Can. J. Math. Vol. **49** (4), 1997 pp. 788–797

TRACE FUNCTIONS IN THE RING OF FRACTIONS OF POLYCYCLIC GROUP RINGS, II

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ABSTRACT. We prove the existence of trace functions in the rings of fractions of polycyclic-by-finite group rings or their homomorphic images. In particular a trace function exists in the ring of fractions of KH, where H is a polycyclic-by-finite group and $\operatorname{char} K > N$, where N is a constant depending on H.

1. **Introduction.** Let K be a field of characteristic zero, H be a polycyclic-by-finite group, A be a semiprime ideal in the group ring KH. The semiprime ring $K[H] = (KH)/A$ is a Goldie ring; let R be its ring of fractions. The existence of trace functions in the matrix rings over the ring of fractions of the group ring KH was established by the author in [6]. In this note we generalize this result by proving the following theorem.

THEOREM 1. *Let K be a field of characteristic zero, A be a semiprime ideal in KH, R be the Goldie ring of fractions of the ring $K[H] = (KH)/A$. Then*

$$1 \notin [R, R]. \tag{1}$$

The relation (1) means then it is impossible to find elements $x_j, y_j \in R$ ($j = 1, 2, \ldots, k$) such that

$$1 = \sum_{j=1}^{k} [x_j, y_j]. \tag{2}$$

It is well known (see [3]–[15]) that the relation (1) implies an existence of a nontrivial trace function in $R_{n\times n}$. Indeed it is easy to see that if $X = (x_{ij})$ is an $n \times n$ matrix over R and $X \in [R_{n\times n}, R_{n\times n}]$ then $(\sum_i x_{ii}) \in [R, R]$. If now R is a K-algebra with a trace function $T\colon R \longrightarrow K$ then we can define $T_n(X) = \sum_i T(x_{ii})$ and if $\operatorname{char} K \neq 0$ then (1) implies that $T(1_{n\times n}) \neq 0$.

Now let $\operatorname{char} K = p > 0$. By applying Theorem 3.12 in [7] we will obtain the following theorem.

THEOREM 2. *Let H be a polycyclic-by-finite group, R be the ring of fractions of KH. Then the relation (1) holds in R provided that $p > N$, where N is a constant depending on H.*

The restriction on the characteristic of K cannot be removed; in Section 8 we consider the case when $\operatorname{char} K = p > 0$, the group H is torsion free and is an extension of an

Partially supported by NSA Grant No. 144–FL181.
Received by the editors November 30, 1996; revised March 10, 1997.
AMS subject classification: Primary 20C07; secondary 16A08, 16A39.

https://doi.org/10.4153/CJM-1997-039-1 Published online by Cambridge University Press

abelian group by a finite p-group. It is known that in this case the group ring KH contains no zero divisors and has a division ring of fractions D; the dimension of D over its center is a power of p. We show that in this case the relation (1) is not true anymore *i.e.* the unit element is a sum of commutators in D. We conjecture however that the following fact is true.

Let $\operatorname{char} K = p > 0$ *and* H *be a polycyclic-by-finite group which contains no finite normal* p*-subgroups,* R *be the ring of fractions of* KH*. Then*

$$[R, R] \neq R.$$

It is worth remarking that this holds in the case when R is the division ring of fractions of a group ring of a torsion free abelian-by-finite group. This can be obtained as a corollary of Lorenz's results [10]. I am grateful to the referee who brought this to my attention.

We prove in Section 7 the following theorem which is a generalization of M. Lorenz's theorem in [9].

THEOREM 3. *Let* H *be a finitely generated nilpotent group,* A *be a semiprime ideal in* KH*,* R *be the ring of fractions of* $(KH)/A$*. Then the relation (1) holds in* R*.*

Lorenz obtained this result for the case when $\operatorname{char} K = 0$.

