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SOME RESULTS ON A COMBINATORIAL PROBLEM OF CORDES

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Abstract

Cordes (1976) introduced the problem of determining the maximum number of resolution classes of a finite set partitioned into equicardinal subsets such that the number of pairs common to any 2 classes is minimized. A later paper of Mullin and Stanton (1976) investigated those conditions under which the configurations were actually BIBD's. They obtained a bound for these special configurations and conjectured it applied in general. We prove this in the present paper. A recursive and a direct construction are also given for a special class of configurations.

1. Introduction

Cordes (1976) introduces the following problem. We are given nk objects where n and k are positive integers. A partition of the objects into n sets each of cardinality k will be called a round. Letting $\sigma(n, k)$ denote the least number of pairs common to the k-sets of any 2 rounds, the Cordes problem then is to determine the maximum number $R(n, k)$ of rounds such that every pair of rounds has precisely $\sigma(n, k)$ pairs in common. In his paper Cordes gives some bounds for $R(n, k)$ and mentions the relationship of the configuration in certain instances to other combinatorial structures such as affine planes and Hadamard matrices.

Cordes has shown that $\sigma(n, k) = ns(2k - ns - n)/2$ where $s = \lfloor k/n \rfloor$ and $\lfloor\ \rfloor$ denotes the greatest integer function. Also $\sigma(n, k)$ is achieved only when each k-set from one round intersects the k-sets in every other round in either s or $s + 1$ elements.

If α denotes the number of blocks in a round that a k-set meets in s elements, then the k-set meets $n - \alpha$ blocks in $s + 1$ elements.

Since $s\alpha + (s + 1)(n - \alpha) = k$ we see that α is dependent only on n and k.

Following Mullin and Stanton (1976), for given n and k we denote a Cordes configuration on w rounds by $C(n, k, w)$.

A balanced incomplete block design (BIBD) is a pair (V, B) where V is a finite set of cardinality v whose elements are called varieties and B is a collection of b subsets of V called blocks, each of cardinality $k < v$, such that each pair of distinct varieties occurs in precisely λ blocks. Letting r denote the number of blocks containing a given variety we write the parameters of such a BIBD as (v, b, r, k, λ). It is easy to show that r is independent of the variety chosen.

If the blocks of a BIBD can be partitioned into subsets, called resolution classes, such that each variety occurs precisely once in each resolution class the BIBD is said to be resolvable (RBIBD).

In their paper Mullin and Stanton prove the following theorem.

THEOREM 1.1. *Given positive integers n, k and w such that any $C(n, k, w)$ is a RBIBD, then $R(n, k) \leqq w$.*

They express w in terms of the parameters n, k, s and α as follows,

$$w = r(n, k) = \frac{(nk - 1)\{k(k - 1) - (k - \alpha)s\}}{k(k - 1)^2 - (nk - 1)(k - \alpha)s}. \tag{1.1}$$

In the following section we actually prove that (1.1) is an upper bound for all Cordes configurations.

We note that (1.1) is in fact the r parameter of an RBIBD obtained from certain Cordes configurations. Although RBIBD's are not obtained from most Cordes configurations it is reasonable to expect that each pair of elements tends to occur with the same frequency, as Mullin and Stanton have noted.

2. An upper bound

For notational convenience we let $k = mn + l$ where $m \geqq 0$ and $0 \leqq l \leqq n - 1$ such that $k > 0$. We shall then obtain an upper bound for $R(n, mn + l)$ by extending an argument of Cordes based on the principle of inclusion and exclusion.

We assume we have a $C(n, mn + l, \delta)$. Let x_i $(1 \leqq i \leqq \delta)$ denote the number of different pairs appearing in exactly i rounds and let $N(a_{i_1} a_{i_2} \cdots a_{i_t})$ denote the number of different pairs appearing in rounds $i_1, i_2, \cdots, i_t$.

