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Nagoya Math. J.
Vol. 99 (1985), 45-62

SOME RESULTS ON HARMONIC ANALYSIS ON COMPACT QUOTIENTS OF HEISENBERG GROUPS

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Heisenberg group $H_{2g+1}(\boldsymbol{R})$ of dimension $2g+1$ is a real nilpotent group defined on $\boldsymbol{R} \times \boldsymbol{R}^g \times \boldsymbol{R}^g$ by the law of composition

$$(x_0, \hat{x}, x) \circ (y_0, \hat{y}, y) = (x_0 + y_0 + \hat{x}^t y, \hat{x} + \hat{y}, x + y),$$

which is isomorphic to the unipotent matrix group

$$\left\{\begin{pmatrix} 1 & \hat{c}_1, \cdots, \hat{c}_g, & c_0 \\ & 1 \quad & c_1 \\ & \ddots & \vdots \\ & \quad 1 & c_g \\ & & 1 \end{pmatrix}\right\} \quad (c_0, \hat{c}_i, c_i \in \boldsymbol{R}, 1 \leqslant i, j \leqslant g).$$

$H_{2g+1}(\boldsymbol{Z})$ means the discrete subgroup of integral elements, and $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$ is the L^2-space of the quotient space

$$H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})$$

with respect to the invariant measure

$$dx_0 d\hat{x} dx = dx_0 d\hat{x}_1 \cdots d\hat{x}_g dx_1 \cdots dx_g .$$

The right action of $H_{2g+1}(\boldsymbol{R})$ induces a unitary representation ρ on $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$:

$$\rho(y_0, \hat{y}, y)\phi(x_0, \hat{x}, x) = \phi((x_0, \hat{x}, x) \circ (y_0, \hat{y}, y)) .$$

For each non-zero real number λ $H_{2g+1}(\boldsymbol{R})$ also acts on the usual L^2-space $L^2(\boldsymbol{R}^g)$ as follows

$$\chi_\lambda(y_0, \hat{y}, y) f(\xi) = \exp(2\pi\lambda\sqrt{-1}(y_0 + \hat{y}^t\xi)) f(\xi + y),$$
$$(f(\xi) \in L^2(\boldsymbol{R}^g), \qquad (y_0, \hat{y}, y) \in H_{2g+1}(\boldsymbol{R})),$$

Since Lebesgue measure is invariant with respect to translations, χ_λ is a

Received March 16, 1984.

unitary representation of $H_{2g+1}(\boldsymbol{R})$.

In the present article, for each theta function $\vartheta(\tau|z)$ of level n, we shall construct a transformation

$$\phi_\vartheta:\ L^2(\boldsymbol{R}^g) \longrightarrow L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$$

such that

i) $\langle \phi_\vartheta(f_1), \phi_\vartheta(f_2)\rangle = \langle f_1, f_2\rangle$ $(f_1, f_2 \in L^2(\boldsymbol{R}^g))$

ii) $\phi_\vartheta \circ \chi_n(y_0, \hat{y}, y) = \rho(y_0, \hat{y}, y) \circ \phi_\vartheta$ $((y_0, \hat{y}, y) \in H_{2g+1}(\boldsymbol{R}))$

ϕ_ϑ is actually given by

$$\begin{aligned}&\phi_\vartheta(\exp(\pi n\sqrt{-1}(\xi\tau^t\xi))\xi^j)(x_0, \hat{x}, x)\\ &\quad = (2\pi n\sqrt{-1})^{-|j|}\exp(\pi n\sqrt{-1}(x\tau^t x + 2x^t x - 2x_0))\\ &\quad\quad \cdot\left(2\pi n\sqrt{-1} + \frac{\partial}{\partial\hat{x}}\right)^j \vartheta(\tau|\hat{x} + x\tau).\end{aligned}$$

Choosing the canonical basis of theta functions

$$\vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z) \qquad (a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ n \geqq 1),$$

we denote by $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ the transformation

$$L^2(\boldsymbol{R}^g) \longrightarrow L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$$

associating with theta function $\vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z)$ and denote

$$\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(L^2(\boldsymbol{R}^g)), \qquad \overline{\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}} = \overline{\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(L^2(\boldsymbol{R}^g))}.$$

Then the decomposition of the unitary representation ρ is given by

$$\begin{aligned}L^2\Big(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})\Big) &= \Big(\bigoplus_{\substack{a\in \boldsymbol{Z}^g/n\boldsymbol{Z}^g\\ n\geqq 1}} \boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\Big)\\ &\oplus\Big(\bigoplus_{\substack{a\in \boldsymbol{Z}^g/n\boldsymbol{Z}^g\\ n\geqq 1}} \overline{\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}\Big) \oplus \Big(\bigoplus_{(\hat{k},k)\in \boldsymbol{Z}^g\times \boldsymbol{Z}^g} \boldsymbol{C}\exp(2\pi n\sqrt{-1}(\hat{k}^t\hat{x} + k^t x))\Big).\end{aligned}$$

The invariant subspaces $\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$, $\overline{\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ $(a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ n \geqq 1)$ are independent of the choice of τ, and $\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi a^t\hat{x})\boldsymbol{H}^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

In the next article, we shall be concerned with an application to quantum mechanics.

Notations.