2. The following fact is well known and its proof is straightforward.

LEMMA 1. *Let* R *be an algebra over a field* K*,* K_1 *be a field extension of* K*. If the relation (1) holds in* R *then it holds also in* $K_1 \otimes R$*.*

LEMMA 2. *Let* $K[G]$ *be an algebra generated by a finite group* G *over a field* K*. If* $\operatorname{char} K$ *does not divide the order* $(G:1)$ *then the relation (1) holds in the ring* $K[G]$*.*

PROOF. Clearly, we can assume that K is algebraically closed. We have in this case

$$K[G] \simeq \sum_{\alpha=1}^{k} K_{m_\alpha \times m_\alpha}, \tag{3}$$

where $m_\alpha | (G:1)$ $(\alpha = 1, 2, \ldots, k)$. The decomposition (3) now reduces the proof to the case when $K[G] \simeq K_{m\times m}$ where m is prime to $\operatorname{char} K$. We observe now that the relation (2) can not hold in $K_{m\times m}$ because the trace of the right side is zero whereas the trace of the left side is $m \neq 0$. This completes the proof.

LEMMA 3. *Let* R *be a ring. Assume that there exists a system of subrings* T_i $(i \in I)$ *and homomorphisms* $\theta_i : T_i \longrightarrow R_i$ $(i \in I)$ *such that for every given elements*

$$r_j \in R \quad (j = 1, 2, \ldots, k) \tag{4}$$

a subring T_i *containing these elements can be found. If the relation (1) holds in every ring* R_i $(i \in I)$ *then it holds also in* R*.*

PROOF. Assume that the relation (2) holds for some elements $x_j, y_j \in R$ $(j = 1, 2, \ldots, k)$. We find a homomorphism $\theta_i : R \longrightarrow R_i$ such that its domain T_i contains all the elements

https://doi.org/10.4153/CJM-1997-039-1 Published online by Cambridge University Press

x_j, y_j $(j = 1, 2, \ldots, k)$ and obtain the following relation for the elements $\bar{x_j} = \theta_i(x_j)$, $\bar{y_j} = \theta_i(y_j)$ $(j = 1, 2, \ldots, k)$ in the ring R_i:

$$\sum_{j=1}^{k} [\bar{x_j}, \bar{y_j}] = 1$$

which contradicts the assumption of the assertion. This completes the proof.

3.

LEMMA 4. *Let G be a soluble group which contains a finite subgroup U of order n such that the quotient group G/U is free abelian of finite rank k. Then G contains a free abelian subgroup V of rank k and of finite index such that all the prime divisors of $(G : V)$ are divisors of n.*

PROOF. Since the commutator subgroup G' is finite we conclude easily that for every element $g \in G$ there exists a number $m(g)$ such that the power $g^{m(g)}$ belongs to the center of G and hence the center has a finite index. We obtain then that there exists a central torsion free subgroup Z of finite index.

Let $V \supseteq Z$ be the subgroup of G which is the inverse image of the Hall n'-subgroup of G/Z (it is worth remarking that the group G/Z is soluble); clearly, all the prime divisors of $(G : V)$ are divisors of n.

Since the group V/Z is an n'-group we obtain from Schur's theorem that V' is a finite n'-group. But $V' \subseteq U$ and hence V' is an n-group. This implies that $V' = 1$, *i.e.* V is abelian. Since V is an extension of a torsion free group Z by an n'-group V/Z all the elements of finite order in V must be n'-elements; once again, since $(U : 1) = n$ we obtain that $V \cap U = 1$ and hence V is isomorphic to a subgroup (of finite index) of G/U. This implies that V is free abelian of rank k and the proof is completed.

LEMMA 5. *Let H be a polycyclic group, F be nilpotent normal subgroup of H. Assume that the order of the torsion subgroup of F is n and the quotient group H/F is free abelian. Then H contains a poly-$\{$infinite cyclic$\}$ subgroup H_1 of finite index such that all the prime divisors of $(H : H_1)$ are divisors of n.*

PROOF. Let c be the nilpotency class of F. Malcev's theorem (see [2]) implies that the group F^{n^c}, generated by the n^c-powers of the elements of F, is torsion free. The quotient group $\bar{H} = H/F^{n^c}$ is an extension of a finite nilpotent group $\bar{F} = F/F^{n^c}$ by a free abelian group $\bar{H}/\bar{F} \simeq H/F$. Hence by Lemma 4 $\bar{H}$ contains a free abelian subgroup $\bar{H}_1$ of finite index $(\bar{H} : \bar{H}_1)$ whose prime divisors divide the number n^c. The inverse image H_1 of $\bar{H}_1$ is a subgroup of H which is an extension of a torsion free nilpotent group F^{n^c} by a free abelian group $\bar{H}_1$; hence H_1 is poly-$\{$infinite cyclic$\}$ and satisfies all the other conclusions of the assertion.