Then

$$\sum_{i=1}^{\delta} N(a_i) = \delta n \binom{mn + l}{2} \tag{2.1}$$

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$$\sum_{\substack{i,j=1\\ i\neq j}}^{\delta} N(a_i a_j) = n\binom{\delta}{2}\left[(n-l)\binom{m}{2} + l\binom{m+1}{2}\right] \tag{2.2}$$

$$\sum N(a_{i_1} a_{i_2} \cdots a_{i_t}) = x_t + \binom{t+1}{t} x_{t+1} + \cdots + \binom{\delta}{t} x_\delta \tag{2.3}$$

If we let K denote the number of pairs which appear in some round then by the principle of inclusion and exclusion

$$\begin{aligned} K &= \sum_i N(a_i) - \sum_{i,j} N(a_i a_j) + \sum_{i,j,k} N(a_i a_j a_k) - \cdots \\ &\quad + (-1)^{\delta+1} N(a_1 a_2 \cdots a_\delta) \\ &= \delta n \binom{mn+l}{2} - n\binom{\delta}{2}\left[(n-l)\binom{m}{2} + l\binom{m+1}{2}\right] \qquad (2.4) \\ &\quad + \left[x_3 + \binom{4}{3}x_4 + \cdots + \binom{\delta}{3}x_\delta\right] \\ &\quad - \left[x_4 + \binom{5}{4}x_5 + \cdots + \binom{\delta}{4}x_\delta\right] + \cdots + (-1)^{\delta+1}x_\delta \end{aligned}$$

after we substitute the expressions from (2.1), (2.2) and (2.3).

Since

$$\begin{aligned} \sum_{i=3}^{t} (-1)^{i+1}\binom{t}{i} &= \binom{t}{0} - \binom{t}{1} + \binom{t}{2} \\ &= \tfrac{1}{2}(t-1)(t-2) \end{aligned}$$

(2.4) becomes

$$\begin{aligned} K &= \delta n\binom{nm+l}{2} - n\binom{\delta}{2}\left[(n-l)\binom{m}{2} + l\binom{m+1}{2}\right] \\ &\quad + \sum_{i=3}^{\delta} \tfrac{1}{2}(i-1)(i-2)x_i \end{aligned} \tag{2.5}$$

Since $K \leqq \binom{n(nm+l)}{2}$ we will estimate a value for $\sum_{i=3}^{\delta}\frac{1}{2}(i-1)(i-2)x_i$ which (for given δ) is less than or equal to the minimum value of the expression. We then obtain an inequality for δ in terms of n, m, l which must hold in order to satisfy the inequality for K.

In order to obtain an estimate for $\sum_{i=3}^{\delta}\frac{1}{2}(i-1)(i-2)x_i$ we need the average occurrence of pairs in the configuration. We denote this by $\bar{\lambda}$. For δ rounds we easily obtain

$$\bar{\lambda} = \frac{\delta(nm + l - 1)}{n^2 m + nl - 1}.$$

LEMMA 2.1. *For any* $C(n, nm + l, \delta)$ *with* $\bar{\lambda} \geqq 1$,

$$\tfrac{1}{2}(\bar{\lambda} - 1)(\bar{\lambda} - 2)\binom{n(nm + l)}{2} \leqq \sum_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)x_i.$$

PROOF. For $1 \leqq \bar{\lambda} \leqq 2$, $\tfrac{1}{2}(\bar{\lambda} - 1)(\bar{\lambda} - 2)\binom{n(nm + l)}{2} \leqq 0$ while $\Sigma_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)x_i$ is always non-negative since $x_i \geqq 0$ for all i. So we consider the case when $\bar{\lambda} > 2$. We let $\bar{\lambda} = a + b/c$ with $a \geqq 2$ and $0 \leqq b < c$ where $c = \binom{n(nm + l)}{2}$, the total number of different possible pairs.

We now show that $\Sigma_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)x_i$ is minimized when the only non-zero x_i are x_a and x_{a+1}. Let $x_0, x_1, \cdots, x_\delta$ be a solution which minimizes $\Sigma_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)x_i$ subject to the necessary constraints that $\Sigma_{i=0}^{\delta} x_i = c$ and $\Sigma_{i=0}^{\delta} i x_i = \delta n \binom{nm + l}{2}$. If $x_i > 0$ and $x_{i+j} > 0$, where $j \geqq 2$, then the new solution given by