$Z_{\geqq 0} = \{\text{non-negative integer}\}$,

$Z^g_{\geqq 0} = \{j = (j_1, \cdots, j_g) | j_i \in Z_{\geqq 0}\}$,

$|j| = j_1 + \cdots + j_g$,

$j \pm \varepsilon_i = (j_1, \cdots, j_{i-1}, j_i \pm 1, j_{i+1}, \cdots, j_g)$,

$$\left(2\pi n\sqrt{-1}\,x + \frac{\partial}{\partial \hat{x}}\right)^j = \left(2\pi n\sqrt{-1}x_1 + \frac{\partial}{\partial \hat{x}_1}\right)^{j_1} \cdots \left(2\pi n\sqrt{-1}x_g + \frac{\partial}{\partial \hat{x}_g}\right)^{j_g}$$

$$\left(x + \ell + \frac{a}{n}\right)^j = \left(x_1 + \ell_1 + \frac{a_1}{n}\right)^{j_1} \cdots \left(x_g + \ell_g + \frac{a_g}{n}\right)^{j_g}.$$

§ 1. Equivariant isomorphisms of $L^2(R^g)$ into $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$

1.1. First we choose a complex symmetric $g \times g$ matrix $\tau = \tau' + \sqrt{-1}\,\tau''$ with positive definite imaginary part τ'', and fix τ once for all. A system of complex coordinates is introduced,

$$z = \hat{x} + x\tau\,, \qquad \bar{z} = \hat{x} + x\bar{\tau}\,. \tag{1.1}$$

Real and complex coordinates are related as follows

$$\frac{\partial}{\partial \hat{x}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\,, \qquad \frac{\partial}{\partial x} = \tau\frac{\partial}{\partial z} + \bar{\tau}\frac{\partial}{\partial z} \tag{1.2}$$

$$\tau\frac{\partial}{\partial \hat{x}} - \frac{\partial}{\partial x} = (\tau - \bar{\tau})\frac{\partial}{\partial \bar{z}} = 2\sqrt{-1}\,\tau''\frac{\partial}{\partial z}\,. \tag{1.3}$$

Let us recollect the definition of theta functions. An entire function $f(z)$ in z is called a theta function of level n with respect to τ, if it satisfies

$$f(z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau{}^t b + 2z{}^t b))f(z) \qquad ((\hat{b}, b) \in Z^g \times Z^g)\,. \tag{1.4}$$

The space $\Theta_0^{(n)}$ of theta functions of level n is a vector space of dimension n^g with a basis consisting of theta series of level n

$$\vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z) = \sum_{\ell \in Z^g} \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + 2z^t\left(\ell + \frac{a}{n}\right)\right)\right) \qquad (a \in Z^g/nZ^g)\,. \tag{1.5}$$

A function $\varphi(u, z)$ in $2g$ complex variables $(u, z) = (u_1, \cdots, u_g, z_1, \cdots, z_g)$ is called an auxiliary theta function of level n (with respect to τ), if it satisfies,

https://doi.org/10.1017/S0027763000021486 Published online by Cambridge University Press

i) $\varphi(u, z)$ is a polynomial in z whose coefficient are entire functions in u.

ii) $\varphi(u + b, z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau{}^t b + 2z{}^t b))\varphi(u, z)$
$((\hat{b}, b) \in Z^g \times Z^g)$.

In the previous article[1] the author proved that the space $\Theta^{(n)}$ of auxiliary theta functions of level n has a basis consisting of auxiliary theta series of level n,

$$(1.6)\quad \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|u, z) = \left(2\pi n\sqrt{-1}u + \frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z)$$
$$= (2\pi n\sqrt{-1})^{|j|} \sum_{\ell \in Z^g}\left(u + \ell + \frac{a}{n}\right)^j$$
$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + 2z^t\left(\ell + \frac{a}{n}\right)\right)\right)$$
$$(a \in Z^g/nZ^g,\ j \in Z^g_{\geq 0}).$$

A mixed theta function of level n (with respect to τ) mean a real analytic function $\varphi(\hat{x}, x)$ in $(\hat{x}, x)$ such that

i) $\varphi(\hat{x}, x)$ is a polynomial in x whose coefficients are entire function in complex variables $z = \hat{x} + x\tau$,

ii) $\varphi(\hat{x} + \hat{b}, x + b) = \exp(-\pi n\sqrt{-1}(b\tau{}^t b + 2(\hat{x} + x\tau){}^t b))\varphi(\hat{x}, x)$
$((\hat{b}, b) \in Z^g \times Z^g)$.

It will be shown soon that the space $\Theta^{(n)}_{\text{mix}}$ of mixed theta functions of level n (with respect to τ) has a basis consisting of mixed theta series of level n,

$$(1.7)\quad \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x, \hat{x} + x\tau)$$
$$= (2\pi n\sqrt{-1})^{|j|} \sum_{\ell \in Z^g}\left(x + \ell + \frac{a}{n}\right)^j$$
$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + 2(\hat{x} + x\tau)^t\left(\ell + \frac{a}{n}\right)\right)\right)$$
$$(a \in Z^g/nZ^g,\ j \in Z^g_{\geq 0},\ n \geq 1),$$

which are the specializations of auxiliary theta series with respect to $(u, z) \mapsto (x, \hat{x} + x\tau)$.

1.2. Let us introduced a family of real analytic functions

1) See [3].

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x_0, \hat{x}, x)$$

$$= (2\pi n\sqrt{-1})^{-|j|}\exp(\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0))\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x, \hat{x} + x\tau)$$

$$= \exp(-2\pi n\sqrt{-1}\,x_0)\sum_{\ell\in Z^g}\left(x + \ell + \frac{a}{n}\right)$$

$$\cdot\exp\left(\pi n\sqrt{-1}\left(\left(x + \ell + \frac{a}{n}\right)\tau^t\left(x + \ell + \frac{a}{n}\right) + 2\hat{x}^t\left(x + \ell + \frac{a}{n}\right)\right)\right)$$

$$(a \in Z^g/nZ^g,\ j \in Z^g,\ n \geqq 1).$$

PROPOSITION 1.1.