LEMMA 6. *Let $K[H]$ be a ring generated by a group H over a field K. Assume that there exists a finite central subgroup $C \subseteq H$ such that $K[H]$ is isomorphic to a suitable cross product $K[H] \simeq K[C] * (H/C)$ where the group H/C is torsion free. Let H_1 be* *a torsion free subgroup of finite index m in H. Then the subalgebra $K[C,H_1]$ generated by the subgroups C and H_1 is isomorphic to the group ring $K[C]H_1$, the subgroup $H_2 = \langle C, H_1\rangle$ is isomorphic to the direct product $C \times H_1$ and $K[H]$ is a (left) free module over $K[C]H_1$ of finite dimension $m_1 = \operatorname{ind}(H : H_2)$ and $m_1 \mid m$.*

PROOF. Since $H_1 \cap C = 1$ the subgroup H_1 can be included into a transversal of C in H; the properties of cross products now imply that $K[C,H_1] \simeq K[C]H_1$; the relation $H_2 \simeq H_1 \times C$ is obvious. If $h_1 = 1, h_2, \ldots, h_{m_1}$ is a transversal of H_2 in H then the elements h_i $(i = 1, 2, \ldots, m_1)$ form a basis of KH over KH_2; clearly, $m_1 \mid m$.

4. We need now a few concepts and results on polycyclic groups. If A is a non-unit torsion free abelian normal subgroup of an arbitrary group H, then the conjugation in H defines in A a structure of ZH-module. This subgroup A is a plinth in H if H and all its subgroups of finite index act rationally irreducible on A. (See Roseblade [13] or Passman [11], Chapter 12). Every infinite polycyclic-by-finite group H has a subgroup of finite index which contains a plinth (see [13] or [11], 12.1.4). It is not difficult to prove the following fact (see [7], Section 3.1).

LEMMA 7. *Let H be a polycyclic-by-finite group. Then it contains a polyplinthic normal subgroup H_0 of finite index, i.e. H_0 is torsion free, and contains a series of nilpotent normal subgroups*

$$H_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{r-1} \supseteq A_r = 1, \tag{5}$$

where A_i/A_{i+1} is a plinth in H_0/A_{i+1}, $A_1 \subseteq C(A_i/A_{i+1})$, the quotient groups $H_0/C(A_i/A_{i+1})$ and H_0/A_1 are free abelian and hence all the groups H_0/A_i $(i = 1, 2, \ldots, s)$ are torsion free.

(Here $C(A_i/A_{i+1})$ is the centralizer of the factor A_i/A_{i+1}, *i.e.* $C(A_i/A_{i+1}) = \{h \in H_0 \mid [h,a] \in A_{i+1}$ for all $a \in A_i\}$.) We would like to point out that this definition of the polyplinthic group differs from the corresponding definition in the book of Shirvany and Wehrfritz [14] (see [14], p. 142).

We will need the following fact which is Theorem 3.12 in [7].

PROPOSITION 1. *Assume that H is a polyplinthic group with a Hirsch number h and K is a finitely generated commutative field of characteristic zero or $p > C(h)$, where $C(h)$ is a constant depending on h. Let D be the division ring of fractions of KH,*

$$x_1, x_2, \ldots, x_s \tag{6}$$

be given non-zero elements of KH and t be a given natural number. Then there exists an ideal $C \subseteq KH$ such that $K[\tilde{H}] = (KH)/C$ is isomorphic to a semiprime subalgebra of a matrix algebra $(K_1)_{m\times m}$, where

$$m = 2^{\beta}\alpha_1\alpha_2\cdots\alpha_\ell q_1^{m_1}q_2^{m_2}\cdots q_\ell^{m_\ell}, \tag{7}$$