$$\bar{x}_i = x_i - 1, \qquad \bar{x}_{i+j} = x_{i+j} - 1,$$
$$\bar{x}_{i+1} = x_{i+1} + 1, \qquad \bar{x}_{i+j-1} = x_{i+j-1} + 1,$$
$$\bar{x}_l = x_l, \qquad l \neq i, i + j, i + 1, i + j - 1$$

yields a value

$$\sum_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)\bar{x}_i < \sum_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)x_i$$

except when $i = 0$, $j = 2$. In this case the 2 expressions have the same value. But $\bar{\lambda} = a + b/c > 2$, so some $x_j > 0$ for $j \geqq 3$. Thus the minimum is obtained when the only possible non-zero x_i are x_a and x_{a+1}. In fact if $b = 0$, the minimum is obtained at $x_a > 0$ and $x_i = 0$, $i \neq a$. We note that the minima obtained are implicitly related to the integrality of the variables x_i.

So we have

$$x_a + x_{a+1} = c, \qquad a x_a + (a + 1)x_{a+1} = ac + b$$

which yields

$$x_a = c - b, \qquad x_{a+1} = b. \tag{2.6}$$

Now

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$$\tfrac{1}{2}(\bar{\lambda}-1)(\bar{\lambda}-2)\binom{n(mn+l)}{2} \tag{2.7}$$
$$=\frac{1}{2}\left(\frac{ac+b-c}{c}\right)\left(\frac{ac+b-2c}{c}\right)c$$
$$=\frac{a^2c^2+2abc-3ac^2+b^2-3bc+2c^2}{2c}.$$

If $a=2$, $b>0$ and so using (2.6)

$$\sum_{i=3}^{\delta}\tfrac{1}{2}(i-1)(i-2)x_i=b. \tag{2.8}$$

Then (2.7) becomes $(b^2+bc)/2c$. If we assume $(b^2+bc)/2c>b$ then $b^2>bc$ which contradicts $0\leqq b<c$. So $(2.7)\leqq(2.8)$ as required.

If $a>2$ then (2.6) yields

$$\sum_{i=3}^{\delta}\tfrac{1}{2}(i-1)(i-2)x_i=\tfrac{1}{2}[(a-1)(a-2)(c-b)+(a)(a-1)(b)] \tag{2.9}$$
$$=\tfrac{1}{2}[a^2c-3ac+2c+2ab-2b].$$

Assuming $(2.7)>(2.9)$ yields

$$a^2c^2+2abc-3ac^2+b^2-3bc+2c^2>a^2c^2-3ac^2+2c^2+2abc-2bc$$

or $b^2>bc$ which again contradicts $0\leqq b<c$. Therefore $(2.7)\leqq(2.9)$ and the proof is complete. □

With the previous lemma we now prove the main result of this section.

THEOREM 2.2.

$$R(n,nm+l)\leqq\frac{(n^2m+nl-1)\{(nm+l-1)(nm+l)-m(nm-n+2l)\}}{(nm+l)(nm+l-1)^2-m(n^2m+nl-1)(nm-n+2l)}.$$

PROOF. If $\bar{\lambda}=\delta(nm+l-1)/(n^2m+nl-1)<1$ then $\delta<(n^2m+nl-1)/(nm+l-1)$. Now if

$$\frac{n^2m+nl-1}{nm+l-1}>\frac{(n^2m+nl-1)\{(nm+l-1)(nm+l)-m(nm-n+2l)\}}{(nm+l)(nm+l-1)^2-m(n^2m+nl-1)(nm-n+2l)}$$

then

$$(nm+l)(nm+l-1)^2-m(n^2m+nl-1)(nm-n+2l)$$
$$>(nm+l)(nm+l-1)^2-m(nm+l-1)(nm-n+2l).$$

So

$$m(n^2m+nl-1)(nm-n+2l)<m(nm+l-1)(nm-n+2l).$$

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If $m = 0$ we have a contradiction. If $m > 0$ we obtain $n^2m + nl - 1 < nm + l - 1$ which implies $n < 1$, again a contradiction. So for $\bar{\lambda} < 1$, $(n^2m + nl - 1)/(nm + l - 1)$ satisfies the result. For $\bar{\lambda} \geqq 1$ we use the expression (2.5) for K where as previously noted $K \leqq \binom{n(mn + l)}{2}$. So

$$\delta n\binom{nm + l}{2} - n\binom{\delta}{2}\left[(n - l)\binom{m}{2} + l\binom{m + 1}{2}\right] + \sum_{i=3}^{\delta} \tfrac{1}{2}(i - 1)(i - 2)x_i \leqq \binom{n(nm + l)}{2}.$$

