\{e_n: n \in N\}$ denote the usual unit vector basis.

Definition 1.1 (Clarkson [**1**]). A Banach space B is *uniformly rotund* (*UR*)

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if and only if for every $0 < \epsilon \leqq 2$, there exists a $\delta > 0$ such that $||\frac{1}{2}(x + y)|| \leqq 1 - \delta$ whenever $x, y \in \Sigma$ and $||x - y|| \geqq \epsilon$.

Definition 1.2. A Banach space B is $UR^{A'}$, where A' is a nonempty subset of B^*, if and only if, for any pair of sequences $\{x_n\}$ and $\{y_n\}$ in Σ, if $||x_n + y_n|| \to 2$ then $f(x_n - y_n) \to 0$ for all f in A'. In particular, (Šmul'yan [**8**]) a Banach space B is *weakly uniformly rotund* (WUR) if B is UR^{B^*}, and B^* is *weak* uniformly rotund* (W^*UR) if B^* is $UR^{Q(B)}$, where $Q: B \to B^{**}$ is the canonical embedding.

Definition 1.3. Let B be a Banach space. For every nonzero element z in B and $0 < \epsilon \leqq 2$, define the *directional modulus of rotundity*, $\delta(\epsilon, z)$, by

$$\delta(\epsilon, z) \equiv \inf \{1 - ||\tfrac{1}{2}(x + y)||: x, y \in \Sigma, x - y = \alpha z, \text{and } ||x - y|| \geqq \epsilon\}.$$

Then B is UR_A, where A is a nonempty subset of B, if and only if $\delta(\epsilon, z) > 0$ for every nonzero element z in A and $0 < \epsilon \leqq 2$. In particular, (Garkavi [**6**]) a Banach space B is *uniformly rotund in every direction* ($URED$) if B is UR_B.

Definition 1.4. A Banach space B is *rotund* (R) if and only if $||x + y|| < 2$ whenever $x, y \in \Sigma$ and $x \neq y$.

The following theorem is due to Zizler [**10**, Proposition 14] and will be used in Section 2. The statement given here is slightly different from that given by Zizler.

THEOREM 1.5. *Let $(B_1, ||\cdot||_1)$ and $(B_2, ||\cdot||_2)$ be Banach spaces and let $T: B_1 \to B_2$ be a continuous linear mapping. Then $|||\cdot|||$, defined for x in B_1 by*

$$|||x||| = (||x||_1^2 + ||Tx||_2^2)^{1/2},$$

is an equivalent norm on B_1. Furthermore, if B_2 is $UR_{T(B_1)}$ and B_1 is $UR_{K(T)}$, where $K(T) \equiv \{x \in B_1: Tx = 0\}$, then $|||\cdot|||$ is URED.

2. Banach spaces that are URWC. In this section the rotundity notions UR, WUR, W^*UR, and $URED$ are all expressed in terms of the directional modulus of rotundity. A general viewpoint is given from which these well known rotundity conditions can be seen in order of strength and from which a missing link in the directional uniform rotundity chain can be identified.

Definition 2.1. Let B be a Banach space and let $\mathscr{A}$ be a nonempty collection of nonempty subsets of $B\backslash\{0\}$. For every $A \in \mathscr{A}$ and $0 < \epsilon \leqq 2$, define

$$\delta(\epsilon, A) \equiv \inf \{\delta(\epsilon, z): z \in A\}.$$

Then B is $UR_{\mathscr{A}}$ if and only if $\delta(\epsilon, A) > 0$ for every $A \in \mathscr{A}$ and $0 < \epsilon \leqq 2$.

THEOREM 2.2. *For a Banach space B, let $\mathscr{A}$ be the collection of all norm closed and bounded nonempty subsets of $B\backslash\{0\}$, let $\mathscr{B}$ be the collection of all weakly closed and bounded nonempty subsets of $B\backslash\{0\}$, let $\mathscr{C}$ be the collection of all weak** *closed and bounded (equivalently, weak* compact) nonempty subsets of* $B^*\backslash\{0\}$, *and let* $\mathscr{D}$ *be the collection of all norm compact nonempty subsets of* $B\backslash\{0\}$. *Then*

(i) *B is UR if and only if B is* $UR_{\mathscr{A}}$,
(ii) *B is WUR if and only if B is* $UR_{\mathscr{B}}$,
(iii) B^* *is W*UR if and only if* B^* *is* $UR_{\mathscr{C}}$,
(iv) *B is URED if and only if B is* $UR_{\mathscr{D}}$.