the numbers q_i are prime and $q_i > t$, $\alpha_i \mid (q_i - 1)$, $(i = 1, 2, \ldots, \ell)$, the group $\tilde{H}$ is finite and ℓ does not exceed the number r, the length of the plinth series (5), $\beta \leq r$ and

$$m_i \leq \varphi(h) \quad (i = 1, 2, \ldots, \ell)$$

where $\varphi(h)$ is an integer valued function of the Hirsch number h; $K_1 = K(\epsilon)$ is a cyclotomic extension of degree $q_1 q_2 \cdots q_\ell$ over K. Furthermore, the images $\tilde{x}_j$ of the elements x_j $(j = 1, 2, \ldots, s)$ are invertible in the ring $(KH)/C$ and the homomorphism $\alpha: KH \longrightarrow K[\tilde{H}]$ is extended to a specialization $\pi: D \longrightarrow K[\tilde{H}]$, i.e. there exists a subring $D \supseteq T \supseteq KH$ and an epimorphism $\pi: T \longrightarrow K[H]$ such that $\ker \pi$ *is a quasiregular ideal at T.*

REMARK. We use in this paper the concept of "specialization" as defined in Passman's article [12].

It follows also from Theorem I and Proposition (2.2) in [7] that for every given number $q = q_i$ in (7)

$$\frac{q-1}{2} = p_1 p_2 \cdots p_s$$

where $p_\alpha > q^{\frac{1}{2h}}$ $(\alpha = 1, 2, \ldots, s)$ are distinct prime numbers.

PROPOSITION 2. *Let H be a polyplinthic group with Hirsch number h, K be a field of finite characteristic $p > \max\bigl(2, C(h)\bigr)$, D be the division ring of fractions of KH. Then the relation (1) holds in D.*

PROOF. Clearly, we can assume that K is finitely generated. Now assume that nonzero elements $r_j \in D$ $(j = 1, 2, \ldots, k)$ be given. Let

$$r_j = a_j b_j^{-1} \quad (j = 1, 2, \ldots, k).$$

Take $t = p^{2h}$ and apply Proposition 1 to the set of elements a_j, b_j $(j = 1, 2, \ldots, k)$. We obtain a specialization $\theta: D \longrightarrow K[\tilde{H}] \subseteq (K_1)_{m\times m}$ such that its domain T contain all the elements a_j, b_j and hence the elements r_j $(j = 1, 2, \ldots, k)$, and p does not divide the number m. The assertion now follows from Lemmas 2 and 3.

PROOF OF THEOREM 2. Lemma 7 implies that H contains a polyplinthic normal subgroup of finite index $m = (H : H_0)$; hence, R is isomorphically imbedded into a matrix ring $D_{m\times m}$, where D is the division ring of fractions of KH_0 (see [14]). Proposition 2 now implies that the relation (1) holds in $D_{m\times m}$ if $\operatorname{char} K > N$, where $N = \max\bigl(2, m, C(h)\bigr)$. Hence it holds in R.

5. We will need the following fact in the proof of Theorem 1.

LEMMA 8. *Let K be a field of finite characteristic p, $K[H]$ be a domain generated by a polycyclic group H over K, R be the division ring of fractions of $K[H]$. Assume that H contains a finite central subgroup C such that the quotient group $\bar{H} = H/C$ is poly-$\{$infinite cyclic$\}$ and $K[H]$ is isomorphic to a suitable cross product*

$$K[H] \simeq K[C] * (H/C).$$

Assume that there exists a nilpotent normal subgroup $F \supseteq C$ such that the quotient group H/F is free abelian. Assume also that the relation (1) holds in the division ring of fractions of $K\overline{H}$. Then it holds also in R.