Replacing the summation by $\frac{1}{2}(\bar{\lambda} - 1)(\bar{\lambda} - 2)\binom{n(nm + l)}{2}$ as allowed by Lemma 2.1, the inequality becomes

$$\begin{aligned}&\frac{\delta n(nm + l)(nm + l - 1)}{2} - \frac{n^2\delta(\delta - 1)m(m - 1)}{4} - \frac{n\delta(\delta - 1)lm}{2}\\ &+ (n^2m + nl)(n^2m + nl - 1)[\delta nm + \delta l - \delta - (n^2m + nl - 1)]\\ &\quad \cdot \frac{[\delta nm + \delta l - \delta - 2(n^2m + nl - 1)]}{4(n^2m + nl - 1)^2}\\ &\leqq \frac{(n^2m + nl)(n^2m + nl - 1)}{2}\end{aligned}$$

Multiplying both sides by $4(n^2m + nl - 1)$ yields

$$\begin{aligned}&(n^2m + nl - 1)\\ &\quad \cdot [2\delta n(nm + l)(nm + l - 1) - n^2\delta(\delta - 1)m(m - 1) - 2n\delta(\delta - 1)lm]\\ &+ (n^2m + nl)\\ &\quad \cdot [(\delta nm + \delta l - \delta)^2 - 3(n^2m + nl - 1)(\delta nm + \delta l - \delta) + 2(n^2m + nl - 1)^2]\\ &\leqq 2(n^2m + nl)(n^2m + nl - 1)^2.\end{aligned}$$

Subtracting $2(n^2m + nl)(n^2m + nl - 1)^2$ from both sides and dividing by δn we obtain

$$\begin{aligned}&(n^2m + nl - 1)[2(nm + l)(nm + l - 1) - n(\delta - 1)m(m - 1) - 2(\delta - 1)lm]\\ &\quad + (nm + l)[(nm + l - 1)^2\delta - 3(nm + l - 1)(n^2m + nl - 1)] \leqq 0.\end{aligned}$$

Collecting terms involving δ we obtain

$$\begin{aligned}&\delta[(nm + l)(nm + l - 1)^2 - nm(m - 1)(n^2m + nl - 1) - 2lm(n^2m + nl - 1)]\\ &\quad \leqq (n^2m + nl - 1)\{(nm + l)(nm + l - 1) - m(nm - n + 2l)\}.\end{aligned}$$

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This simplifies to

$$\delta \leqq \frac{(n^2m + nl - 1)\{(nm + l)(nm + l - 1) - m(nm - n + 2l)\}}{(nm + l)(nm + l - 1)^2 - m(n^2m + nl - 1)(nm - n + 2l)}$$

and the proof is complete. □

Letting $k = nm + l$, $s = m$ and $\alpha = n - l$ we obtain the suggested bound of Mullin and Stanton (1976), that

$$R(n, k) \leqq \frac{(nk - 1)\{k(k - 1) - (k - \alpha)s\}}{k(k - 1)^2 - (nk - 1)(k - \alpha)s} = r(n, k).$$

So although pairs do not occur with equal frequency the size of a $C(n, k, \delta)$ is bounded by considerations that pairs occur with the same average frequency.

3. Some improvements on the bound

In the previous section we obtained an upper bound for $R(n, nm + l)$ by estimating the minimum value of $\Sigma_{i=3}^{\delta}\frac{1}{2}(i - 1)(i - 2)x_i$ with $\frac{1}{2}(\bar{\lambda} - 1)(\bar{\lambda} - 2)\cdot\binom{n(nm + l)}{2}$. In many cases this estimate is low and the bound can be improved.

For specified m, n and l, let δ be the largest integer less than or equal to the bound of Theorem 2.2.