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x_0, \hat{x}, x) = \exp(2\pi\sqrt{-1})a^t\hat{x})\phi_j\begin{bmatrix} 0 \\ 0 \end{bmatrix}\left((\tau|x_0, \hat{x}, x)\circ\left(0, 0, \frac{a}{n}\right)\right), \tag{1.8}$$

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|(b_0, \hat{b}, b)\circ(x_0, \hat{x}, x)) = \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x_0, \hat{x}, x) \tag{1.9}$$

$$((b_0, \hat{b}\ b) \in H_{2g+1}(Z),\ a \in Z^g/nZ^g,\ j \in Z^g_{\geqq 0},\ n \geqq 1).$$

Proof. (1.8) is an immediate consequence of the definition of $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x_0, \hat{x}, x)$. For each $(b_0, \hat{b}, b)$ in $H_{2g+1}(Z)$ we have

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}((b_0, \hat{b}, b)\circ(x_0, \hat{x}, x))$$

$$= \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(b_0 + x_0 + \hat{b}^t x, \hat{b} + \hat{x}, b + x)$$

$$= \exp(2\pi n\sqrt{-1}(b_0 + x_0 + \hat{b}^t x))\sum_{\ell\in Z^g}\left(x + b + \ell + \frac{a}{n}\right)^j$$

$$\cdot\exp\left(\pi n\sqrt{-1}\left(\left(x + b + \ell + \frac{a}{n}\right)\tau^t\left(x + b + \ell + \frac{a}{n}\right)\right.\right.$$

$$\left.\left. + 2(\hat{x} + \hat{b})^t\left(x + b + \ell + \frac{a}{n}\right)\right)\right)$$

$$= \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x_0, \hat{x}, x).$$

Proposition 1.1 means that $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x_0, \hat{x}, x)$, $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|\hat{x}_0, x)$ are real analytic functions on the quotient space $H_{2g+1}(Z)\backslash H_{2g+1}(R)$.

1.3. Lie algebra $\mathfrak{h}_{2g+1}(R)$ of left invariant vector fields on $H_{2g+1}(R)$ has a basis

$$D_0 = -\frac{\partial}{\partial x_0}, \quad \hat{D}_i = \frac{\partial}{\partial \hat{x}_i}, \quad D_i = \frac{\partial}{\partial x_i} + \hat{x}_i \frac{\partial}{\partial x_0} \qquad (1 \leqq i \leqq g)$$

such that

$$[D_0, \hat{D}_i] = [D_0, D_i] = [D_i, D_k] = [\hat{D}_i, \hat{D}_k] = 0$$

$$[D_i, \hat{D}_k] = \begin{cases} D_0 & (i = k) \\ 0 & (i \neq k) . \end{cases}$$

THEOREM 1.1.

$$D_0 \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) = 2\pi n \sqrt{-1} \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) , \tag{1.10}$$

$$\hat{D}_i \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) = 2\pi n \sqrt{-1} \phi_{j+\varepsilon_i}^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) , \tag{1.11}$$

$$\begin{aligned} D_i \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) &= j_i \phi_{j-\varepsilon_1}^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) \\ &\quad + 2\pi n \sqrt{-1} \sum_{p=1}^{g} \tau_{ip} \phi_{j+\varepsilon_p}^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) \\ &\quad (1 \leqq i \leqq g,\ a \in Z^g/nZ^g,\ j \in Z^g_{\geqq 0},\ n \geqq 1) . \end{aligned} \tag{1.12}$$

Proof. From the definition of $\phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x)$ it follows

$$\begin{aligned} & D_0 \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) \\ &= -\frac{\partial}{\partial x_0} \Big\{ (2\pi n \sqrt{-1})^{-|j|} \\ &\quad \cdot \exp (\pi n \sqrt{-1} (x \tau {}^t x + 2 \hat{x} {}^t x - 2 x_0)) \vartheta_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x, \hat{x} + x\tau) \Big\} \\ &= 2\pi n \sqrt{-1} \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) , \end{aligned}$$

$$\begin{aligned} & \hat{D}_i \phi_j^{(n)} \begin{bmatrix} a/n \\ 0 \end{bmatrix} (\tau | x_0, \hat{x}, x) \\ &= \frac{\partial}{\partial \hat{x}_i} \Big\{ \exp (-2\pi n \sqrt{-1} x_0) \sum_{\ell \in Z^g} \left(x + \ell + \frac{a}{n} \right)^j \\ &\quad \cdot \exp \left(\pi n \sqrt{-1} \left(\left(x + \ell + \frac{a}{n} \right) \tau {}^t \left(x + \ell + \frac{a}{n} \right) + 2 \hat{x} {}^t \left(x + \ell + \frac{a}{n} \right) \right) \right) \Big\} \\ &= 2\pi n \sqrt{-1} \exp (-2\pi n \sqrt{-1} x_0) \sum_{\ell} \left(x + \ell + \frac{a}{n} \right)^{j + \varepsilon_i} \end{aligned}$$

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$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x+\ell+\frac{a}{n}\right)\tau^t\left(x+\ell+\frac{a}{n}\right)+2\hat{x}^t\left(x+\ell+\frac{a}{n}\right)\right)\right)$$

$$= 2\pi n\sqrt{-1}\,\phi^{(n)}_{j+\varepsilon_i}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)\,,$$

$$D_i\phi^{(n)}_j\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)$$

$$=\left(\frac{\partial}{\partial x_i}+\hat{x}_i\frac{\partial}{\partial x_0}\right)\left\{\exp\left(-2\pi n\sqrt{-1}x_0\right)\sum_{\ell\in Z^g}\left(x+\ell+\frac{a}{n}\right)^j\right.$$

$$\left.\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x+\ell+\frac{a}{n}\right)\tau^t\left(x+\ell+\frac{a}{n}\right)+2\hat{x}^t\left(x+\ell+\frac{a}{n}\right)\right)\right)\right\}$$

$$=\exp\left(-2\pi n\sqrt{-1}x_0\right)\sum_{\ell\in Z^g}\left\{j_i\left(x+\ell+\frac{a}{n}\right)^{j-\varepsilon_i}\right.$$

$$\left.+2\pi n\sqrt{-1}\left(-\hat{x}_i+\sum_{p=1}^{g}\tau_{ip}\left(x_p+\ell_p+\frac{a_p}{n}\right)+\hat{x}_i\right)\left(x+\ell+\frac{a}{n}\right)^j\right\}$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x+\ell+\frac{a}{n}\right)\tau^t\left(x+\ell+\frac{a}{n}\right)+2\hat{x}^t\left(x+\ell+\frac{a}{n}\right)\right)\right)$$

$$=j_i\phi^{(n)}_{j-\varepsilon_i}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)+2\pi n\sqrt{-1}\sum_{p=1}^{g}\phi^{(n)}_{j+\varepsilon_p}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)\,.$$