PROOF. It is well known that the group C is cyclic: in fact, it is a finite subgroup of a field $K[C]$. Furthermore, the order of C is prime to p. Now apply Lemma 5 and obtain a poly-$\{$infinite cyclic$\}$ subgroup $H_1 \subseteq H$ such that the index $m = (H : H_1)$ is prime to p. Lemma 6 now implies that the index m_1 of the subgroup $H_2 = \langle C, H_1 \rangle = C \times H$ is prime to p and that $K[H]$ has a faithful representation of degree m_1 over the group ring $K[C]H_1$, hence R has a faithful representation of degree m_1 over the ring of fractions S of $K[C]H_1$. Finally, H_1 is a subgroup of H which does not intersect C; hence, it is isomorphic to a subgroup of the quotient group $\overline{H} = H/C$. Since the relation (1) holds in the division ring of fractions of $K\overline{H}$ it must hold, via Lemma 1, in the ring of fractions of $K[C]\overline{H}$ and hence in its subring S. Finally, since R is imbedded isomorphically in $S_{m_1 \times m_1}$ and $(p, m_1) = 1$ we obtain that the relation (1) holds in R.

6. We will prove in this section Theorem 1. Clearly, we can assume that the ideal A in Theorem 1 is prime and faithful. We need first the following fact which is statement ii) in Proposition 3 in [8].

PROPOSITION 3. *Let A be a prime ideal of KH, R be the Goldie ring of fractions of the ring $K[H] = (KH)/A$. Then R has a finite left dimension over a division subring D, which is isomorphic to the ring of fractions of a domain $K[H_1]$, where H_1 is a torsion free normal subgroup of finite index in H. Furthermore, $K[H_1]$ is isomorphic to a suitable cross product*

$$K[H_1] \simeq K[C] * (H_1/C)$$

where C is a central subgroup of H_1 and the group H_1/C is poly-$\{$infinite cyclic$\}$.

PROOF OF THEOREM 1. Since Proposition 3 implies that R has a faithful matrix representation over D we can assume in the proof of Theorem 1 that in fact $H_1 = H$, *i.e.* $K[H]$ is a domain,

$$K[H] \simeq K[C] * (H/C), \tag{8}$$

and C is central in H. Furthermore, we can find a normal subgroup $H_0 \supseteq C$ of finite index in H such that the group H_0/C is polyplinthic. The representation (8) implies that

$$K[H] \simeq K[H_0] * (H/H_0) \tag{9}$$

where the group H/H_0 is finite. We conclude from (9) that D has a finite left dimension over the division subring D_0, generated by $K[H_0]$. Once again, we see that *we can assume that the group H/C in (8) is polyplinthic*; this implies, in particular, that there exists a nilpotent normal subgroup $F \supseteq C$ such that H/F is free abelian.

Now let K_i be an arbitrary finitely generated subfield of K. The ideal $A_i = A \cap K_iH$ is completely prime in K_iH and the ring $K_i[H] = (K_iH)/A_i$ is a subring of $K[H]$; $K_i[H]$

generate a division subring $D_i \subseteq D$. Since D is a direct limit of the division subrings D_i *we reduced the proof to the case when the field K is finitely generated*.

Let K_0 be an arbitrary finitely generated subring of K such that K is the field of fractions of K_0. Once again, the ideal $A_0 = A \cap K_0H$ is completely prime in K_0H and D is isomorphic to the division ring of fractions of the ring $K_0[H] = (K_0H)/A_0$; it is easy to see also that

$$K_0[H] \simeq K_0[C] * (H/C) = S * \bar{H} \tag{10}$$

where S is a finitely generated central subring and $\bar{H} = H/C$ is polyplinthic. Let h be the Hirsch number of $\bar{H}$.

We pick now in S an infinite system of maximal ideals A_i ($i \in I$) such that $\bigcap_{i \in I} A_i = 0$ and the quotient rings $S_i = S/A_i$ are finite fields of characteristics $p_i > \max\bigl(2, C(h)\bigr)$ ($i \in I$), where $C(h)$ is the same as in Proposition 2. Every ideal A_i generates in $S * \bar{H}$ a completely prime ideal $(A_i) = A_i * \bar{H}$. This ideal is localizable by Roseblade's Theorem 11.2.9 in [11]. Let $T_i = (S*\bar{H})M_i^{-1}$ where $M_i = (S*\bar{H}) \setminus (A_i*\bar{H})$. Then $T_i/J(T_i)$ is a division ring D_i, which is isomorphic to the ring of fractions of $(K_0[H])/(A_i)$. On the other hand we have for the ring $K_0[H]/(A_i)$ a representation

$$K_0[H]/(A_i) \simeq K_i[H_i], \tag{11}$$

and

$$K_i[H_i] \simeq S_i * \bar{H} \tag{12}$$

where K_i is an image of the ring K_0 and H_i is the image of the group H. Since $p_i > C(h)$ the representations (11) and (12) show that every division ring D_i satisfies the conditions of Lemma 8 and hence the relation (1) holds in D_i.