Let

$$\bar{\lambda}^* = \lfloor \delta(nm + l - 1)/(n^2m + nl - 1) \rfloor . \tag{3.1}$$

As shown in Lemma 2.1, the minimum occurs when the only non-zero x_i are $x_{\bar{\lambda}^*}$ and $x_{\bar{\lambda}^*+1}$. Thus we solve the following two equations for integers b and c

$$b\bar{\lambda}^* + c(\bar{\lambda}^* + 1) = nk\binom{nm + l}{2} \tag{3.2}$$

and

$$b + c = \binom{n(nm + l)}{2} \tag{3.3}$$

We then calculate K from (2.5) with $x_i = 0$ except for $x_{\bar{\lambda}^*} = b$ and $x_{\bar{\lambda}^*+1} = c$. If $K \leqq \binom{n(nm + l)}{2}$ then $R(n, nm + l) \leqq \delta$, otherwise $R(n, nm + l) \leqq \delta - 1$ and we repeat the procedure replacing δ by $\delta - 1$. We denote the bound obtained by $r^*(n, nm + l)$. The above technique was used as an algorithm in a computer program to calculate $r^*(n, nm + l)$ for small

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values of n, m and l. The results are listed in Table 3.3 at the end of this section. Also listed, for comparison, is $r(n, nm+l)$ the bound of Theorem 2.2.

Since the technique that determines $r^*(n, nm+l)$ uses the actual minimum of $\sum_{i=3}^{\delta}\frac{1}{2}(i-1)(i-2)x_i$ instead of an estimate, $r(n, nm+l) - r^*(n, nm+l) \geqq 0$.

In fact, this difference can be arbitrarily large in certain instances.

THEOREM 3.1.

$$\lim_{p\to\infty}[r(2p, 3p) - r^*(2p, 3p)] = \infty.$$

PROOF. Letting $n = 2p$, $m = 1$ and $l = p$ Theorem 2.2 gives

$$r(2p, 3p) = \frac{(6p^2-1)[(3p-1)(3p)-2p]}{3p(3p-1)^2-(6p^2-1)(2p)} = \frac{54p^3-30p^2-9p+5}{15p^2-18p+5}$$

Using the procedure outlined above we obtain $r^*(2p, 3p) = 3p+2$ for $p > 4$. So

$$r(2p, 3p) - r^*(2p, 3p) = \frac{9p^3-6p^2+12p-5}{15p^2-18p+5}$$

and the result follows. □

THEOREM 3.2. $R(2, 2m+1) \leqq 4m+4$ for $m \geqq 2$.

PROOF. From Theorem 2.2

$$R(2, 2m+1) \leqq \frac{(4m+1)[2m(2m+1)-m(2m)]}{(2m+1)(2m)^2-m(4m+1)(2m)}$$

which reduces to

$$R(2, 2m+1) \leqq 4m+5+\frac{1}{m} \quad \text{or} \quad R(2, 2m+1) \leqq 4m+5.$$

Letting $\delta = 4m+5$, solving (3.1), (3.2) and (3.3) yields

$$\bar{\lambda}^* = \lfloor (4m+5)(2m)/(4m+1) \rfloor = 2m+1$$

so $b = 4m+2$ and $c = (2m+1)(4m-1)$. Thus (2.5) gives

$$K = (4m+5)2\binom{2m+1}{2} - 2\binom{4m+5}{2}\left[\binom{m}{2}+\binom{m+1}{2}\right]$$
$$+\tfrac{1}{2}(2m)(2m-1)2(2m+1)+\tfrac{1}{2}(2m+1)(2m)(2m+1)(4m-1)$$

which simplifies to

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$$K = 8m^2 + 7m > \binom{4m+2}{2}.$$

Therefore $\delta \leqq 4m+4$ and solving as above for $\delta = 4m+4$ we obtain $K = m(8m+5) < \binom{4m+2}{2}$, so $R(2, 2m+1) \leqq 4m+4$. □

Using the same method it can be shown that $R(2, 2m) \leqq 4m-1$ as has also been shown by Cordes (1976).