COROLLARY 1.1.1.

$$\left(D_i-\sum_{p=1}^{g}\tau_{ip}\hat{D}_p\right)\phi^{(n)}_j\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)=j_i\phi^{(n)}_{j-\varepsilon_i}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)\,, \tag{1.13}$$

$$\left(\hat{D}_iD_i-\sum_{p=1}^{g}\tau_{ip}\hat{D}_i\hat{D}_p\right)\phi^{(n)}_j\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x) \tag{1.14}$$

$$=2\pi n\sqrt{-1}\,j_i\phi^{(n)}_j\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x,\hat{x},x)\,.$$

$$(1\leqq i\leqq g,\ a\in Z^g/nZ^g,\ j\in Z^g_{\geqq 0}.\ n\geqq 1)\,.$$

These are direct consequences of (1.11), (1.12).

COROLLARY. $\vartheta^{(n)}_j\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x,\hat{x}+x\tau)$ $(a\in Z^g/nZ^g,\ j\in Z^g_{\geqq 0},\ n\geqq 1)$ *are linearly independent.*

Proof. For each $\hat{a}\in Z^g/nZ^g$ we have

$$\vartheta^{(n)}_j\begin{bmatrix}a/n\\0\end{bmatrix}\left(\tau|x,\hat{x}+\frac{\hat{a}}{n}+x\tau\right)$$

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$$= (2\pi n\sqrt{-1})^{|j|} \sum_{\ell \in Z^g} \left(x + \ell + \frac{a}{n}\right)^j$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + 2\left(\hat{x} + \frac{\hat{a}}{n}\right)^t\left(\ell + \frac{a}{n}\right)\right)\right)$$

$$= \exp\left(2\pi\sqrt{-1}\,\frac{\hat{a}^t a}{n}\right) \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x, \hat{x} + x\tau).$$

Hence by virtue of (1.10), (1.13), (1.14) and the above relation we conclude the linearly independence of

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x)$$

$$= (2\pi n\sqrt{-1})^{-|j|} \exp(\pi n\sqrt{-1}\,(x\tau^t x + 2\hat{x}^t x - 2x_0))\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x, \hat{x} + x\tau).$$

Denote

$$\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \text{the completion of the vector space}$$

$$\text{spanned by } \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) \qquad (j \in \boldsymbol{Z}_{\geqq 0}^g)$$

$$\overline{\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}} = \text{the complex conjugate of } \boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}.$$

THEOREM 1.2. $\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$, $\overline{\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ *are irreducible invariant subspace of* $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$ *with respect to the unitary representation* ρ:

$$\rho(y_0, \hat{y}, y)\phi(x_0, \hat{x}, x) = \phi((x_0, \hat{x}, x) \circ (y_0, \hat{y}, y))$$

such that

$$H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi a^t \hat{x}) H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\rho(y_0, 0, 0)\phi = \exp(-2\pi n\sqrt{-1}\, y_0)\phi \qquad \left(\phi \in \boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\right),$$

$$\rho(y_0\, 0, 0)\bar{\phi} = \exp(2\pi n\sqrt{-1}\, y_0)\bar{\phi} \qquad \left(\bar{\phi} \in \overline{\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}\right).$$

Proof. Theorem 1.1 states that the Lie algebra representation $d\rho$ of $\mathfrak{h}_{2g}(\boldsymbol{R})$ preserves $\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$, $\overline{\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ and

$$d\rho(D_0)\phi = -\frac{\partial}{\partial x_0}\phi = 2\pi n\sqrt{-1}\,\phi \qquad \left(\phi \in H_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\right),$$

$$d\rho(D_0)\bar{\phi} = \frac{\partial}{\partial x_0}\phi = -2\pi n\sqrt{-1}\,\phi \qquad \left(\bar{\phi} \in H_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\right).$$

Since $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x) = \exp(2\pi\sqrt{-1}\,a^t\hat{x})\phi_j^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau\,|\,(x_0, \hat{x}, x)\circ(0, 0, a/n))$, we complete the proof of Theorem 1.2.

1.4. Let $L^2(\boldsymbol{R}^g, \mu_{\tau''}^{(n)})$ be the L^2-space of $\boldsymbol{R}^g$ with respect to the measure

$$\pi_{\tau''}^{(n)}(d\xi) = \exp(-2\pi n\xi\tau''\xi)\,d\xi\,,$$

where $n \geqq 1$.

LEMMA 1.1. *The transformation*

$$f(\xi) \longmapsto \exp(-\pi n\sqrt{-1}\,\xi\tau^t\xi)f(\xi) \tag{1.15}$$

is an isomorphism of Hilbert space $L^2(\boldsymbol{R}^g, \mu_{\tau''}^{(n)})$ *onto* $L^2(\boldsymbol{R}^g)$.

Proof. Since $\tau - \bar{\tau} = 2\sqrt{-1}\,\tau''$, we have

$$\int_{\boldsymbol{R}^g} \overline{\exp(-\pi n\sqrt{-1}\,\xi\tau^t\xi)}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)f_2(\xi)\,d\xi$$
$$= \int_{\boldsymbol{R}^g}\exp(-2\pi n\xi\tau''^t\xi)\overline{f_1(\xi)}f_2(\xi)\,d\xi\,.$$

Hence the transformations (1.15) is an isomorphism of $L^2(\boldsymbol{R}, \mu_{\tau''}^{(n)})$ onto $L^2(\boldsymbol{R}^g)$.

