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SIMPLE EIGENVALUES AND BIFURCATION FOR A MULTIPARAMETER PROBLEM

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1. Introduction

We are concerned with the problem of bifurcation of solutions of a non-linear multiparameter problem at a simple eigenvalue of the linearised problem.

Let X and Y be real Banach spaces, and let A, B_i, $i=1,\dots,n\in B(X,Y)$. Let $\mathcal{N}$: $R^n\times X\to Y$ be a non-linear mapping. We consider the equation

$$M(\lambda,x):=L(\lambda)x+\mathcal{N}(\lambda,x)=0 \tag{1.1}$$

where

$$L(\lambda):=A-\sum_{i=1}^{n}\lambda_i B_i \tag{1.2}$$

and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in R^n$ is an n-tuple of spectral parameters.

Such non-linear multiparameters problems have been the subject of much recent work (see e.g. [**1, 2, 5**]). The case $n=1$ is covered by the work of Crandall and Rabinowitz [**3**]. However, for $n>1$, different definitions of the notion of a simple eigenvalue have been given, and it is with this that we are mostly concerned.

In Section 2 we discuss different concepts of a simple eigenvalue of a multiparameter operator (1.2) that have appeared in the literature. We propose a generalised definition and give an illustrative example. Lemmas 2.5 and 2.6 are concerned with the nature of the multiparameter simple eigenvalue and its associated eigenvector.

Section 3 considers the non-linear problem and the main result, Theorem 3.1, shows the existence of non-trivial solutions of (1.1) bifurcating from simple eigenvalues of the linearised operator (1.2) at points where the non-linear term satisfies some standard conditions.

2. Definitions of a simple eigenvalue

Extensions to a multiparameter setting of the notion of a simple eigenvalue of a linear operator have been made by various authors. Shearer [**7**] gives the following definition of a simple eigenvalue of a two-parameter family of operators,

$$\mathcal{F}=\{g(\lambda)\in B(X,Y)\,|\,\lambda=(\lambda_1,\lambda_2)\in R^2\}$$

where $g\in C^r(R^2,B(X,Y))$, $r\geqq 1$, the set of r times continuously (Fréchet) differentiable mappings of R^2 into $B(X,Y)$.

Definition 2.1. $\lambda^0=(\lambda_1^0,\lambda_2^0)\in R^2$ is called a simple eigenvalue for $\mathscr{F}$ if

(i) $g(\lambda^0)\in B(X,Y)$ is Fredholm with index zero and

$$\dim N(g(\lambda^0))=\operatorname{codim} R(g(\lambda^0))=1;$$

(ii) there exists $(\alpha_1,\alpha_2)\in R^2$ such that

$$(\alpha_1 D_{\lambda_1}g(\lambda^0)+\alpha_2 D_{\lambda_2}g(\lambda^0))x_0\notin R(g(\lambda^0))$$

where $x_0\in N(g(\lambda^0))$ and $D_{\lambda_i}g$ denotes the Fréchet derivative of g with respect to λ_i, $i=1,2$.

For $g(\lambda):=A-\lambda_1B_1-\lambda_1B_2$, A, B_1, $B_2\in B(X,Y)$ part (ii) of this definition reduces to

(ii)′ there exists $(\alpha_1,\alpha_2)\in R^2$ such that

$$\alpha_1B_1x_0+\alpha_2B_2x_0\notin R(A_1-\lambda_1^0B_1-\lambda_2^0B_2)$$

where $x_0\in N(A_1-\lambda_1^0B_1-\lambda_2^0B_2)$.

A natural extension of this definition to a linear n-parameter operator (1.2) is as follows:

$\lambda^0=(\lambda_1^0,\dots,\lambda_n^0)\in R^n$ is a simple eigenvalue of (1.2) if

(i) $L(\lambda^0)$ is Fredholm with index zero and

$$\dim N(L(\lambda^0))=\operatorname{codim} R(L(\lambda^0))=1;$$

(ii) there exists $(\alpha_1,\alpha_2,\dots,\alpha_n)\in R^n$ such that

$$\sum_{i=1}^{n}\alpha_iB_ix_0\notin R(L(\lambda^0))$$

where $x_0\in N(L(\lambda^0))$.

