ON THE STEPANOV-ALMOST PERIODIC SOLUTION OF A SECOND-ORDER OPERATOR DIFFERENTIAL EQUATION

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1. Introduction

Suppose X is a Banach space and J is the interval $-\infty < t < \infty$. For $1 \leqq p < \infty$, a function $f \in L^p_{\text{loc}}(J; X)$ is said to be Stepanov-bounded or S^p-bounded on J if

$$\| f \|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \| f(s) \|^p ds \right]^{1/p} < \infty \tag{1.1}$$

(for the definitions of almost periodicity and S^p-almost periodicity, see Amerio-Prouse (**1**, pp. 3 and 77).

Let $\mathscr{L}(X, X)$ be the Banach space of all bounded linear operators on X into itself, with the uniform operator topology.

Our theorem is as follows.

Theorem. *Suppose f: $J \to X$ is an S^p-almost periodic continuous function $(1 \leqq p < \infty)$, and B: $J \to \mathscr{L}(X, X)$ is almost periodic with respect to the norm of $\mathscr{L}(X, X)$. Then any S^p-almost periodic solution of the second-order operator differential equation*

$$u''(t) = B(t)u(t) + f(t) \quad \text{on } J \tag{1.2}$$

is also almost periodic from J to X.

2. Proof of Theorem

By (1.2), we have the representation

$$u'(t) = u'(0) + \int_0^t [B(s)u(s) + f(s)]ds \quad \text{on } J.$$

Since B is almost periodic from J to $\mathscr{L}(X, X)$, we have

$$\sup_{t \in J} \| B(t) \| = K < \infty. \tag{2.1}$$

Further, since u is S^p-almost periodic from J to X, it is S^p-bounded on J.

Now, given $\varepsilon > 0$, suppose that τ is an ε-almost period of B and also an

ε-S^p-almost period of u (see Amerio-Prouse (**1**, pp. 10, 77 and 78)). Then we have

$$\left[\int_t^{t+1} \| B(s+\tau)u(s+\tau)-B(s)u(s) \|^p ds\right]^{1/p}$$

$$\leqq \left[\int_t^{t+1} \| B(s+\tau)-B(s) \|^p . \| u(s+\tau) \|^p ds\right]^{1/p} + \left[\int_t^{t+1} \| B(s) \|^p . \| u(s+\tau)-u(s) \|^p ds\right]^{1/p}$$

$$\leqq \varepsilon \| u \|_{S^p}+K\varepsilon \quad \text{on } J, \text{ by (1.1) and (2.1).}$$

Thus it follows that $B(t)u(t)$ is S^p-almost periodic from J to X. Consequently, $B(t)u(t)+f(t)$ is S^p-almost periodic from J to X. Hence, by Amerio-Prouse (**1**, Theorem 8, p. 79), u' is uniformly continuous on J.

Now consider a sequence $\{\psi_n(t)\}_{n=1}^{\infty}$ of infinitely differentiable non-negative functions on J such that

$$\psi_n(t)=0 \quad \text{for} \quad |t| \geqq n^{-1}, \quad \int_{-n^{-1}}^{n^{-1}} \psi_n(t)dt=1. \tag{2.2}$$

The convolution between u and ψ_n is defined by

$$(u*\psi_n)(t)=\int_J u(t-s)\psi_n(s)ds=\int_J u(s)\psi_n(t-s)ds.$$

Since u' is uniformly continuous on J, given $\eta>0$, there exists $\delta>0$ such that

$$\| u'(t_1)-u'(t_2) \| \leqq \eta \quad \text{for} \quad t_1, t_2 \in J \quad \text{with} \quad | t_1-t_2 | \leqq \delta.$$

So we have, for $| t_1-t_2 | \leqq \delta$,

$$\| (u'*\psi_n)(t_1)-(u'*\psi_n)(t_2) \| \leqq \int_{-n^{-1}}^{n^{-1}} \| u'(t_1-s)-u'(t_2-s) \| \psi_n(s)ds$$

$$\leqq \eta \int_{-n^{-1}}^{n^{-1}} \psi_n(s)ds=\eta, \text{ by (2.2).}$$

Therefore $u'*\psi_n$ is uniformly continuous on J for $n = 1, 2, \ldots$.

Since u is S^p-almost periodic (and hence is S^1-almost periodic) from J to X, we can show that $u*\psi_n$ is almost periodic from J to X for $n = 1, 2, \ldots$.

Moreover, we have

$$(u*\psi_n)'(t)=(u'*\psi_n)(t) \quad \text{on } J.$$

Consequently, by Amerio-Prouse (**1**, Theorem 6, p. 6), $u'*\psi_n$ is almost periodic from J to X for $n = 1, 2, \ldots$.

Now we observe that, by the uniform continuity of u' on J, the sequence of convolutions $(u'*\psi_n)(t)$ converges to $u'(t)$ uniformly on J. So u' is almost periodic from J to X. Hence u is uniformly continuous on J (u' being bounded

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on J). Thus, by Amerio-Prouse (**1**, Theorem 7, p. 78), u is almost periodic from J to X. This completes the proof of the theorem.

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