	$n = 3$		$n = 4$		$n = 5$		$n = 6$		$n = 7$	
$mn + l$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$
4	7.9	7								
5	9.3	8	8.1	6						
6	8.5	7	10.3	9	8.8	7				
7	12.3	11	10.8	9	11.1	9	9.6	8		
8	13.8	13	10.3	9	12.3	11	12.0	9	10.5	9
9	13.0	13	13.5	13	12.6	11	13.6	13	12.9	10
10	16.8	16	15.5	14	12.3	11	14.3	13	14.6	12
11	18.3	17	16.1	14	15.1	13	14.4	13	15.8	15
12	17.5	16	15.7	14	17.2	16	14.2	13	16.3	15
13	21.3	20	18.8	17	18.5	17	16.8	15	16.4	15
14	22.8	22	20.8	19	18.8	17	19.0	17	16.2	15
15	22.0	22	21.5	21	18.5	17	20.6	19	18.7	16
16	25.8	25	21.0	21	21.3	21	21.4	20	20.9	18
17	27.3	26	24.1	22	23.4	22	21.6	20	22.6	22
18	26.5	25	26.1	25	24.7	23	21.4	20	23.8	22
19	30.3	29	26.8	25	25.1	23	24.0	22	24.4	23
20	31.8	31	26.3	25	24.8	23	26.2	25	24.5	23
21	31.0	31	29.5	29	27.6	26	27.7	26	24.3	23
22	34.8	34	31.4	30	29.7	28	28.6	26	26.9	24
23	36.3	35	32.1	30	30.9	29	28.8	27	29.0	27
24	35.5	34	31.7	30	31.3	31	28.6	26	30.7	29
25	39.3	38	34.8	33	31.0	31	31.2	31	31.9	30
26	40.8	40	36.8	35	33.8	32	33.4	32	32.5	30
27	40.0	40	37.4	37	35.9	34	34.9	33	32.7	30
28	43.8	43	37.0	37	37.2	36	35.8	33	32.5	30
29	45.3	44	40.1	38	37.6	36	36.0	34	35.0	32
30	44.5	43	42.1	41	37.3	36	35.8	33	37.2	36

(table continued on next page)

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	$n=8$		$n=9$		$n=10$		$n=11$		$n=12$	
$mn+l$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$	$r(s)$	$r^*(s)$
9	11.5	10								
10	13.8	11	12.4	11						
11	15.7	12	14.7	12	13.3	12				
12	17.0	15	16.6	13	15.6	13	14.3	13		
13	17.9	17	18.2	15	17.6	14	16.6	14	15.3	14
14	18.2	17	19.3	19	19.3	15	18.6	15	17.5	15
15	18.3	17	19.9	19	20.6	17	20.3	16	19.6	16
16	18.1	17	20.2	19	21.4	21	21.8	18	21.4	17
17	20.6	18	20.3	19	22.0	21	22.8	20	22.9	18
18	22.8	20	20.1	19	22.2	21	23.5	23	24.1	20
19	24.6	22	22.5	20	22.2	21	24.0	23	25.0	22
20	26.0	25	24.7	22	22.1	21	24.2	23	25.6	25
21	26.9	25	26.6	23	24.5	22	24.2	23	26.0	25
22	27.3	26	28.1	26	26.6	24	24.1	23	26.2	25
23	27.5	26	29.2	28	28.5	25	26.4	24	26.2	25
24	27.3	26	29.9	28	30.1	27	28.6	25	26.1	25
25	29.7	27	30.3	29	31.4	31	30.5	27	28.4	26
26	31.9	29	30.4	29	32.3	31	32.2	28	30.5	27
27	33.7	33	30.3	29	32.9	31	33.6	30	32.5	29
28	35.0	34	32.6	30	33.3	32	34.7	34	34.2	30
29	35.9	34	34.8	32	33.3	32	35.4	34	35.7	32
30	36.4	34	36.6	34	33.2	32	36.0	35	36.9	34

Table 3.3

4. Other results

The class of configurations $C(2, k, R(2, k))$ are interesting due to their relationship with other combinatorial structures. For $k = 2m$ Cordes has shown the following.

THEOREM 4.1. $R(2,2m) \leqq 4m - 1$ *with equality if and only if there exists a Hadamard matrix of order* $4m$.