Now assume that elements (4) in D are given. Apply Proposition 1(i) from [8] and obtain that there exists a cofinite subset $I_1 \subseteq I$ such that for every $i \in I_1$ the elements (4) belong to the subring $T_i \subseteq D$, where T_i is the domain of the specialization $\theta_i : D \longrightarrow D_i$. Theorem 1 now follows from Lemma 3 and the proof is completed.

7. We will need in the proof of Theorem 3 the following fact which is proved in [5] (see [5], Corollary 1.2 or Proposition 2.8).

LEMMA 9. *Let H be a finitely generated torsion free nilpotent group, K be an arbitrary field, Δ be the division ring of fractions of KH and*

$$x_j \quad (j = 1, 2, \ldots, n) \tag{12}$$

be given non-zero elements of Δ. *Let* $q \neq \operatorname{char} K$ *be a given prime number. Then there exists a specialization* $\pi : \Delta \longrightarrow K[\tilde{G}]$ *such that its domain T contains the elements (12),* $K[\bar{G}]$ *is a finite dimensional simple algebra generated by a finite q-group* $\tilde{G} = \pi(G)$ *and* $\ker \pi$ *is the Jacobson radical* $S(T)$ *of* T.

PROOF OF THEOREM 3. Let H be a finitely generated nilpotent group, A be a prime ideal of KH, R be the ring of fractions of KH; we can assume that the ideal A is faithful.

Theorem 1 makes possible to assume that char $K = p > 0$ and Lemma 1 reduces the proof to the case when $K = Z_p$. Let $C = \Delta(h) = \{h \in H \mid h$ has a finite number of conjugates in $H\}$. We obtain now from Zalesskii's Theorem 11.4.5 in [11] that $(KH)/A$ is isomorphic to a suitable cross product

$$K[H] \simeq K[C] * (H/C) \tag{13}$$

where the group H/C is torsion free nilpotent and the group C is abelian-by-finite. Since $K[H]$ is prime the ring $K[C]$ contains no nilpotent ideals. But the group C is finitely generated abelian-by-finite; hence $K[C]$ is a PI-ring and we obtained that $K[C]$ is semisimple.

Let Q be a primitive ideal of $K[C]$; since C is abelian-by-finite and we assumed that $K = Z_p$ we obtain that $K[C]/Q$ is a finite dimensional algebra over K generated by a finite group $\bar{C}$, the image of C; since the algebra $K[\bar{C}]$ is simple and $\bar{C}$ is nilpotent it must be a p'-group. Since H acts as a finite group on $C = \Delta(H)$, we conclude that the orbit $h^{-1}Qh(h \in H)$ of Q must be finite because the image of C in $K[G]/Q$ is finite. Let $Q_1 = Q, Q_2, \ldots, Q_r$ be the orbit of Q and

$$B = \bigcap_{\alpha=1}^{r} Q_\alpha.$$

The ideal $B \subseteq K[C]$ is H-invariant and the quotient algebra $K[C]/B$ is semisimple artinian and generated by a finite p'-group $\tilde{C}$, the image of C; the group $\tilde{C}$ is a subdirect product of the groups $\bar{C}_\alpha$, the images of C in $K[C]/Q_\alpha$ $(\alpha = 1, 2, \ldots, r)$.

We take now an arbitrary system of primitive ideals $Q_i \subseteq K[C]$ $(i \in I)$ with intersection zero. Let $B_i = \bigcap_{h \in H} h^{-1}Q_i h$. Then $\bigcap_{i \in I} B_i = 0$ and every ideal B_i is H-invariant. We consider now the system of ideals $(B_i) = B_i(K[H]) \subseteq K[H]$ $(i \in I)$. Since H is nilpotent every ideal (B_i) $(i \in I)$ is polycentral and it can be localized in $K[H]$ by Roseblade's Theorem 11.2.9 in [11].