Proof. (i). This equivalence is immediate since $\Sigma \in \mathscr{A}$, and B is *UR* if and only if $\delta(\epsilon, \Sigma) > 0$ for each $0 < \epsilon \leqq 2$.

(ii). If B is not $UR_{\mathscr{B}}$, then there exist $\epsilon > 0$ and $A \in \mathscr{B}$ such that $\delta(\epsilon, A) = 0$. For each $n \in N$, choose $z_n \in A$ and $x_n, y_n \in \Sigma$ such that $x_n - y_n = \alpha_n z_n$, $||x_n - y_n|| \geqq \epsilon$, and $||x_n + y_n|| \geqq 2 - 1/n$. Since $\{z_n\} \subset A$, the sequence $\{||z_n||\}$ is bounded and bounded away from zero; since $\epsilon \leqq |\alpha_n|\, ||z_n|| \leqq 2$, the same is true of the sequence $\{\alpha_n\}$. Since $z_n \nrightarrow 0$ weakly, it follows that $x_n - y_n \nrightarrow 0$ weakly, and hence B is not *WUR*.

Conversely, if B is not *WUR*, then there exist $\epsilon > 0, f \in \Sigma'$, and sequences $\{x_n\}$ and $\{y_n\}$ in Σ such that $||x_n + y_n|| \to 2$ and $f(x_n - y_n) \geqq \epsilon$ for all n. Let $A = \{x \in B: f(x) \geqq \epsilon\} \cap 2U$. Note that $A \in \mathscr{B}$, $||x_n - y_n|| \geqq \epsilon$, and $x_n - y_n \in A$. Hence

$$0 = \inf_n \delta(\epsilon, x_n - y_n) \geqq \delta(\epsilon, A) \geqq 0,$$

and B is not $UR_{\mathscr{B}}$.

(iii). The proof of (iii) is analogous to that of (ii).

(iv). If B is not $UR_{\mathscr{D}}$, then there exist $\epsilon > 0$ and $A \in \mathscr{D}$ such that $\delta(\epsilon, A) = 0$. Choose sequences $\{x_n\}$, $\{y_n\}$, and $\{\alpha_n\}$ as in the proof of (ii). By taking subsequences if necessary, we may assume that $\alpha_n \to \alpha \neq 0$ and $z_n \to z \in A$. Hence $x_n - y_n = \alpha_n z_n \to \alpha z \neq 0$, and B is not *URED* by Theorem 1 of [**4**].

Conversely, if B is $UR_{\mathscr{D}}$, then clearly B is *URED*. This completes the proof of Theorem 2.2.

In view of the preceding theorem the following definition arises quite naturally.

Definition 2.3. Let B be a Banach space and let $\mathscr{W}$ be the collection of all weakly compact nonempty subsets of $B\backslash\{0\}$. Then B is *uniformly rotund in weakly compact sets of directions* (*URWC*) if B is $UR_{\mathscr{W}}$.

As immediate consequences of Theorem 2.2, (1) the properties *WUR* and *URWC* coincide in reflexive Banach spaces, and (2) the properties *URWC* and *URED* coincide in Banach spaces with the Schur property (that is, in spaces in which norm and weak convergence of sequences coincide). Employing the same technique as in the proof of Theorem 2.2, it can be shown that if B is $UR^{A'}$, where A' is a subset of B^* that is total over B, then B is *URWC*. It is trivial that *URWC* is a formally stronger notion than *URED*.

The following theorem gives some equivalent formulations of the property *URWC*. That condition (i) is equivalent to *URWC* follows from the techniques given in Garkavi [**6**], Zizler [**10**], and the proof of Theorem 2.2. Note that (ii) is equivalent to (i) since $2(||x_n||^2 + ||y_n||^2) - ||x_n + y_n||^2 \to 0$ implies $||x_n|| - ||y_n|| \to 0$.