However, Hale [5] (see also [2]) gives the following definition of a simple eigenvalue of (1.2):

Definition 2.2. $\lambda^0=(\lambda_1^0,\lambda_2^0,\dots,\lambda_n^0)\in R^n$ is a simple eiganvalue of (1.2) if

(i) $L(\lambda^0)\in B(X,Y)$ is Fredholm with index $1-n$,

and

$$\dim N(L(\lambda^0))=1,\ \operatorname{codim} R(L(\lambda^0))=n;$$

(ii) if $Y_0:=\operatorname{Span}\{B_ix_0,\ i=1,\dots,n\}$, where $x_0\in N(L(\lambda^0))$, then $Y=Y_0\oplus R(L(\lambda^0))$.

Remarks.

(1) If $X=Y=R^n$, then every linear operator $L(\lambda)$: $R^n \to R^n$ has index zero. Thus, in this case, Hale's concept of a simple eigenvalue requires that $n=1$. This is rather restrictive for finite dimensional problems.

(2) The rationale of Hale's definition is that simple eigenvalues are isolated points in the parameter space, whereas Shearer's definition gives rise to curves in the parameter space.

(3) In the literature use has been made of both definitions. For example, Zachman [**9**] implicitly uses Definition 2.1, while Turyn [**8**] uses Definition 2.2.

Here we propose the following:

Definition 2.3. $\lambda^0=(\lambda_1^0,\ldots,\lambda_n^0)\in R^n$ is a generalised simple eigenvalue (G-simple eigenvalue) of (1.2) if

(i) $\dim N(L(\lambda^0))=1, 0<\operatorname{codim} R(L(\lambda^0))=m\leqq n$;

(ii) $B_i x_0 \notin R(L(\lambda^0))$, $i=1,\ldots,n$, where $x_0\in N(L(\lambda^0))$ and

$$Y=\operatorname{span}\{B_i x_0,\ i=1,\ldots,n\}\oplus R(L(\lambda^0)).$$

This is exactly Definition 2.2 if $m=n$. On the other hand, although we allow $\operatorname{codim} R(L(\lambda^0))=1$ as in Definition 2.1, the direct sum in condition (ii) is stronger than the condition required by Shearer. Further refinements in this area are presently being pursued.

As a simple example of Definition 2.3 consider the following:

Example 2.4. Let $X=R^2$, $Y=R^3$,

$$A=\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\quad B_1=\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix},\quad B_2=\begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix},\quad B_3=\begin{bmatrix}0&0\\1&1\\-1&0\end{bmatrix}.$$

For $\lambda^0=(0,0,0)$,

$$N(L(\lambda^0))=\operatorname{span}\{e_1\},\ e_1:=(1,0)^T;$$

$$R(L(\lambda^0))=\operatorname{span}\{E_1\},\ E_1:=(1,0,0)^T.$$

$B_1e_1=E_2:=(0,1,0)^T$; $B_2e_1=E_3:=(0,0,1)^T$, $B_3e_1=E_2-E_3$. It is easily seen that $\lambda^0=(0,0,0)$ is a G-simple eigenvalue of

$$L(\lambda)=A-\lambda_1B_1-\lambda_2B_2-\lambda_3B_3.$$

However, we should note that for $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)\in R^3$

$$L(\boldsymbol{\lambda})x=0, \quad x\neq 0$$

$$\Leftrightarrow \begin{bmatrix} x_2 \\ -(\lambda_1+\lambda_3)x_1-\lambda_3 x_2 \\ (-\lambda_2+\lambda_3)x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad x=(x_1,x_2)\neq 0$$

$$\Leftrightarrow x_2=0, \quad \lambda_1=-\lambda_2=-\lambda_3.$$

Thus we have an eigencurve though $\boldsymbol{\lambda}^0=(0,0,0)$: the straight line $\lambda_1=-\lambda_2=-\lambda_3$, and each point on this line is a G-simple eigenvalue with the same eigenvector. In this sense we have a G-simple eigencurve.