Pick now some $i \in I$ and consider the ring $K[H_i] = K[H]/(B_i)$ and its ring of fractions R_i. We will prove that the relation (1) holds in the ring R_i. Since for every $i \in I$ the ideal (B_i) is localizable we see that there exists a specialization $\theta_i \colon R \longrightarrow R_i$ and once again as in the proof of Theorem 1 Theorem 3 will follow from Lemma 3.

The ring $K[H_i]$ is isomorphic to a suitable cross product

$$K[H_i] \simeq K[C_i] * (H_i/C_i),$$

where $\bar{H}_i = H_i/C_i$ is a finitely generated torsion free nilpotent group and C_i is a finite p'-group, say of order m_i. Let c_i be the nilpotency class of H_i. Once again, as above, the group $U_i = H_i^{m_i^{c_i}}$ is torsion free, it generates over $K[C_i]$ a subring, isomorphic to the group ring $K[C_i]U_i$ and $K[H_i]$ is isomorphic to a suitable cross product

$$K[H_i] \simeq (K[C_i]U_i) * (H_i/V_i),$$

where $V_i \simeq C_i \times U_i$, the index $(H_i : V_i)$ is finite and prime to p; hence, once again, the proof is reduced to the case when R_i is isomorphic to the ring of fractions of the group ring $K[C_i]U_i$. Since the field K is algebraically closed we have

$$K[C_i] \simeq K_{n_1 \times n_1} + K_{n_2 \times n_2} + \cdots + K_{n_r \times n_r} \tag{14}$$

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where $n_j \mid (C_i : 1)$ and hence $p \nmid n_j$ $(j = 1, 2, \ldots, r)$.

The decomposition (14) implies that the group ring $K[C_i]U_i$ is isomorphic to a direct sum of group rings over the matrix rings $K_{n_j \times n_j}$ $(j = 1, 2, \ldots, r)$ and the ring of fractions of $K[C_i]U_i$ is a direct sum of the rings of fractions of the group rings $(K_{n_j \times n_j})U_i$ $(j = 1, 2, \ldots, r)$; but for every given j the ring of fractions of $(K_{n_j \times n_j})U_i$ is isomorphic to the matrix ring of degree n_j over the ring of fractions of KU_i. Since U_i is a torsion free nilpotent group the assertion now follows from Lemmas 3 and 9.

8. Let H be a torsion free group which is an extension of an abelian group U by a cyclic group of order p^k. It is known that under these conditions the group H must be poly-{infinite cyclic} (see, for instance, [4]), the group ring KH is an Ore domain; let D be the division ring of fractions of KH. Since the quotient group H/U is cyclic of order p^k we obtain easily that D is a cyclic algebra of dimension p^{2n} $(n < k)$ over its center. It follows now from Proposition 0.3 in [1] that 1 is a commutator in D, *i.e.* the condition (1) does not hold in division rings of fractions of group rings of poly-{infinite cyclic} groups when $\operatorname{char} K = p > 0$.

An explicit example can be constructed in the following way. Let H be a semidirect product of two infinite cyclic groups, *i.e.* $H = \langle g, h \mid ghg^{-1} = h^{-1} \rangle$. The group H is an extension of the free abelian subgroup $U = \langle g^2, h \rangle$ by a cyclic group of order 2. We consider its group ring KH over an arbitrary field K of characteristic 2 and observe that

$$[g, hg^{-1}] = ghg^{-1} - h = h^{-1} - h$$

is a central element of KH. Now denote $z = h^{-1} - h$ and obtain that in the ring of fractions D the unit element is a commutator

$$[gz^{-1}, hg^{-1}] = 1.$$

ACKNOWLEDGEMENT. The work on this paper was completed when the author spent the Fall semester of 1995 in the University of Wisconsin at Madison. I am grateful to Professor D. S. Passman for fruitful discussions.

I am greatly indebted to the referee for his useful remarks which helped me to avoid an error in the first version of the proof of Theorem 1 and to improve the exposition.

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