THEOREM 2.4. *For a Banach space B, each of the following statements is equivalent to the statement that B is URWC.*

(i) *If $\{x_n\}$ and $\{y_n\}$ are sequences in Σ such that $||x_n + y_n|| \to 2$ and $x_n - y_n \to z$ weakly, then $z = 0$.*

(ii) *If $\{x_n\}$ and $\{y_n\}$ are sequences in B such that $\{x_n\}$ is bounded, $2(||x_n||^2 + ||y_n||^2) - ||x_n + y_n||^2 \to 0$, and $x_n - y_n \to z$ weakly, then $z = 0$.*

In Theorem 1.5, if $T: B_1 \to B_2$ is one-to-one and if B_2 is *URED*, then $||| \cdot |||$ is *URED*. This method of injecting a space into a space which possesses a certain rotundity property and then pulling that property back to the domain space via $||| \cdot |||$ was introduced by Clarkson [**1**] when he showed that every separable Banach space has an equivalent norm that is *R*. Clarkson's result was improved by Zizler [**10**] who used Theorem 1.5 to show that every separable Banach space has an equivalent norm that is *URED*. The property *URWC* can also be pulled back by Clarkson's method, as is shown by the following theorem.

THEOREM 2.5. *Let $(B_1, ||\cdot||_1)$ and $(B_2, ||\cdot||_2)$ be Banach spaces and let $T: B_1 \to B_2$ be a one-to-one continuous linear mapping. If B_2 is URWC, then $||| \cdot |||$, defined for $x \in B_1$ by*

$$|||x||| = (||x||_1^2 + ||Tx||_2^2)^{1/2},$$

is an equivalent norm on B_1 that is URWC.

Proof. That $||| \cdot |||$ is an equivalent norm on B_1 is immediate. To show that $||| \cdot |||$ is *URWC*, let $\{x_n\}$ and $\{y_n\}$ be sequences in B_1 such that $\{x_n\}$ is bounded,

$$2(|||x_n|||^2 + |||y_n|||^2) - |||x_n + y_n|||^2 \to 0,$$

and $x_n - y_n \to z$ weakly. Then $\{Tx_n\}$ is bounded in B_2,

$$2(||Tx_n||_2^2 + ||Ty_n||_2^2) - ||Tx_n + Ty_n||_2^2 \to 0,$$

and $Tx_n - Ty_n \to Tz$ weakly. Since B_2 is *URWC*, it follows that $Tz = 0$, and hence $z = 0$. This shows that $||| \cdot |||$ is *URWC*.

The following corollary strengthens a result of Zizler [**10**].

COROLLARY 2.6. *If B is a Banach space such that B^* contains a countable total set over B (for example, if B is separable), then B has an equivalent norm that is URWC.*

Proof. Let $\{f_i\}$ in Σ' be total over B. Then, as noted by Zizler [**10**], the mapping given by $Tx = \{f_i(x)/2^i\}$ is a one-to-one continuous linear mapping of B into the uniformly rotund space l^2.

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The following examples show that the notion *URWC* is distinct from the other directional uniform rotundity notions defined in Section 1. Specifically, the property *URWC* is weaker than *WUR*, weaker than *W*UR* in a conjugate space, and stronger than *URED*.

Example 2.7. Since $C[0, 1]$ is separable, it has an equivalent norm that is *URWC* by Corollary 2.6. But $C[0, 1]$ has no equivalent norm that is *WUR*, as noted by Zizler [**10**]. Note this example shows that the property *WUR* cannot be pulled back by the method discussed before Theorem 2.5.

Example 2.8. An equivalent conjugate norm is defined on l^1 that is *URED*, and hence *URWC* since l^1 has the Schur property, but not *W*UR*.

