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Nagoya Math. J.  
Vol. 54 (1974), 1-6

## N-MANIFOLDS CARRYING BOUNDED BUT NO DIRICHLET FINITE HARMONIC FUNCTIONS

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Among the most remarkable results in the theory of harmonic functions on Riemann surfaces is the strictness of the inclusion relations  $O_G < O_{HB} < O_{HD}$ , established by Ahlfors [1, 2], Royden [2, 4], and Tôki [8] two decades ago. Subsequently the strictness of the relations  $O_G < O_{HP} < O_{HB}$  was shown and a somewhat simpler proof of  $O_{HB} < O_{HD}$  given by Sario [5] and Tôki [9]. Here  $O_G$  is the class of parabolic surfaces, and  $O_{HP}, O_{HB}, O_{HD}$  stand for the classes of surfaces which do not carry nonconstant harmonic functions which are positive, bounded, or Dirichlet finite, respectively. The corresponding nonstrict inclusion relations extend readily to Riemannian manifolds of any dimension, and so does the strictness of  $O_G < O_{HP} < O_{HB}$  (see e.g. Sario-Schiffer-Glasner [7] and Sario-Nakai [6]). In contrast, the strictness of  $O_{HB} < O_{HD}$  has remained an open problem. The purpose of the present paper is to submit an example which solves the problem in the affirmative for an arbitrary dimension  $N$ . In the process we also obtain complete characterizations of the Poincaré  $N$ -ball in  $O_{HB}$  and  $O_{HD}$ . This manifold plays an important role in the harmonic and biharmonic classification theory.

1. For  $N \geq 3$ , consider the  $N$ -ball  $B_\alpha^N = \{|x| < 1, x = (x^1, \dots, x^N), ds\}$  with the Poincaré-type metric  $ds = (1 - |x|^\alpha)^{-1} |dx|$ ,  $\alpha$  constant.

**THEOREM 1.**  $B_\alpha^N \notin O_{HB} \Leftrightarrow \alpha < 1/(N - 2)$ ,  $N \geq 3$ .

*Proof.* For the necessity, observe that a radial function  $h(r)$ ,  $r = |x|$ , is harmonic if and only if

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Received November 16, 1972.

The work was sponsored by the U.S. Army Research Office-Durham, Grant DA-ARO-D-31-124-71-G181, University of California, Los Angeles.

MOS classification 31B05.

$$\Delta h(r) = -(1-r^2)^{-N\alpha} r^{1-N} \frac{d}{dr} [(1-r^2)^{(N-2)\alpha} r^{N-1} h'(r)] = 0,$$

that is,

$$h(r) = \text{const} \int_a^r (1-r^2)^{-(N-2)\alpha} r^{1-N} dr.$$

Since  $B_\alpha^N \in O_G$  if and only if  $|h(r)| \rightarrow \infty$  as  $r \rightarrow 1$ , the parabolicity is characterized by  $\alpha \geq 1/(N-2)$ . In view of  $O_G \subset O_{HB}$ , we have the necessity. To prove the sufficiency, let  $\{S_{nm}\}$  be the spherical harmonics. First we find an  $f_n(r)$  such that  $f_n S_{nm}$  is harmonic on  $B_\alpha^N$ . The equation  $\Delta(f_n S_{nm}) = 0$  gives

$$(1-r^2)r^2 f_n''(r) + \{N-1 - [(N-1) + 2(N-2)\alpha]r^2\} r f_n'(r) - n(n+N-2)(1-r^2)f_n(r) = 0.$$

The origin is a regular singular point, and there exists a solution of the form  $f_n(r) = \sum_{i=0}^{\infty} c_{n,i} r^{n+i}$  with  $c_{n,0} = 1$ . A recursion formula for the coefficients is obtained in the usual manner and yields (cf. [3])

$$c_{n,2i} = \prod_{j=1}^i \frac{(n+2j-2)[n+2j+N-4+2(N-2)\alpha] - n(n+N-2)}{(n+2j)(n+2j+N-2) - n(n+N-2)},$$

$c_{n,2i+1} = 0$  for  $i \geq 0$ . We shall show that  $f_n S_{nm} \in HB(B_\alpha^N)$  if  $\alpha < 1/(N-2)$ .