Since the set of monomials $\{\xi^j\,|\,j \in \boldsymbol{Z}_{\geqq 0}^g\}$ is a basis of $L^2(\boldsymbol{R}^g, \mu_{\tau''}^{(n)})$, the set of functions $\{\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j\,|\,j \in \boldsymbol{Z}_{\geqq 0}^g\}$ is a basis of $L^2(\boldsymbol{R}^g)$.

LEMMA 1.2.

$$\int_{H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})} \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\,dx_0d\hat{x}dx \tag{1.16}$$
$$= \begin{cases} \displaystyle\int_{\boldsymbol{R}^g} y^{j+k}\exp(-2\pi ny\tau''^ty)\,dy & (n = m,\ a \equiv C \bmod n) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let us first integrate on fibers

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x),\ \phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\right\rangle$$
$$= \int_{H_{2g+1}(\boldsymbol{Z})\backslash H_{g+1}(\boldsymbol{R})} \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau/x, \hat{x}, x)dx_0d\hat{x}dx$$

$$= \sum_{\ell,\ell' \in Z^g} \int_{Z^g\backslash R^g} \int_{Z^g\backslash R^g} \left\{ \int_{Z^g\backslash R} \exp(-2\pi\sqrt{-1}(-n+m)x_0) \right.$$
$$\cdot dx_0 \left(x + \ell + \frac{a}{n}\right)^j \left(x + \ell' + \frac{c}{m}\right)^k$$
$$\cdot \exp\left(\pi\sqrt{-1}\left(\ell + \frac{a}{n}\right)\bar{\tau}^t\left(\ell + \frac{a}{n}\right) + m\left(\ell' + \frac{c}{m}\right)\tau^t\left(\ell' + \frac{c}{m}\right)\right)$$
$$\left. \cdot \exp 2\pi\sqrt{-1}\left(\hat{x}\left(-n^t\left(\ell + \frac{a}{n}\right) + m^t\left(\ell' + \frac{c}{m}\right)\right)\right)\right\} d\hat{x}dx\,.$$

This means that

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle = 0 \qquad (n \neq m)\,,$$

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(n)}\begin{bmatrix} c/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle = 0 \qquad (a \not\equiv c \bmod n)$$

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle$$
$$= \sum_{\ell \in Z^g} \int_{Z^g\backslash Z^g} \left(x + \ell + \frac{a}{n}\right)^{j+k} \exp -2\pi n\left(x + \ell + \frac{a}{n}\right)\tau''\left(x + \ell + \frac{a}{n}\right)dx$$
$$= \int_{R^g} y^{j+k} \exp(-2\pi n y \tau''^t y)\, dy\,.$$

LEMMA 1.3. *The transformations of* $L^2(R^g, \mu_{\tau''}^{(n)})$ *onto* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *given by*

(1.17) $$\xi_j \longmapsto \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0\, \hat{x}, x) \qquad (j \in Z^g_{\geqq 0})$$

is an isomorphism of Hilbert spaces.

This is an immediate consequence of Lemma 1.2.

We define unitary action of $H_{2g+1}(R)$ on $L^2(R)$ by

(1.18) $$\chi_n(x_0, \hat{x}, x) f(\xi) = \exp(-2\pi n\sqrt{-1}(x_0 + x^t\xi)) f(\xi + \hat{x})\,.$$

THEOREM 1.3. *Let* $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *be the transformation of* $L^2(R^g)$ *onto* $H_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *given by*

(1.19) $$\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\,\xi^j)(x_0, \hat{x}, x)$$
$$= \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \qquad (j \in Z^g_{\geqq 0})\,.$$

Then $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *is an isomorphism of Hilbert space* $L^2(\boldsymbol{R}^g)$ *to Hilbert space* $\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *such that*

$$\rho(y_0, \hat{y}, y) \circ \phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \circ \chi_n(y_0, \hat{y}, y) \qquad ((y_0, \hat{y}, y) \in H_{2g+1}(\boldsymbol{R}))\,, \tag{1.20}$$

$$\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi\sqrt{-1}\, a^t\hat{x})\rho\Big(0, 0, \frac{a}{n}\Big)\phi_\tau^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.21}$$

Proof. By virtue of Lemmas 1.2 and 1.3 $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ is an isomorphism of Hilbert spaces. Let us prove the equivariance of $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$. From the action

$$\chi_n(x_0, \hat{x}, x)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) = \exp(-2\pi n\sqrt{-1}\,(x_0 + x^t\xi))\exp(\pi n\sqrt{-1}(\xi + \hat{x})\tau^t(\xi + \hat{x}))(\xi + \hat{x})^j\,,$$

we have

$$\begin{aligned}
&d\chi_n(D_0)(\exp(\pi n\sqrt{-1}\,\xi\tau^t)\xi^j) \\
&\quad = -\frac{\partial}{\partial x_0}\Big|_{x_0=0}(\chi_n(x_0, \hat{x}, x) - 1)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\
&\quad = -2\pi n\sqrt{-1}\,\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j\,, \\
&d\chi_n(D_i)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\
&\quad = -2\pi n\sqrt{-1}\,\xi_i\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j \\
&\quad = -2\pi n\sqrt{-1}\,\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^{j+\varepsilon_i}\,, \\
&d\chi_n(\hat{D}_i)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\
&\quad = \exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\Big(2\pi n\sqrt{-1}\,\Big(\sum_{p=1}^{g}\tau_{ip}\xi_p\Big)\xi^j + j_i\xi^{j-\varepsilon_i}\Big) \\
&\quad = 2\pi\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^{j+\varepsilon}p + j_i\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^{j-\varepsilon_i}\,.
\end{aligned}$$

