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ON NORMAL SUBSPACES

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In this paper the anti-normal property is studied. A space is anti-normal if its only normal subspaces are those whose cardinalities require them to be normal. It is shown that every topological space of at least four elements contains a normal three point subspace from which it follows that there is only one non-trivial anti-normal space.

This paper answers the question raised in [2], concerning the characterization of the anti-normal spaces. In so doing, it provides yet another topological feature which distinguishes normality from every other separation property.

In this note, we particularize the discussion of topological anti-properties to the class of anti-normal spaces. The general discussion is given in the papers of Bankston [1] and Reilly and Vamanamurthy [2], and we follow their set-theoretic and notational conventions.

A topological space X is *anti-normal* if and only if the only normal subspaces of X are those whose cardinalities require them to be normal. The spectrum of a topological class K, is the class of cardinal numbers κ such that any topology on a set of power κ lies in K. In [2] it was shown that spec (normal) = $\{0, 1, 2\}$, but the authors were unable to obtain a characterization of the class of anti-normal spaces. Here, we are able to give a rather surprising characterization of the anti-normal spaces, especially in view of the results for other classes defined by separation properties in [2, §2]. Indeed, by proving that any topological

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space having at least four elements contains a normal three point subspace, we can show that there is only one non-trivial anti-normal space, up to homeomorphism.

Our first proposition is part of the folklore.

PROPOSITION 1. *Let* $X = \{a, b, c\}$ *be a three point set. There are nine homeomorphism classes of topologies on* X *, of which only one is not normal. This topology is characterized by one open and not closed singleton, and two closed and not open singletons.*

Proof. The homeomorphism classes are as follows:

(i) discrete, one member,

(ii) indiscrete, one member,

(iii) $\{\emptyset, X, \{a\}\}$, three members,

(iv) $\{\emptyset, X, \{a, b\}\}$, three members,

(v) $\{\emptyset, X, \{a\}, \{a, b\}\}$, six members,

(vi) $\{\emptyset, X, \{a\}, \{b, c\}\}$, three members,

(vii) $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, three members,

(viii) $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, six members,

(ix) $\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, three members. □

We observe that the classes (i), (ii) and (vi) are regular and normal, the classes (iii), (iv), (v), (vii) and (viii) are normal but not R_0 , and class (ix) is neither normal nor R_0 . (A topological space (X, T) is R_0 if and only if $x \in U \in T$ implies $T\ \mathrm{cl}\{x\} \subset U$.)

PROPOSITION 2. *Spectrum (normal)* $= \{0, 1, 2\} = 3$.

Proof. Any topology on a set of at most two points is normal. From Proposition 1, we have a three point non-normal space. If X has more than three points, let $X = \{a, b, c\} \cup E$, where the union is disjoint. Then (X, T) is not normal where

$$T = \{\emptyset, X, \{a\} \cup E, \{a, b\} \cup E, \{a, c\} \cup E\} . \quad \square$$

THEOREM 1. *Let* X *be a topological space having at least four elements. Then* X *contains a normal three point subspace.*

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Proof. It suffices to consider the case where X has exactly four points; otherwise replace X by any four point subspace. Suppose $X = \{a, b, c, d\}$. We consider three cases.

I. Suppose X has a three point open subset, say $\{a, b, c\}$. If $\{a, b, c\}$ is normal, we are through. Otherwise, it is non-normal, so by Proposition 1 (ix), two doubleton subsets of $\{a, b, c\}$ are open in $\{a, b, c\}$, say $\{a, b\}$ and $\{a, c\}$. Since $\{a, b, c\}$ is open in X , $\{a, b\}$ and $\{a, c\}$ are open in X . Hence, $\{b\}$ and $\{c\}$ are open in $\{b, c, d\}$, and thus $\{b, c, d\}$ is a normal three point subspace of X .

II. Suppose X has a two point open subset, say $\{a, b\}$. If $\{a, b, c\}$ is normal, we are through. Otherwise, it is non-normal, so by Proposition 1 (ix), at least one of $\{a, c\}$ and $\{b, c\}$ is open in $\{a, b, c\}$, say $\{a, c\}$. Thus either $\{a, c\}$ or $\{a, c, d\}$ is open in X . In either case we obtain a three point open subset of X , namely $\{a, b, c\} = \{a, b\} \cup \{a, c\}$ in the former, and $\{a, c, d\}$ in the latter. Thus case I contains case II.

III. Finally suppose that X contains neither two point nor three point open subsets. Hence, either X is indiscrete or X contains only one proper open set and this open set is a singleton, say $\{a\}$. In either of these cases X contains an indiscrete, and hence normal, three point subspace, for example, $\{b, c, d\}$. □

THEOREM 2. *There is only one non-trivial anti-normal space (up to homeomorphism), namely the three point non-normal space.*

Proof. By Proposition 2, the three point non-normal space is anti-normal. If X has more than three points, then by Theorem 1, it has a normal three point subspace, and hence X is not anti-normal.

References

[1] Paul Bankston, "The total negation of a topological property", *Illinois J. Math.* 23 (1979), 241-252.

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[2] I.L. Reilly and M.K. Vamanamurthy, "Some topological anti-properties", *Illinois J. Math.* (to appear).

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