There exists an  $i_0 = i_0(N, \alpha)$  such that for  $j \geq i_0$  and  $\alpha < 1/(N-2)$ ,  $c_{n,2j}/c_{n,2j-2}$  is positive and dominated by

$$\frac{(n+2j-2)[n+2j+N-4+2(N-2)\alpha]}{(n+2j)(n+2j+N-2)}.$$

For  $i \geq i_0$ ,

$$\begin{aligned} |c_{n,2i}| &< c \prod_{j=1}^i \frac{n+2j-2}{n+2j} \cdot \prod_{j=i_0}^i \frac{n+2j+N-4+2(N-2)\alpha}{n+2j+N-2} \\ &= \frac{cn}{n+2i} \prod_{j=i_0}^i \left( \frac{2-2(N-2)\alpha}{n+2j+N-2} \right). \end{aligned}$$

Here and later,  $c$  is a positive constant, not always the same. We now impose on  $i_0$  the additional condition that for  $j \geq i_0$

$$0 < \frac{2-2(N-2)\alpha}{n+2j+N-2} < 1.$$

Then

$$\begin{aligned} \sum_{j=i_0}^i \log \left( 1 - \frac{2 - 2(N - 2)\alpha}{n + 2j + N - 2} \right) &< -2[1 - (N - 2)\alpha] \\ &\times \sum_{j=i_0}^i \frac{1}{n + 2j + N - 2} < -[1 - (N - 2)\alpha] \int_{i_0}^{i+1} \frac{dx}{n + x + N - 2} \\ &= \log \left( \frac{n + i_0 + N - 2}{n + i + N - 1} \right)^{1 - (N - 2)\alpha} \end{aligned}$$

Therefore

$$\prod_{j=i_0}^i \left( 1 - \frac{2 - 2(N - 2)\alpha}{n + 2j + N - 2} \right) < \left( \frac{n + i_0 + N - 2}{i} \right)^{1 - (N - 2)\alpha}$$

and

$$|c_{n,2i}| < \frac{c}{n + 2i} \left( \frac{1}{i} \right)^{1 - (N - 2)\alpha} < c \left( \frac{1}{i} \right)^{2 - (N - 2)\alpha}.$$

We conclude that

$$|f_n S_{mn}| < \sum_{i=0}^{\infty} |c_{n,2i}| r^{n+2i} < c \sum_{i=i_0}^{\infty} \left( \frac{1}{i} \right)^{2 - (N - 2)\alpha}.$$

This is finite for  $\alpha < 1/(N - 2)$ , and the proof Theorem 1 is complete.

2. To discuss the existence of *HD*-functions, we first prove:

**LEMMA.** For  $\alpha \leq 0$ ,  $f'_1$  is bounded; for  $\alpha > 0$ ,  $f'_1(r) < \text{const} (1 - r^2)^{-(N - 2)\alpha}$ .

*Proof.* If  $\alpha < 0$ , it is immediate from the estimates in the proof of Theorem 1 that  $f'_1$  is bounded. If  $\alpha = 0$ ,  $ds = |dx|$  (the Euclidean metric), and  $f'_1 = 1 < \infty$ . If  $\alpha > 0$ ,  $c_{1,2i} > 0$  for all  $i$ , and

$$\begin{aligned} c_{1,2i} &= \prod_{j=1}^i \frac{(2j - 1)[2(N - 2)\alpha + N - 3 + 2j] - (N - 1)}{(2j + 1)(2j + N - 1) - (N - 1)} \\ &< \prod_{j=1}^i \frac{(2j - 1)[2(N - 2)\alpha + N - 3 + 2j]}{(2j + N)2j}. \end{aligned}$$

For  $N$  even,

$$\begin{aligned} c_{1,2i} &< \prod_{j=1}^i \frac{2j}{2j + N} \cdot \prod_{j=1}^i \left( \frac{2j - 1}{2j} \cdot \frac{2(N - 2)\alpha + N - 2 + 2j - 1}{2j} \right) \\ &< \frac{2 \cdot 4 \cdots N}{(2i + 2)(2i + 4) \cdots (2i + N)} \\ &\times \prod_{j=1}^i \left( \frac{2j - 1}{2j} \cdot \frac{2(N - 2)\alpha + N - 2 + 2j - 2}{2j - 1} \right) \end{aligned}$$

$$= \frac{2 \cdot 4 \cdots N}{(2i + 2)(2i + 4) \cdots (2i + N)} \prod_{j=1}^i \frac{(N - 2)\alpha + \frac{1}{2}(N - 2) + j - 1}{j}.$$

Hence

$$\begin{aligned} f_1'(r) &= 1 + \sum_{i=1}^{\infty} (2i + 1)c_{1,2i}r^{2i} < N + \sum_{i=1}^{\infty} (2i + N)c_{1,2i}r^{2i} \\ &< N \left[ 1 + 2 \cdot 4 \cdots (N - 2) \sum_{i=1}^{\infty} \left( \frac{1}{(2i + 2)(2i + 4) \cdots (2i + N - 2)} \right. \right. \\ &\quad \left. \left. \times \prod_{j=1}^i \frac{(N - 2)\alpha + \frac{1}{2}(N - 2) + j - 1}{j} \right) r^{2i} \right]. \end{aligned}$$

On the other hand,

$$(1 - r^2)^{-[(N-2)\alpha + \frac{1}{2}(N-2)]} = 1 + \sum_{i=1}^{\infty} \left( \prod_{j=1}^i \frac{(N - 2)\alpha + \frac{1}{2}(N - 2) + j - 1}{j} \right) r^{2i}.$$