Hence by virtue of (1.10), (1.11), (1.12) we have

$$\begin{aligned}
&\Big(\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \circ d\chi_n(D_0)\Big)(\xi^j\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0, \hat{x}, x) \\
&\quad = 2\pi n\sqrt{-1}\,\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x, \hat{x}, x) \\
&\quad = d\rho(D_0)\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\xi^j\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0, \hat{x}, x)
\end{aligned}$$

$$\left(\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ d\chi_n(-D_i)\right)(\xi^j \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x)$$
$$= 2\pi n\sqrt{-1}\,\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\xi^{j+\varepsilon_i} \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x)$$
$$= 2\pi n\sqrt{-1}\,\phi_{j+\varepsilon_i}^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0, \hat{x}, x) = d\rho(\hat{D}_i)\phi_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)$$
$$= \left(d\rho(\hat{D}_i)\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\right)(\xi^j \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x),$$

$$\left(\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ d\chi_n(\hat{D}_i)\right)(\xi_j \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x)$$
$$= j_i\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\xi^{j-\varepsilon_i} \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x) + 2\pi n\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}$$
$$\cdot(\xi^{j+\varepsilon_p} \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x)$$
$$= j_i\phi_{j-\varepsilon_i}^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0, \hat{x}, x) + 2\pi n\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\phi_{j+\varepsilon_p}^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)$$
$$= d\rho(D_i)\phi_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)$$
$$= d\rho(D_i)\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\xi^j \exp(\pi n\sqrt{-1}\,\xi\tau{}^t\xi))(x_0, \hat{x}, x)\,.$$

This means

$$d\rho^{\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)}\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix} = \phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ d\chi_n\,,$$

and thus

$$\rho^{\left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right)}\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix} = \phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ\chi_n\,,$$

where

$$\rho^{\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)}(x_0, \hat{x}, x) = \rho\left((x_0, \hat{x}, x)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right).$$

§ 2. Decomposition of unitary representation ρ

2.1. An irreducible unitary representation ρ_λ of $H_{2g+1}(\boldsymbol{R})$ is characterized by a real number λ such that

$$\rho_\lambda(y_0, 0, 0)\phi = \exp(-2\pi\lambda\sqrt{-1}\,y_0)\phi$$

provided $\lambda \neq 0$. If $\lambda = 0$, then it is characterized by a pair $(\hat{k}, k) \in \boldsymbol{R}^g \times \boldsymbol{R}^g$ of vectors as follows

$$\rho_{\hat{k},k}(y_0, \hat{y}, y)\phi = \exp(2\pi\sqrt{-1}(\hat{k}^t\hat{y} + k^t y)\phi .$$

For each integer b_0 and $\phi(x_0, \hat{x}, x)$ in $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$

$$\begin{aligned}\phi(x_0, \hat{x}, x) &= \phi((b_0, 0, 0)\circ(x_0, \hat{x}, x)) = \phi((x_0, \hat{x}, x)\circ(b_0, 0, 0)) \\ &= \rho_\lambda(b_0, 0, 0)\phi(x_0, \hat{x}, x) ,\end{aligned}$$

hence for every irreducible factor ρ_λ of ρ, λ must be an integer.

LEMMA 2.1. *Let $\phi(x_0, \hat{x}, x)$ be a real analytic function on $H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})$ such that*

i) $\exp(2\pi n\sqrt{-1}\,x_0)\phi(x_0, \hat{x}, x)$ *is independent on* x_0,

ii) $\left(D_i - \sum_{p=1}^{g}\tau_{ip}\hat{D}_p\right)\phi(x_0, \hat{x}, x) = 0 \qquad (1 \leqq i \leqq g).$

Then

$$\psi(\hat{x}, x) = \exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x))\phi(x_0, \hat{x}, x)$$

is a theta function of level n in $z = \hat{x} + x\tau$ with respect to τ.

Proof. For each $(b_0, \hat{b}, b)$ in $H_{2g+}(\boldsymbol{Z})$ we have

$$\begin{aligned}&\exp(-\pi n\sqrt{-1}((x+b)\tau^t(x+b) + 2(\hat{x}+\hat{b})^t(x+b) - 2b_0 - 2x - 2\hat{b}^t x)) \\ &\quad\cdot\phi((b_0, \hat{b}, b)\circ(x_0, \hat{x}, x)) \\ &= \exp(-\pi n\sqrt{-1}(b\tau^t b + 2(\hat{x} + x\tau)^t b)) \\ &\quad\cdot\exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0))\phi(x_0, \hat{x}, x) .\end{aligned}$$

Hence we have the difference relation:

$$\psi(\hat{x} + \hat{b}, x + b) = \exp(-\pi n\sqrt{-1}(b\tau^t b + 2(\hat{x} + x\tau)^t b)\psi(\hat{x}, x) .$$

From the relation

$$\tau\frac{\partial}{\partial\hat{x}} - \frac{\partial}{\partial x} = (\tau - \bar{\tau})\frac{\partial}{\partial\bar{z}} ,$$

in order to prove $(\partial/\partial\bar{z}_i)\psi(\hat{x}, x) = 0$ $(1 \leqq i \leqq g)$, it is sufficient to show

$$\left(\tau\frac{\partial}{\partial\hat{x}} - \frac{\partial}{\partial x}\right)\psi(\hat{x}, x) = 0 .$$

$$\begin{aligned}\left(\sum_{p=1}^{g}\tau_{ip}\frac{\partial}{\partial\hat{x}_p} - \frac{\partial}{\partial x_i}\right)\psi(\hat{x}, x) &= \left(\sum_{p=1}^{g}\tau_{ip}\frac{\partial}{\partial\hat{x}_p} - \frac{\partial}{\partial x_i} - \hat{x}_i\frac{\partial}{\partial x_0}\right)\psi(\hat{x}, x) \\ &= \left(\sum_{p=1}^{g}\tau_{ip}\frac{\partial}{\partial\hat{x}_p} - \frac{\partial}{\partial x_i} - \hat{x}_i\frac{\partial}{\partial x_0}\right) \\ &\quad\cdot(\exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0)))\phi(x_0, \hat{x}, x)\end{aligned}$$

$$+ \exp(-\pi n\sqrt{-1}(x\tau{}^t x + 2\hat{x}{}^t x - 2x_0))\Big(\sum_{p=1}^{g} \tau_{ip}\hat{D}_p - D_i\Big)\phi(x_0, \hat{x}, x)$$
$$= 0 .$$