Define $T: l^1 \to l^2$ by $T(x^1, x^2, \ldots) = (x^2, x^3, \ldots)$. Then T is a continuous linear mapping. For $x \in l^1$, define

$$||x||_C = (||x||_1^2 + ||Tx||_2^2)^{1/2}.$$

Then $||\cdot||_C$ is an equivalent norm on l^1. Note that T is an adjoint mapping (It is, in fact, S^* where $S: l^2 \to c_0$ is given by $S(y^1, y^2, \ldots) = (0, y^1, y^2, \ldots)$.) and hence T is weak*-weak* continuous. It follows that $||\cdot||_C$ is a conjugate norm.

To show that $||\cdot||_C$ is *URED* it suffices by Theorem 1.5 to show that $||\cdot||_1$ is $UR_{\{e_1\}}$. For this, let $\epsilon > 0$ be given and let $x = (x^j)_{j=1}^\infty$ and $y = (y^j)_{j=1}^\infty$ in l^1 be such that $||x||_1 = ||y||_1 = 1$ and $x - y = \epsilon e_1$. Then $x^j = y^j$ for all $j \geqq 2$, and hence $x^1 = -y^1$. It follows that

$$||\tfrac{1}{2}(x + y)||_1 = \sum_{j=2}^\infty |x^j| = ||x||_1 - |x^1| = 1 - \epsilon/2,$$

and hence $||\cdot||_1$ is $UR_{\{e_1\}}$.

To see that $||\cdot||_C$ is not *W*UR*, let $x_n = e_1$ and $y_n = (e_2 + \ldots + e_{n+1})/n$ for each $n \in N$. Then $||x_n||_C = 1$, $||y_n||_C \to 1$, and $||x_n + y_n||_C \to 2$. But $x_n - y_n \to e_1$ weak*, and hence $||\cdot||_C$ is not *W*UR*.

Example 2.9. An equivalent norm is defined on l^2 that is *URED* but not *URWC*.

For $x = (x^j)_{j=1}^\infty$ in l^2, define

$$||x||_F = |x^1| + \left(\sum_{j=2}^\infty |x^j|^2\right)^{1/2}.$$

Then $||\cdot||_F$ is a norm on l^2, and it is equivalent to $||\cdot||_2$ since $||\cdot||_2 \leqq ||\cdot||_F \leqq 2||\cdot||_2$. Let $\{\alpha_j\}_{j=2}^\infty$ be a sequence of positive real numbers such that $\alpha_j \to 0$. Define $T: l^2 \to l^2$ by $T(x^1, x^2, \ldots) = (\alpha_2 x^2, \alpha_3 x^3, \ldots)$. Then T is a continuous linear mapping. For $x \in l^2$, define

$$||x||_A = (||x||_F^2 + ||Tx||_2^2)^{1/2}.$$

Then $||\cdot||_A$ is an equivalent norm on l^2.

To show that $||\cdot||_A$ is *URED* it suffices by Theorem 1.5 to show that $||\cdot||_F$ is $UR_{\{e_1\}}$. For this, let $\epsilon > 0$ be given and let $x = (x^j)_{j=1}^{\infty}$ and $y = (y^j)_{j=1}^{\infty}$ in l^2 be such that $||x||_F = ||y||_F = 1$ and $x - y = \epsilon e_1$. Then $x^j = y^j$ for all $j \geqq 2$, and hence $x^1 = -y^1$. It follows that

$$||\tfrac{1}{2}(x+y)||_F = \left(\sum_{j=2}^{\infty} |x^j|^2\right)^{1/2} = ||x||_F - |x^1| = 1 - \epsilon/2,$$

and hence $||\cdot||_F$ is $UR_{\{e_1\}}$.

To see that $||\cdot||_A$ is not *URWC*, let $x_n = e_1$ and $y_n = e_n$ for each $n \geqq 2$. Then $||x_n||_A = 1$, $||y_n||_A \to 1$, and $||x_n + y_n||_A \to 2$. But $x_n - y_n \to e_1$ weakly, and hence $||\cdot||_A$ is not *URWC*.

If a Banach space is *URWC*, then clearly each of its subspaces is also *URWC*. The remainder of this section is devoted to the investigation of the inheritance of the property *URWC* by product spaces and quotient spaces.