In general, if $\boldsymbol{\lambda}^0\in R^n$ is a G-simple eigenvalue of (1.2), then the coset $\Gamma:=\boldsymbol{\lambda}^0+K$, where K is some well defined $n-m$ dimensional subspace of R^n will consist entirely of eigenvalues with the same eigenvector: this follows from the implied linear dependence of $B_i x_0$, $i=1,\dots,n$, where $x_0\in N(L(\boldsymbol{\lambda}^0))$. To see this, we re-order the parameters so that $\{B_i x_0,\ i=1,\dots,m\}$ forms a basis for $Y_0:=\operatorname{Span}\{B_i x_0,\ i=1,\dots,n\}$. Then $X=X_0\oplus X_1$, $X_0:=N(L(\boldsymbol{\lambda}^0))=\operatorname{span}\{x_0\}$

$$Y=Y_0\oplus Y_1, \quad Y_1:=R(L(\boldsymbol{\lambda}^0)). \tag{2.1}$$

Now

$$B_j x_0=\sum_{i=1}^{m}\alpha_{ij}B_i x_0 \quad \text{for some } \alpha_{ij},\ i=1,\dots m,\ j=m+1,\dots n,$$

so that

$$Ax_0-\sum_{i=1}^{n}\lambda_i^0 B_i x_0=0$$

$$\Leftrightarrow Ax_0-\sum_{i=1}^{n}\lambda_i(\mathbf{t})B_i x_0=0 \quad \mathbf{t}=(t_{m+1},\dots t_n)\in R^{n-m} \tag{2.2}$$

where

$$\boldsymbol{\lambda}(\mathbf{t})=\boldsymbol{\lambda}^0+\sum_{j=m+1}^{n}t_j\boldsymbol{\alpha}_j \tag{2.3}$$

and

$$\boldsymbol{\alpha}_j=(\alpha_{1j},\alpha_{2j},\dots\alpha_{mj},0,\dots,0,-1,0,\dots,0),$$

where -1 appears in the jth position. The equation (2.2) shows that x_0 is an eigenvector corresponding to the eigenvalue $\lambda(\mathbf{t})$ for all $\mathbf{t}\in R^{n-m}$. The set

$$\Gamma:=\{\lambda(\mathbf{t})\,|\,\mathbf{t}\in R^{n-m}\} \tag{2.4}$$

is a coset as described above.

What is not claimed is that each point in Γ is a G-simple eigenvalue since the dimension of the null space and the co-dimension of the range may vary, and further property (ii) in Definition 2.3 may not be satisfied. The next two lemmas consider the nature of eigenvalues of (1.2) close to the G-simple eigenvalue $\boldsymbol{\lambda}^0$.

Lemma 2.5. *Let $\boldsymbol{\lambda}^0$ be a G-simple eigenvalue of* (1.2) *with corresponding eigenvector x_0. Given $\varepsilon>0$, there exists $\delta>0$ such that, if $\boldsymbol{\lambda}\in R^n$ and $x=x_0+x_1\in X$, $x_1\in X_1$(see* (2.1)) *satisfy*

$$|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|:=\operatorname{Max}|\lambda_i-\lambda_i^0|<\delta$$

and

$$L(\boldsymbol{\lambda})x=0$$

then

$$\|x_1\|<\varepsilon\|x_0\|. \tag{2.5}$$

Proof.

$$L(\boldsymbol{\lambda})(x_0+x_1)=0\Leftrightarrow L(\boldsymbol{\lambda}^0)x_1=\sum_{i=1}^{n}(\lambda_i-\lambda_i^0)B_i(x_0+x_1)$$

Since $L(\boldsymbol{\lambda}^0)$ is a bijection of X_1 onto $Y_1:=R(L(\boldsymbol{\lambda}^0))$, there exists $c>0$ such that

$$\|L(\boldsymbol{\lambda}^0)x_1\|\geqq c\|x_1\|\quad x_1\in X_1.$$

Therefore

$$c\|x_1\|\leqq\left\|\sum_{i=1}^{n}(\lambda_i-\lambda_i^0)B_i(x_0+x_1)\right\|$$

$$\leqq K|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|(\|x_0\|+\|x_1\|),\ K=\sum_{i=1}^{n}\|B_i\|$$

$$\Rightarrow(c-K|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|)\|x_1\|\leqq K|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|\,\|x_0\|$$

$$\Rightarrow\|x_1\|\leqq\frac{K|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|}{c-K|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|}\|x_0\|\ \text{provided}\ |\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|<\frac{c}{K}.$$

Thus (2.5) hold for $|\boldsymbol{\lambda}-\boldsymbol{\lambda}^0|<\delta:=c\varepsilon/K(1+\varepsilon)$.