Hence $\psi(\hat{x}, x)$ is an entire function in $z = \hat{x} + x\tau$ satisfying the difference equation for theta function of level n, and thus $\psi(\hat{x}, x)$ must be a theta function of level n.

THEOREM 2.1. *Let* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *be the completion of the vector space spanned by* $\left\{\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \mid j \in Z^g_{\geqq 0}\right\}$ *and* $\overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ *be the complex conjugate of* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$. *Then* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *and* $\overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ *are irreducible invariant subspaces of* $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ *such that*

$$\rho(y_0, 0, 0)\phi(x_0, \hat{x}, x) = \exp(-2\pi n\sqrt{-1}\, y_0)\phi(x_0, \hat{x}, x) \tag{2.1}$$
$$\rho(y_0, 0, 0)\overline{\phi(x_0, \hat{x}, x)} = \exp(-2\pi n\sqrt{-1}\, y_0)\overline{\phi(x_0, \hat{x}, x)}$$
$$\Big(\phi \in H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\Big),$$

and the decomposition of ρ *is given by*

$$L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R)) \tag{2.2}$$
$$= \Big(\bigoplus_{\substack{a \in Z^g/nZ^g \\ n \geqq 1}} H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\Big) \oplus \Big(\bigoplus_{\substack{a \in Z^g/nZ^g \\ n \geqq 1}} \overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}\Big)$$
$$\Big(\bigoplus_{(\hat{k}, k) \in Z^g \times Z^g} C \exp(2\pi\sqrt{-1}(\hat{k}{}^t\hat{x} + k^t x))\Big).$$

The invariant subspaces

$$H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi a^t\hat{x})H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}} = \exp(-2\pi a^t\hat{x})\overline{H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

are independent of the choice of τ.

Proof. Since the space A of real an analytic functions on $H_{2g+1}(Z)\backslash H_{2g+1}(R)$ is dense in $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ and A is invariant for $D_0, \hat{D}_i, D_i$ $(1 \leqq i \leqq g)$, i.e. for the action of $H_{2g+1}(R)$, it is sufficient to decompose A. Let W be an irreducible invariant subspace of A such that $\rho(y_0, 0, 0)\phi(x_0, \hat{x}, x) = \exp(-2\pi n\sqrt{-1}\, y_0)\phi(x_0, \hat{x}, x)$ $(y_0 \in R, \phi \in \overline{W})$. If $n = 0$, then there exists $(\hat{k}, k) \in Z^g \times Z^g$ such that

$$W = C \exp(2\pi\sqrt{-1}(\hat{k}{}^t\hat{x} + k^t x)).$$

If n is negative, we replace W by $\overline{W}$. So we may assume that n is a positive integer. Let ν be a $\mathfrak{h}_{2g+1}(\boldsymbol{R})$-module isomorphism of $\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \cap A$ onto $\overline{W}$, i.e., a linear isomorphism satisfying

$$D_0 \circ \nu = \nu \circ D_0, \quad \hat{D}_i \circ \nu = \nu \circ \hat{D}_i, \quad D_i \circ \nu = \nu \circ D_i \quad (1 \leqq i \leqq g).$$

Since $\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \cap A$ contains the element $\phi_0^{(n)}$ satisfying

$$\Big(D_i - \sum_{d=1}^{g} \tau_{ip}\hat{D}_p\Big)\phi_0^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) = 0 \qquad (1 \leqq i \leqq g).$$

There exists an element $\phi_0(x_0, \hat{x}, x)$ in $\overline{W}$ such that

$$\Big(D_i - \sum_{p=1}^{g} \tau_{ip}\hat{D}_p\Big)\phi_0(x_0, \hat{x}, x) = 0 \qquad (1 \leqq i \leqq g).$$

From $\phi_0(x_0, \hat{x}, x) = \rho(x_0, 0, 0)\phi_0(0, \hat{x}, x) = \exp(-2\pi n\sqrt{-1}\,x_0)\phi_0(0, \hat{x}, x)$ we see that $\phi_0(x_0, \hat{x}, x)$ satisfies the conditions in Lemma 2.1, and thus

$$\phi_0(x_0, \hat{x}, x) = \exp(\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0)) \sum_{b \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g} \alpha_b \vartheta(n)\begin{bmatrix} b/n \\ 0 \end{bmatrix}(\tau \mid \hat{x} + x\tau)$$

with constants α_b. Hence

$$\phi_0(x_0, \hat{x}, x) \in \bigoplus_{b \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g} \boldsymbol{H}^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix} = \boldsymbol{H}^{(n)}.$$

On the other hand W is spanned by $\hat{D}^j\phi_0 (j \in \boldsymbol{Z}^g_{\geqq 0})$, and thus

$$W \subset \bigoplus_{b \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g} \boldsymbol{H}^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix}.$$