We refer the reader to Wallis, Street, Wallis (1972) for the definition and an account of Hadamard matrices. For $k = 2m + 1$, less is known. Cordes showed that $R(2,3) = 10$ and Theorem 3.2 proves that $R(2,2m+1) \leqq 4m + 4$ for $m \geqq 2$. We obtain a lower bound for $R(2,2m+1)$ in the following.

THEOREM 4.2.

$$R(2,2m+1) \geqq R(2,2m).$$

PROOF. It is straightforward to verify that adding a new element x to one block in each round and adding a new element y to the other block in each round transforms a $C(2,2m,w)$ into a $C(2,2m+1,w)$. □

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Generalizing a recursive technique of Cordes one obtains the following result.

THEOREM 4.3.

$$R(2,4n+1) \geqq 2\min[R(2,2n), R(2,2n+1)] + 1.$$

PROOF. Take a $C(2,2n,\alpha)$ and choose one block from each of the α rounds. We form the $\alpha \times 4n$ incidence matrix A for these blocks as follows. $A = (a_{ij})$ where $a_{ij} = 1$ if object j is in the block chosen from round i. Otherwise $a_{ij} = 0$. Similarly we take a $C(2,2n+1,\beta)$ and choose one block from each of the β rounds. We then form the $\beta \times (4n+2)$ incidence matrix B. We shall assume without a loss of generality that $\alpha < \beta$. We then truncate B to B_α, where B_α is obtained from B by dropping rows $\alpha+1, \alpha+2, \cdots, \beta$.

We now form the incidence matrix

$$C = \left(\begin{array}{c|c} A & B_\alpha \\ \hline A & B_\alpha^* \\ \hline 111\ldots1 & 1000\ldots0 \end{array}\right)$$

where B_α^* is obtained from B_α by interchanging 0's and 1's. It is straightforward to verify that C is a $(2\alpha+1)\times(8n+2)$ matrix with row sums being $4n+1$. It is also easy to see that any 2 rows of C have either $2n$ or $2n+1$ 1's in common. Thus we can interpret C as the incidence matrix of a set of blocks, one from each round, of a $C(2,4n+1,2\alpha+1)$. □

Using the technique above one can also prove

THEOREM 4.4.

$$R(2,4n-1) \geqq 2\min[R(2,2n-1), R(2,2n)] + 1.$$

Since a Hadamard matrix of order 2^n always exists and $R(2,3) = 10$ (see Cordes (1976)) we use Theorem 4.4 recursively to obtain

THEOREM 4.5.

$$2^{n+1} - 1 \leqq R(2,2^n - 1) \leqq 2^{n+1} \quad \text{for} \quad n \geqq 3.$$

It is possible by methods of differences to obtain lower bounds for some configurations with $k = 2m+1$. For a block $B = \{v_1, v_2, \cdots, v_{2m+1}\}$ we form all possible differences $\pm(v_i - v_j) \bmod 4m+2$. For $i \neq j$ one obtains each residue l modulo $4m+2$ α_l times. It is easy to see that for $l \neq 2m+1, B$ and $B + l = \{v_1 + l, v_2 + l, \cdots, v_{2m+1} + l\}$ have precisely α_l elements in common. Also B and $B + (2m+1)$ have precisely $2\alpha_{2m+1}$ elements in common since $2m+1 = -(2m+1)$ modulo $4m+2$.

Now if B intersects $B+1, B+2, \cdots, B+(4m+2)$ in either m or $m+1$

elements then B^c, the complement of B in the set of residues modulo $4m+2$ intersects each block $B+1, \cdots, B+(4m+2)$ in the remaining m or $m+1$ elements. Similarly for $B^c+1, B^c+2, \cdots, B^c+(4m+2)$ we find the desired intersection with B and B^c of either m or $m+1$ elements. Since the differences in B are the same as in $B+l$, the existence of a block B, in which each difference l occurs m or $m+1$ times except $2m+1$ for which $2\alpha_{2m+1}$ is m or $m+1$, implies the existence of a $C(2, 2m+1, 4m+2)$. By taking B and B^c as the first round and forming $4m+1$ other rounds by generating $B+i$, B^c+i for $i = 1, 2, \cdots, 4m+1$ we obtain such a $C(2, 2m+1, 4m+2)$.