We use the following notation introduced by Binding [1]:

$$\text{for} \quad \boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_n)\in R^n$$

$$\boldsymbol{\lambda}_m:=(\lambda_1,\dots,\lambda_m)\in R^m, \quad \boldsymbol{\mu}_m:=(\lambda_{m+1},\dots,\lambda_n)\in R^{n-m},$$

so that

$$\boldsymbol{\lambda}=(\boldsymbol{\lambda}_m,\boldsymbol{\mu}_m).$$

Lemma 2.6. *Let $\boldsymbol{\lambda}^0$ be a G-simple eigenvalue of* (1.2), *and let* Γ *be defined by* (2.3)–(2.4). *Then there exists a neighbourhood $U\subset R^n$ of $\boldsymbol{\lambda}^0$ such that $\Gamma\cap U$ consists entirely of G-simple eigenvalues of* (1.2) *with* $\dim N(L(\boldsymbol{\lambda}))=1$, $\operatorname{codim} R(L(\boldsymbol{\lambda}))=m\,\forall\,\boldsymbol{\lambda}\in\Gamma\cap U$, *and U contains no other eigenvalues of* (1.2).

Proof. Consider the mapping $\psi: R^m\times R^{n-m}\times X_1\to Y$ defined by

$$\psi(\boldsymbol{\lambda}_m,\boldsymbol{\mu}_m,x_1)=L(\boldsymbol{\lambda})(x_0+x_1) \tag{2.6}$$

where x_0 is the eigenvector of (1.2) corresponding to $\boldsymbol{\lambda}^0$. ψ is continuous and

$$\psi(\boldsymbol{\lambda}_m^0,\boldsymbol{\mu}_m^0,0)=L(\boldsymbol{\lambda}^0)x_0=0.$$

Taking the Fréchet derivative of (2.6) with respect to $(\boldsymbol{\lambda}_m,x_1)$ we obtain

$$[(D_{(\boldsymbol{\lambda}_m,x_1)}\psi)(\boldsymbol{\lambda}_m^0,\boldsymbol{\mu}_m^0,0)](\boldsymbol{\lambda}_m,x_1)=-\sum_{i=1}^{m}\lambda_iB_ix_0+L(\boldsymbol{\lambda}^0)x_1$$

and it follows that

$$(D_{(\boldsymbol{\lambda}_m,x_1)}\psi)(\boldsymbol{\lambda}_m^0,\boldsymbol{\mu}_m^0,0)\in B(R^m\times X_1,Y)$$

is a linear homeomorphism. From the implicit function theorem (see e.g. [4]), there exists a δ-neighbourhood $V_\delta\subset R^{n-m}$ of $\boldsymbol{\mu}_m^0$ and unique continuous functions $\boldsymbol{\lambda}_m^*: V_\delta\to R^m$ and $z^*: V_\delta\to X_1$ such that

$$\boldsymbol{\lambda}_m^*(\boldsymbol{\mu}_m^0)=\boldsymbol{\lambda}_m^0, \quad z^*(\boldsymbol{\mu}_m^0)=0$$

and

$$\psi(\boldsymbol{\lambda}_m^*(\boldsymbol{\mu}_m),\boldsymbol{\mu}_m,z^*(\boldsymbol{\mu}_m))=0\;\forall\,\boldsymbol{\mu}_m\in V_\delta$$

i.e.

$L(\boldsymbol{\lambda}_m^*(\boldsymbol{\mu}_m),\boldsymbol{\mu}_m)(x_0+z^*(\boldsymbol{\mu}_m))=0\;\forall\,\boldsymbol{\mu}_m\in V_\delta$. It follows from the above discussion that $\{(\boldsymbol{\lambda}_m^*(\boldsymbol{\mu}_m),\boldsymbol{\mu}_m)\,|\,\boldsymbol{\mu}_m\in V_\delta\}\subset\Gamma$ and that $z^*(\boldsymbol{\mu}_m)=0\;\forall\,\boldsymbol{\mu}_m\in V_\delta$.

From stability theory for Fredholm operators (see [6]) it follows that for sufficiently small $|\lambda-\lambda^0|$, $\dim N(L(\lambda))\leqq\dim N(L(\lambda^0))=1$, so that for sufficiently small δ

$$N(L(\lambda_m^*(\mu_m),\mu_m))=\operatorname{span}\{x_0+z^*(\mu_m)\}=\operatorname{span}\{x_0\}. \tag{2.7}$$

Further by the stability of the index,

$$\operatorname{codim} R(L(\lambda_m^*(\mu_m),\mu_m))=\operatorname{codim} R(L(\lambda^0))=m\ \forall\mu_m\in V_\delta.$$

Now, define $\phi(\lambda): R^m\times X_1\to Y$ by

$$\phi(\lambda)(\mathbf{a},x_1)=\sum_{i=1}^{m}a_iB_ix_0+L(\lambda)x_1,$$

where $\mathbf{a}=(a_1,\dots,a_m)\in R^m$.