From the relation

$$\begin{aligned} &\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\Big(\tau \mid x_0, \hat{x} + \frac{\hat{a}}{n}, x\Big) \\ &\quad = \exp\Big(2\pi\sqrt{-1}\Big(\hat{a}^t x + \frac{1}{n}\hat{a}^t a\Big)\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \\ &\qquad (\hat{a} \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ j \in \boldsymbol{Z}^g_{\geqq 0}) \end{aligned}$$

we observe that an element ϕ in

$$\bigoplus_{b \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g} \boldsymbol{H}^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix} \quad \text{belongs to} \quad \boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$$

if and only if

https://doi.org/10.1017/S0027763000021486 Published online by Cambridge University Press

$$\phi\left(x_0, \hat{x} + \frac{\hat{a}}{n}, x\right) = \exp\left(2\pi\sqrt{-1}\left(\hat{a}^t x + \frac{1}{n}\hat{a}^t a\right)\right)\phi(x_0, \hat{x}, x)$$
$$(\hat{a} \in Z^g/nZ^g) .$$

On the other hand $\bigoplus_{b \in Z^g/nZ^g} H^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix}$ are independent on the choice of τ, hence each $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ is independent on the choice of τ. From (1.18) we have

$$H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi\sqrt{-1}\, a^t\hat{x})\, H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

COROLLARY 2.2.1. *For a non-zero integer n, ρ_n means the irreducible unitary representation such that*

$$\rho_n(y_0, 0, 0)\phi = \exp(-2\pi n\sqrt{-1}\, y_0)\phi \qquad (y_0 \in R) ,$$

then the multiplicity $m_{\rho:\rho_n}$ of ρ_n in ρ is given by

$$m_{\rho:\rho_n} = |n|^g . \tag{2.3}$$

Proof. The space of theta functions of level n is a vector space of dimension n^g, hence by virtue of Theorem 2.1 we have (2.3).

2.2. Using Hermitian polynomials we shall first construct an orthogonal basis of $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ associating with τ, and define a natural unitary representation of $Sp_{2g}(R)$ on $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$.

Hermitian polynomials $H_n(v)$ in one variable v are defined by the generating function

$$\exp(-(s^2 - 2sv)) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(v) , \tag{2.4}$$

which satisfy the orthogonal relation:

$$\int_{-\infty}^{\infty} H_n(v)H_m(v)e^{-v^2}dv = \begin{cases} 2^n n!\sqrt{\pi} & (n = m) \\ 0 & (n \neq m) . \end{cases} \tag{2.5}$$

For $j = (j_1, \cdots, j_g) \in Z^g_{\geqq 0}$ and $x = (x_1, \cdots, x_g)$ we denote

$$H_j(x) = H_{j_1}(x_1) \cdots H_{j_g}(x_g) \tag{2.6}$$

then Hermitian polynomials in many variables satisfy the orthogonal relation

$$(2.7)\qquad \int_{R^g} H_j(x)H_k(x)\exp(-x^t x)\,dx = \begin{cases} 2^{|j|}j!\pi^{g/2} & (j=k) \\ 0 & (j\neq k)\,. \end{cases}$$

Since $\tau''=(1/2\sqrt{-1})(\tau-\bar{\tau})$ is positive definite, we can define the unique square root $\sqrt{\tau''}$. The composite functions

$$H_j(x\sqrt{2\pi\tau''})\qquad (j\in Z^g_{\geqq 0})$$

satisfy the orthogonal relation:

$$(2.8)\qquad \int_{R^g} H_j(x\sqrt{2\pi\tau''})H_k(x\sqrt{2\pi\tau''})\exp(-2\pi n x\tau''^t x)dx$$
$$= \begin{cases} \dfrac{2^{|j|}j!}{2^{g/2}\sqrt{\det\tau''}} & (j=k) \\ 0 & (j\neq k)\,. \end{cases}$$

THEOREM 2.2. *Putting*

$$(2.9)\qquad H^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0,\hat{x},x)$$
$$= \frac{2^{g/4}(\det\tau'')^{1/4}}{2^{|j|/2}\sqrt{j!}}\exp(-2\pi n\sqrt{-1}\,x_0)\sum_{\ell\in Z^g} H_j\Big(\Big(x+\ell+\frac{a}{n}\Big)\sqrt{2\pi\tau''}\Big)$$
$$\cdot\exp\Big(\pi n\sqrt{-1}\Big(\Big(x+\ell+\frac{a}{n}\Big)\tau^t\Big(x+\ell+\frac{a}{n}\Big)+2\hat{x}^t\Big(x+\ell+\frac{a}{n}\Big)\Big)$$
$$(a\in Z^g/nZ^g,\ j\in Z^g_{\geqq 0},\ n\geqq 1)\,,$$

we obtain an orthonormal basis

$$\Big\{H^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0,\hat{x},x)\,|\,j\in Z^g_{\geqq 0}\Big\}\quad \textit{of}\quad H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}.$$

Proof. From the orthogonal relation for $H_j(x\sqrt{2\pi\tau''})$ and Lemma 1.2 it follows the orthogonal relation

$$\int_{H_{2g+1}(Z)\backslash H_{2g+1}(R)} \overline{H^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0,\hat{x},x)}\,H^n_k\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0,\hat{x},x)\,dx_0 d\hat{x}dx$$
$$= \begin{cases} 1 & (j=k) \\ 0 & (j\neq k)\,. \end{cases}$$

COROLLARY 2.2.1. $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ has an orthonormal basis

$$(2.12)\qquad \Big\{H^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0,\hat{x},x),\ \overline{H^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0,\hat{x},x)},$$
$$\exp(2\pi\sqrt{-1}\,(\hat{k}^t\hat{x}+k^t x))\,|\,a\in Z^g/nZ^g,$$
$$j\in Z^g_{\geqq 0},\ n\geqq 1,\ (\hat{k},k)\in Z^g\times Z^g)\Big\}\,,$$

https://doi.org/10.1017/S0027763000021486 Published online by Cambridge University Press

such that

$$(2.13)\quad H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) = \exp(2\pi\sqrt{-1}\,\hat{a})\, H_j^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \circ \left(0, 0, \frac{a}{n}\right)\right).$$

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