THEOREM 2.10. *Let X be a full function space on an index set S and let $\{B_s: s \in S\}$ be a collection of Banach spaces.*

(i) *If X is $UR^{C'}$, where C' is the set of evaluation functionals on X, and if every B_s is URWC, then the product space $P_X B_s$ is URWC.*

(ii) *If X is reflexive, then $P_X B_s$ is URWC if and only if X and B_s are all URWC.*

Proof. (i). Let $\{x_n\}$ and $\{y_n\}$ be sequences in $P_X B_s$ such that $||x_n|| = ||y_n|| = 1$, $||x_n + y_n|| \to 2$, and $x_n - y_n \to z$ weakly. For each $n \in N$, define $f_n(s) = ||x_n(s)||$ and $g_n(s) = ||y_n(s)||$ for all $s \in S$. Then $\{f_n\}$ and $\{g_n\}$ are sequences in X such that $||f_n|| = ||g_n|| = 1$ and $||x_n + y_n|| \leqq ||f_n + g_n|| \leqq 2$. It follows that $||f_n + g_n|| \to 2$. Since X is $UR^{C'}$, it follows that $(f_n - g_n)(s) \to 0$, that is, $||x_n(s)|| - ||y_n(s)|| \to 0$ for each $s \in S$. Similarly, letting $h_n = f_n + g_n$ and $k_n(s) = ||x_n(s) + y_n(s)||$ for each $s \in S$, it follows that $||x_n(s)|| + ||y_n(s)|| - ||x_n(s) + y_n(s)|| \to 0$ for each $s \in S$. Also, $x_n(s) - y_n(s) \to z(s)$ weakly for each $s \in S$, since the natural projection $P_X B_s \to B_s$ is a continuous linear mapping. Since for each $s \in S$ the sequence $\{x_n(s)\}$ is bounded and B_s is *URWC*, it follows that $z(s) = 0$. Hence $z = 0$ and $P_X B_s$ is *URWC*.

(ii). If $P_X B_s$ is *URWC*, then X and B_s are all *URWC* since each is isometrically isomorphic to a subspace of $P_X B_s$. The reverse implication follows from the first part of this theorem since a reflexive *URWC* space is already $UR^{C'}$.

Day, James, and Swaminathan [**4**] showed that the property *URED* is not inherited by quotient spaces. Since their example also applies to our situation, it is included here. Let Γ be an uncountable set; Day [**3**] showed that $m(\Gamma)$ is not isomorphic to a rotund space. But if S is dense in the unit sphere of $m(\Gamma)$, then $m(\Gamma)$ is isomorphic to a quotient space of $l^1(S)$. The example is completed by noting that $l^1(S)$ has an equivalent *URED* norm (the inclusion mapping $l^1(S) \to l^2(S)$ is continuous).

Since the properties *URED* and *URWC* coincide for $l^1(S)$, and since *URED* implies R, the above example also shows that the properties *URWC* and R are not inherited by all quotient spaces. However, Klee [**7**] showed that if B is R and if H is a reflexive subspace of B, then the quotient space B/H is R. That the property *URWC* is also inherited by such quotient spaces is shown by the following two results.

LEMMA 2.11. *Let B be a Banach space, H be a reflexive subspace, and π: $B \rightarrow B/H$ be the canonical quotient mapping. If $\tilde{W}$ is a weakly compact subset of B/H then $W = \pi^{-1}(\tilde{W}) \cap 2U$ is a weakly compact subset of B.*

Proof. Note that W is weakly closed and bounded since π is weak-weak continuous. Let $\{x_n\}$ be a sequence in W. Since $\tilde{W}$ is weakly compact, we may assume that $\pi(x_n) \rightarrow \pi(x)$ weakly for some $x \in B$ (by taking a subsequence if necessary). Hence $f(x_n - x) \rightarrow 0$ for all $f \in H^{\perp}$. Since $\{Q(x_n - x)\}$ is bounded in B^{**}, where Q: $B \rightarrow B^{**}$ is the canonical embedding, there exists a subnet $\{x_{n_i} - x\}_{i \in I}$ of $\{x_n - x\}$ and $x_0 \in B^{**}$ such that $Q(x_{n_i} - x) \rightarrow x_0$ weak* in B^{**}. In particular, $f(x_{n_i} - x) \rightarrow f(x_0)$ for all $f \in H^{\perp}$. It follows that $f(x_0) = 0$ for all $f \in H^{\perp}$. Therefore $x_0 \in H^{\perp\perp} = H$, since H is reflexive, and hence $\{x_{n_i}\}_{i \in I}$ converges weakly to $x + x_0$, an element of W. Thus every sequence in W has a weakly convergent subnet, and the proof is complete.