By the definition of a G-simple eigenvalue, $\phi(\lambda^0)$ is an isomorphism. Since $\phi(\cdot)$ is continuous, it follows that, for $|\lambda-\lambda^0|$ sufficiently small, $\phi(\lambda)$ is also an isomorphism, so that

$$Y=R(L(\lambda))+\operatorname{span}\{B_ix_0, i=1,\dots,m\}$$

for $|\lambda-\lambda^0|$ sufficiently small. If, in addition $(\lambda_m^*(\mu_m),\mu_m)\in\Gamma$, then $\operatorname{codim} R(L(\lambda))=m$ and

$$Y=R(L(\lambda))\oplus\operatorname{span}\{B_ix_0, i=1,2,\dots,m\}.$$

It follows from (2.7) that, for $U=\{\lambda\in R^n\,|\,|\lambda-\lambda^0|$ sufficiently small$\}$, $\Gamma\cap U$ consists entirely of G-simple eigenvalues.

In addition, the uniqueness result of the implicit function theorem shows that there exists $\varepsilon>0$ such that, if

$$|\lambda_m-\lambda_m^0|<\varepsilon,\quad \|x_1\|<\varepsilon,\quad |\mu_m-\mu_m^0|<\delta$$

and

$$\psi(\lambda_m,\mu_m,x_1)=0$$

then

$$\lambda_m=\lambda_m^*(\mu_m),\quad x_1=z^*(\mu_m).$$

Finally, we must show that there is a neighbourhood of $\lambda^0=(\lambda_m^0,\mu_m^0)$ which contains no eigenvectors which cannot be written in the form x_0+x_1, $x_1\in X_1$.

Let $x_0^0=(x_0/\|x_0\|)$, so that $\|x_0^0\|=1$. From Lemma 2.5, there exists $\Delta<\min(\varepsilon,\delta)$ such that, if λ is an eigenvalue of (1.2) with $|\lambda-\lambda^0|<\Delta$ and $N(L(\lambda))=\operatorname{span}\{ax_0^0+x_1\}$, $a\in R, x_1\in X_1$, then

$$\|x_1\|<\varepsilon\|ax_0^0\|=\varepsilon|a|\Rightarrow a\neq 0.$$

Therefore, $x_0^0+(x_1/a)\in N(L(\lambda))$, $|\lambda_m-\lambda_m^0|<\varepsilon$, $\|x_1/a\|=\varepsilon$, and $|\mu_m-\mu_m^0|<\delta$, which together with $\psi(\lambda_m,\mu_m,x_1/a)=0$ imply

$$\lambda=(\lambda_m,\mu_m)\in\Gamma.$$

3. Bifurcation at a G-simple eigenvalue

Our main result is that given some standard conditions on the non-linear term in (1.1), a G-simple eigenvalue of (1.2) is a bifurcation point for the non-linear problem.

Theorem 3.1. *Let $\lambda^0\in R^n$ be a G-simple eigenvalue of* (1.2) *and let $\mathcal{N}:R^n\times X\to Y$ satisfy*

C1: *$\mathcal{N}\in C^r(R^n\times X,Y)$, the space of r-times continuously Fréchet differentiable mappings, $r\geqq 2$;*

C2: $\mathcal{N}(\lambda,0)=0$;

C3: $D_x\mathcal{N}((\lambda_m,\mu_m^0),0)=0$.

Then $(\lambda^0,0)\in R^n\times X$ is a bifurcation point for solutions of (1.1) *and there exists a set of solutions*

$$\{(\lambda,x)=((\lambda_m^*(u,\mu_m),\mu_m),x_m^*(u,\mu_m))|u\in(-\delta,\delta)\subset R\quad \text{for some } \delta>0;\ |\mu-\mu_m^0|<\varepsilon \text{ for some } \varepsilon>0\}$$

where $\lambda_m^*:R\times R^{n-m}\to R^m$ and $x^*:R\times R^{n-m}\to X$ are C^{r-1} mappings.