Repeating  $\frac{1}{2}(N - 2)$  times the process of first multiplying through by  $r$  and then integrating, we obtain

$$\begin{aligned} &\text{const } (1 - r^2)^{-(N-2)\alpha} + \text{polynomial} \\ &= \frac{r^{N-2}}{2 \cdot 4 \cdots (N - 2)} \left[ 1 + 2 \cdot 4 \cdots (N - 2) \right. \\ &\quad \left. \times \sum_{i=1}^{\infty} \left( \frac{1}{(2i + 2)(2i + 4) \cdots (2i + N - 2)} \right. \right. \\ &\quad \left. \left. \times \prod_{j=1}^i \frac{(N - 2)\alpha + \frac{1}{2}(N - 2) + j - 1}{j} \right) r^{2i} \right], \end{aligned}$$

which entails  $f_1'(r) < \text{const } (1 - r^2)^{-(N-2)\alpha}$ .

For  $N$  odd, the proof is the same except that now

$$c_{1,2i} < \frac{1 \cdot 3 \cdots N}{(2i + 1)(2i + 3) \cdots (2i + N)} \prod_{j=1}^i \frac{(N - 2)\alpha + \frac{1}{2}(N - 1) + j - 1}{j}.$$

**THEOREM 2.**  $B_\alpha^N \notin O_{HD} \Leftrightarrow |\alpha| < 1/(N - 2), N \geq 3.$

*Proof.* We have

$$D(f_1 S_{1m}) = \int_0^1 (c_1 f_1'(r)^2 + c_2 r^{-2} f_1(r)^2) (1 - r^2)^{(N-2)\alpha} r^{N-1} dr$$

where  $c_1$  and  $c_2$  are positive constants. Since  $f_1$  and  $f_1'$  are bounded for  $\alpha \leq 0$ ,

$$D(f_1 S_{1m}) < c \int_0^1 (1 - r^2)^{(N-2)\alpha} r^{N-3} dr < \infty$$

for  $-1/(N - 2) < \alpha \leq 0$ . By the lemma,

$$D(f_1 S_{1m}) < \int_0^1 [d_1(1 - r^2)^{-2(N-2)\alpha} + d_2 r^{-2}] (1 - r^2)^{(N-2)\alpha} r^{N-1} dr < \infty$$

for  $0 < \alpha < 1/(N - 2)$ . Hence,  $B_\alpha^N \notin O_{HD}$  if  $|\alpha| < 1/(N - 2)$ . In view of  $B_\alpha^N \in O_G \subset O_{HD}$  for  $\alpha \geq 1/(N - 2)$ , it remains to consider  $\alpha \leq -1/(N - 2)$ . Let  $h$  be a nonconstant harmonic function. By means of the eigenfunction expansion of the restriction of  $h$  to  $|x| = r < 1$ , we obtain

$$h = \sum_{n=0}^\infty \sum_{m=1}^{m_n} f_n a_{nm} S_{nm}$$

on all of  $B_\alpha^N$  with absolute and uniform convergence on compact subsets and with  $a_{nm} \neq 0$  for some  $(n, m)$ ,  $n > 0$ . By the Dirichlet orthogonality of spherical harmonics,

$$\begin{aligned} D(h) &\geq a_{nm}^2 D(f_n S_{nm}) > a_{nm}^2 \int_{B_\alpha^N} (1 - r^2)^{-2\alpha} \left( \frac{\partial f_n S_{nm}}{\partial r} \right)^2 dV \\ &> c \int_0^1 f_n'(r)^2 (1 - r^2)^{(N-2)\alpha} dr . \end{aligned}$$

Here  $f_n'(r)$  converges as  $r \rightarrow 1$ , since the  $c_{n,2i}$  are of constant sign for sufficiently large  $i$ . By virtue of  $f_n S_{nm} \neq \text{const}$ , we have  $f_n'(r) \neq 0$  for  $r > 0$ , and  $\lim_{r \rightarrow 1} f_n'(r) \neq 0$ . Consequently

$$D(h) > c \int_{\frac{1}{2}}^1 (1 - r^2)^{(N-2)\alpha} dr ,$$

which implies  $B_\alpha^N \in O_{HD}$  for  $\alpha \leq -1/(N - 2)$ .

**3.** Let  $O_{HB}^N, O_{HD}^N$  be the classes of Riemannian  $N$ -manifolds in  $O_{HB}, O_{HD}$ , respectively.

**THEOREM 3.**  $O_{HB}^N < O_{HD}^N$  for every  $N$ .

*Proof.* The strict inclusion  $O_{HB}^2 < O_{HD}^2$  for Riemannian 2-manifolds is a trivial consequence of that for Riemann surfaces. In fact, endowing a Riemann surface by a conformal metric turns it into a Riemannian 2-manifold without affecting the harmonicity or the Dirichlet integral of a function on it.

For  $N \geq 3$ , the theorem follows from Theorems 1 and 2.

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