THEOREM 2.12. *If B is a Banach space that is URWC and if H is a reflexive subspace, then the quotient space B/H is URWC.*

Proof. If B/H is not *URWC*, then there exist $\tilde{z} \neq 0$ and sequences $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ in B/H such that $||\tilde{x}_n|| = ||\tilde{y}_n|| = 1$, $||\tilde{x}_n + \tilde{y}_n|| \rightarrow 2$, and $\tilde{x}_n - \tilde{y}_n \rightarrow \tilde{z}$ weakly. For each $n \in N$, choose $x_n \in \pi^{-1}(\tilde{x}_n)$ and $y_n \in \pi^{-1}(\tilde{y}_n)$ such that $||x_n|| = ||y_n|| = 1$ (such elements exist since H is reflexive). Since $||\tilde{x}_n + \tilde{y}_n|| \leqq ||x_n + y_n|| \leqq 2$, it follows that $||x_n + y_n|| \rightarrow 2$. Let $\tilde{W} = \{\tilde{x}_n - \tilde{y}_n\} \cup \{\tilde{z}\}$ and $W = \pi^{-1}(\tilde{W}) \cap 2U$. Since $\tilde{W}$ is weakly compact in B/H, it follows from Lemma 2.11 that W is weakly compact in B, and hence there exist subsequences $\{x_n'\}$ and $\{y_n'\}$ of $\{x_n\}$ and $\{y_n\}$ respectively and z in W such that $x_n' - y_n' \rightarrow z$ weakly. Note that $\pi(z) = \tilde{z}$ and hence $z \neq 0$. But now $\{x_n'\}$ and $\{y_n'\}$ are sequences in B such that $||x_n'|| = ||y_n'|| = 1$, $||x_n' + y_n'|| \rightarrow 2$, and $x_n' - y_n' \rightarrow z$ weakly, contrary to the hypothesis that B is *URWC*.

3. Concluding remarks and some problems. Fakhoury [**5**] has introduced the following notion of directionalized uniform rotundity: a Banach space B is *uniformly rotund in the direction of a subspace H* if $\delta(\epsilon, \Sigma(H)) > 0$ for every $0 < \epsilon \leqq 2$, where $\Sigma(H)$ denotes the unit sphere of H. Some attention is given to the case in which H ranges over all finite dimensional subspaces of B as a generalization of the property *URED* (see [**5**, Corollary 2.4 and Theorem 4.4]). However, since $\Sigma(H)$ is norm compact whenever H is finite dimensional, it follows from Theorem 2.2 that these two notions coincide.

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Day, James, and Swaminathan [**4**] showed that $c_0(\Gamma)$ does not have an equivalent norm that is *URED* if Γ is uncountable. However, Troyanski [**9**] has shown that if B is a non-separable Banach space with a symmetric basis and if B is not isomorphic to $c_0(\Gamma)$ for some uncountable set Γ, then B has an equivalent norm that is *URED*. His proof actually shows that such a space B has an equivalent norm that is *URWC*. In particular, a non-separable, reflexive Banach space with a symmetric basis has an equivalent norm that is *WUR*. This leads to our first question.

Problem 1. Does there exist a Banach space that is *URED* but has no equivalent norm that is *URWC*? In particular, is there a reflexive Banach space that is *URED* but has no equivalent norm that is *WUR*?

Employing the technique in the proof of Theorem 2.12, it can be shown that if B is a Banach space that is *URED* and if H is a finite dimensional subspace, then the quotient space B/H is *URED*. This leads to our second question (an affirmative answer to this one seems unlikely).

Problem 2. If B is a Banach space that is *URED* and if H is a reflexive subspace, then is the quotient space B/H necessarily *URED*?

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