Proof. The results follow by an application of the Liapunov–Schmidt method (see [2]). Let Q_0 and Q_1 be the projections of Y onto Y_0 and Y_1 respectively (see (2.1)). Then

$$M(\lambda,x)=0 \tag{3.1}$$

$$\Leftrightarrow Q_1M(\lambda,x)=0 \quad\text{and}\quad Q_0M(\lambda,x)=0, \tag{3.2}$$

the so-called auxiliary equation and bifurcation equation respectively.

The auxiliary equation takes the form

$$Q_1L(\lambda)x_1+Q_1\mathcal{N}((\lambda_m,\mu_m),x_0+x_1)=0 \tag{3.3}$$

where $x=x_0+x_1$, $x_i\in X_i$ $i=0,1$.

Consider the mapping $\psi:R^m\times R^{n-m}\times X_0\times X_1\to Y_1$ defined by

$$\psi(\lambda_m,\mu_m,x_0,x_1)=Q_1L(\lambda)x_1+Q_1\mathcal{N}((\lambda_m,\mu_m),x_0+x_1). \tag{3.4}$$

Using C_2 and C_3 we obtain

$$\psi(\lambda_m^0,\mu_m^0,0,0)=0,$$

$$D_{x_1}\psi(\lambda_m^0,\mu_m^0,0,0)=Q_1L(\lambda^0).$$

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Since $Q_1L(\lambda^0)$ is a linear homeomorphism of X_1 onto Y_1, it follows from the implicit function theorem that there exists a neighbourhood $U \subset R^m \times R^{n-m} \times X_0$ of $(\lambda_m^0, \mu_m^0, 0)$, and a unique mapping $z^* \in C^r(U, X_1)$ such that

$$z^*(\lambda_m^0, \mu_m^0, 0) = 0$$

and

$$\psi(\lambda_m, \mu_m, x_0, z^*(\lambda_m, \mu_m, x_0)) = 0$$

i.e.

$$Q_1L(\lambda)z^*(\lambda_m, \mu_m, x_0) + Q_1\mathcal{N}((\lambda_m, \mu_m), x_0 + z^*(\lambda_m, \mu_m, x_0)) = 0. \tag{3.5}$$

Since, from C2, the point $(\lambda_m, \mu_m, 0, 0)$ satisfies (3.3) and, by the implicit function theorem, z^* is unique, it follows that

$$z^*(\lambda_m, \mu_m, 0) = 0 \ \forall (\lambda_m, \mu_m, 0) \in U. \tag{3.6}$$

Differentiation of (3.5) with respect to x_0 and using C3 and (3.6) gives

$$Q_1L(\lambda_m, \mu_m^0)D_{x_0}z^*(\lambda_m, \mu_m^0, 0) = 0.$$

Since for $|\lambda_m - \lambda_m^0|$ sufficiently small $Q_1L(\lambda_m, \mu_m^0)$ is a homeomorphism of X_1 onto Y_1 we can conclude that

$$D_{x_0}z^*(\lambda_m, \mu_m^0, 0) = 0 \text{ for } |\lambda_m - \lambda_m^0| \text{ sufficiently small.}$$

This may require the neighbourhood U to be restricted.

Differentiating (3.6) repeatedly with respect to μ_m gives

$$D_{\mu_m}^k z^*(\lambda_m, \mu_m, 0) = 0, \quad 1 \leqq k \leqq r. \tag{3.7}$$

Therefore $z^* \in C^r(U, X_1)$ satisfies

(1) $z^*(\lambda_m, \mu_m^0, 0) = 0$,

(2) $D_{x_0}z^*(\lambda_m, \mu_m^0, 0) = 0$,

(3) $D_{\mu_m}^k z^*(\lambda_m, \mu_m^0, 0) = 0, \quad 1 \leqq k \leqq r$,

and so by Taylor's theorem

$$z^*(\lambda_m, \mu_m, x_0) = O(\|x_0\|(\|x_0\| + |\mu_m - \mu_m^0|)) \quad \text{as} \quad \|x_0\|, |\mu_m - \mu_m^0| \to 0.$$