With this approach a computer search generated the following difference blocks for $m = 2, 4, 5, 6$ and showed none existed for $m = 3$,

$$m = 2 \qquad B = \{1, 2, 3, 4, 7\}$$

$$m = 4 \qquad B = \{1, 2, 3, 4, 5, 7, 10, 14, 15\}$$

$$m = 5 \qquad B = \{1, 2, 3, 4, 5, 7, 8, 10, 14, 15, 18\}$$

$$m = 6 \qquad B = \{1, 2, 3, 4, 5, 6, 8, 11, 14, 16, 18, 22, 23\}.$$

If B contains m even (odd) integers and $m+1$ odd (even) integers then we can add a new round in which 1 block contains the even integers and the other block contains the odd integers from the residues modulo $4m+4$. We note that this is the case in the blocks displayed for $m = 2, 4$ and 5.

Another construction is possible using the method of differences and a well-known result of Bose. The configurations obtained are also relatively close to the upper bound of Theorem 3.2. We shall form a difference block B with the elements being taken from $GF(p^\alpha)$ where $p^\alpha \equiv 1 \bmod 4$. Since $p \neq 2$ if the non-zero field element y occurs as a difference λ_y times then B and $B+y$ will have precisely λ_y elements in common. The following result is taken from Bose (1947).

LEMMA 4.6. *Let $p^\alpha = 4n+1$ where p is a prime and write the non-zero elements of $GF(p^\alpha)$ which are squares (the quadratic residues) as $x^0, x^2, \cdots, x^{4n-2}$ (x is a primitive element). Then among the totality of differences of the quadratic residues every non-zero quadratic residue occurs $n-1$ times and every quadratic non-residue occurs n times.*

Letting $B = \{\infty, x^0, x^2, \cdots, x^{4n-2}\}$ we see that B intersects $B+y$ ($y \in GF(p^\alpha)\backslash\{0\}$) in n or $n+1$ elements since $\infty + y = \infty$. Letting the blocks $B, B+x^0, B+x, \cdots, B+x^{4n-1}$ represent one block from each round of a $C(2, 2n+1, 4n+1)$ defined on the variety set $GF(p^\alpha) \cup \{\infty\}$ we have the following result.

THEOREM 4.7. *If* $p^\alpha = 4n+1$ *then* $4n+1 \leqq R(2, 2n+1) \leqq 4n+4$.

Cordes showed that $R(2,3) = 10$. Using this, the difference blocks given earlier, and the recursive constructions, the following small values are bounded as shown below

$$11 \leqq R(2,5) \leqq 12$$

$$15 \leqq R(2,7) \leqq 16$$

$$19 \leqq R(2,9) \leqq 20$$

$$23 \leqq R(2,11) \leqq 24$$

$$26 \leqq R(2,13) \leqq 28$$

Now $R(2,5) = 12$ as the following 12 rounds shows,

1 2 3 4 5	6 7 8 9 10
1 2 3 9 10	4 5 6 7 8
1 4 5 9 10	2 3 6 7 8
1 2 4 6 10	3 5 7 8 9
1 2 4 7 8	3 5 6 9 10
1 2 5 7 9	3 4 6 8 10
2 3 4 7 9	1 5 6 8 10
2 3 5 8 10	1 4 6 7 9
2 4 5 6 9	1 3 7 8 10
3 4 5 7 10	1 2 6 8 9
1 3 4 8 9	2 5 6 7 10
1 3 5 6 7	2 4 8 9 10

The above $C(2,5,12)$ was obtained by an exhaustive computer search. Also obtained in this fashion was a $C(3,4,7)$ which originally motivated Cordes. This shows that $R(3,4) = 7$ and is shown below,

1 2 3 4	*a b c d*	*A B C D*
1 2 *a A*	3 *b c B*	4 *d C D*
1 3 *b C*	2 *c d D*	4 *a A B*
1 2 *d B*	4 *a b D*	3 *c A C*
1 4 *c B*	3 *a d C*	2 *b A D*
2 4 *c C*	1 *b d A*	3 *a B D*
3 4 *d A*	1 *a c D*	2 *b B C*

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