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The bifurcation equation becomes

$$Q_0L(\lambda)(x_0+z^*(\lambda_m,\mu_m,x_0))+Q_0\mathcal{N}((\lambda_m,\mu_m),x_0+z^*(\lambda_m,\mu_m,x_0))=0$$

$$\Leftrightarrow -\sum_{i=1}^{m}(\lambda_i-\lambda_i^0)B_ix_0-\sum_{i=m+1}^{n}(\lambda_i-\lambda_i^0)B_ix_0+Q_0L(\lambda)z^*(\lambda_m,\mu_m,x_0)$$

$$+Q_0\mathcal{N}((\lambda_m,\mu_m),\ x_0+z^*(\lambda_m,\mu_m,x_0))=0. \tag{3.8}$$

Let $x_0=ux_0^0$, $\|x_0^0\|=1$, $u\in R$. Using the basis vectors $B_ix_0^0$, $i=1,\dots,m$, the bifurcation function

$$F=(F_1,F_2,\dots,F_m):R^m\times R^{n-m}\times R\to R^m$$

is defined by

$$\sum_{i=1}^{m}F_i(\lambda_m,\mu_m,u)B_ix_0^0:=-u\sum_{i=1}^{m}(\lambda_i-\lambda_i^0)B_ix_0^0+\sum_{i=1}^{m}G_i(\lambda_m,\mu_m,u)B_ix_0^0$$

$$:=-u\sum_{i=1}^{m}(\lambda_i-\lambda_i^0)B_ix_0^0-u\sum_{i=m+1}^{n}(\lambda_i-\lambda_i^0)B_ix_0^0+Q_0L(\lambda)z^*(\lambda_m,\mu_m,ux_0^0)$$

$$+Q_0\mathcal{N}((\lambda_m,\mu_m),ux_0^0+z^*(\lambda_m,\mu_m,ux_0^0)).$$

It follows that

$$F(\lambda_m,\mu_m,u)=-u(\lambda_m-\lambda_m^0)+G(\lambda_m,\mu_m,u)$$

where

$$G=(G_1,G_2,\dots,G_m):R^m\times R^{n-m}\times R\to R^m,$$

$$G(\lambda_m,\mu_m,0)=0,$$

$$D_uG(\lambda_m,\mu_m^0,0)=0,$$

and

$$D_{\mu_m}^kG(\lambda_m,\mu_m,0)=0,\quad 1\leqq k\leqq r.$$

Thus we can write

$$G(\lambda_m,\mu_m,u)=u\tilde{G}(\lambda_m,\mu_m,u)$$

where $\tilde{G}\in C^{r-1}(R^m\times R^{n-m}\times R,R^m)$, and the bifurcation equation reduces to

$$H(\lambda_m,\mu_m,u):=-(\lambda_m-\lambda_m^0)+\tilde{G}(\lambda_m,\mu_m,u)=0 \tag{3.9}$$

where

$$\tilde{G}(\lambda_m^0, \mu_m^0, 0) = 0 \text{ and } D_{\lambda_m}\tilde{G}(\lambda_m^0, \mu_m^0, 0) = 0.$$

Therefore

$$H(\lambda_m^0, \mu_m^0, 0) = 0 \quad \text{and} \quad D_{\lambda_m}H(\lambda_m^0, \mu_m^0, 0) = -\mathrm{Id}_m,$$

where Id_m denotes the identity mapping on R^m, and so by the implicit function theorem, there exist a neighbourhood

$$V \subset R^{n-m} \times R \text{ of } (\mu_m^0, 0) \quad \text{and a unique function} \quad \lambda_m^* \in C^{r-1}(V, R^m) \quad \text{such that}$$

$$\lambda_m^*(\mu_m, u) = \lambda_m^0 + 0(|u|, |\mu_m - \mu_m^0|)$$

and

$$H(\lambda_m^*(\mu_m, u), \mu_m, u) = 0 \; \forall (\mu_m, u) \in V.$$

Thus (1.1) has a non-trivial solution $((\lambda_m^*, \mu_m), x^*) \in R^n \times X$ given by

$$\lambda_m^*(\mu_m, u) = \lambda_m^0 + 0(|u|, |\mu_m - \mu_m^0|)$$

$$x^* = ux_0^0 + z^*(\lambda_m^*(\mu_m, u), \mu_m, ux_0^0)$$

$$= ux_0^0 + 0(|u|(|u| + |\mu_m - \mu_m^0|))$$

for $(\mu_m, u) \in V$ such that $(\lambda_m^*(\mu_m, u), \mu_m, ux_0^0) \in U$.

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