Glasgow Math. J. **50** (2008) 47–53. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507003916. Printed in the United Kingdom

GENERATORS OF REGULAR SEMIGROUPS

SHMUEL KANTOROVITZ
Bar-Ilan University 52900 Ramat-Gan, Israel
e-mail: kantor@math.biu.ac.il

(Received 23 October, 2006; revised 10 June, 2007; accepted 24 June, 2007)

Abstract. A regular semigroup (cf. [**4**, p. 38]) is a C_0-semigroup $T(\cdot)$ that has an extension as a holomorphic semigroup $W(\cdot)$ in the right halfplane $\mathbb{C}^+$, such that $||W(\cdot)||$ is bounded in the "unit rectangle" $Q := (0, 1] \times [-1, 1]$. The important basic facts about a regular semigroup $T(\cdot)$ are: (i) it possesses a *boundary group* $U(\cdot)$, defined as the limit $\lim_{s\to 0+} W(s + i\cdot)$ in the strong operator topology; (ii) $U(\cdot)$ is a C_0-group, whose generator is iA, where A denotes the generator of $T(\cdot)$; and (iii) $W(s + it) = T(s)U(t)$ for all $s + it \in \mathbb{C}^+$ (cf. Theorems 17.9.1 and 17.9.2 in [**3**]). The following *converse theorem* is proved here. Let A be the generator of a C_0-semigroup $T(\cdot)$. If iA generates a C_0-group, $U(\cdot)$, then $T(\cdot)$ is a regular semigroup, and its holomorphic extension is given by (iii). This result is related to (but *not included* in) known results of Engel (cf. Theorem II.4.6 in [**2**]), Liu [**7**] and the author [**6**] for holomorphic extensions into *arbitrary* sectors, of C_0-semigroups that are *bounded in every proper subsector*. The method of proof is also different from the method used in these references. Criteria for generators of regular semigroups follow as easy corollaries.

2000 *Mathematics Subject Classification.* 47D03, 47D05

1. Introduction. The study of boundary values of holomorphic semigroups motivates the following definition.

DEFINITION 1.1 ([**4**, p. 38]). A *regular semigroup* of operators on the Banach space X is a C_0-semigroup $T(\cdot)$ on X, that has an extension as a holomorphic semigroup $W(\cdot)$ in the open halfplane $\mathbb{C}^+ := \{s + it; s > 0,\ t \in \mathbb{R}\}$ such that $||W(\cdot)||$ is bounded in the "unit rectangle"

$$Q := \{s + it; 0 < s \leq 1,\ -1 \leq t \leq 1\}. \tag{1}$$

Recall that a *holomorphic semigroup* in the right halfplane is a function $W(\cdot) : \mathbb{C}^+ \cup \{0\} \to B(X)$ with the following properties:

(a) $W(\cdot)$ is holomorphic in $\mathbb{C}^+$;
(b) $W(\cdot)$ is strongly continuous at 0; and
(c) $W(0) = I$ and $W(z + w) = W(z)W(w)$ for all $z, w \in \mathbb{C}^+$.

A regular semigroup $T(\cdot)$ is characterized by the existence of a (unique) holomorphic extension $W(\cdot)$ in $\mathbb{C}^+$, that possesses a *boundary group* $U(\cdot)$, defined as the limit

$$U(t) := \lim_{s\to 0+} W(s + it) \qquad (t \in \mathbb{R}) \tag{2}$$

in the strong operator topology.

The group $U(\cdot)$ is a C_0-group commuting with $W(\cdot)$, and

$$W(s+it) = T(s)U(t) \qquad (s>0;\ t\in\mathbb{R}). \tag{3}$$

Furthermore, if A denotes the (infinitesimal) generator of the regular semigroup $T(\cdot)$, then iA is the generator of the associated boundary group $U(\cdot)$. Cf. [**3**, Theorems 17.9.1 and 17.9.2].

Conversely, we shall prove in Section 2 that if $T(\cdot)$ is a C_0-semigroup, whose generator A is such that iA generates a C_0-group, $U(\cdot)$, then $T(\cdot)$ is a regular semigroup. When this is the case, the unique holomorphic extension of $T(\cdot)$ to $\mathbb{C}^+$ is given by (3).

As a consequence, we obtain conditions on an operator A that are necessary and sufficient for it to be the generator of a regular semigroup.

For convenience, we use [**5**] as a reference for needed facts about semigroups.

We wish to thank the referee for his useful suggestions (especially for the remark following Theorem 2.1 and a simplification of the proof of Corollary 3.6).

2. Characterization of regular semigroups.

THEOREM 2.1. *Let $T(\cdot)$ be a C_0-semigroup, and let A be its generator. Then $T(\cdot)$ is a regular semigroup if and only if iA is the generator of a C_0-group, $U(\cdot)$. In this case, the (unique) holomorphic extension of $T(\cdot)$ to $\mathbb{C}^+$ is given by $W(s+it) = T(s)U(t)$, $(s>0; t\in\mathbb{R})$.*

(The associated boundary group is necessarily $U(\cdot)$.)

This result is *not* contained in [**2**, Theorem II.4.6], that concerns the holomorphic extension into a sector of a *bounded* C_0-semigroup, that is *bounded* in each proper subsector (cf. [**2**, Definition II.4.5]). Note the equivalent Condition (e) in [**2**, Theorem II.4.6] ("A is sectorial"), that includes in particular the requirement that the resolvent set of A contains a sector of half opening *strictly greater than* $\pi/2$ (cf. [**2**, Definition II.4.1]), while no such requirement is made here.

Proof. We discussed the necessity part in Section 1. To prove the sufficiency part, suppose that iA generates a C_0-group $U(\cdot)$. By [**5**, Theorem 1.1] applied to the C_0-semigroups $\{T(t); t\geq 0\}$, $\{U(t); t\geq 0\}$, and $\{U(-t; t\geq 0\}$, there exist constants $a, b\geq 0$ and $M, N\geq 1$ such that

$$||T(s)|| \leq M\,e^{as}; \qquad ||U(t)|| \leq N\,e^{b|t|} \qquad (s\geq 0; t\in\mathbb{R}). \tag{4}$$

We *define* $W(\cdot)$ on $\mathbb{C}^+$ by (3).

Fix x in the domain $D(A)$ of A, and consider the X-valued function on $\mathbb{C}^+$

$$g(s+it) := W(s+it)Ax = T(s)U(t)Ax = T(s)AU(t)x \qquad (s>0;\ t\in\mathbb{R}). \tag{5}$$

(Cf. [**5**, Theorem 1.2].)

Observe that g is strongly continuous in $\mathbb{C}^+$, because if $s+it,\ s'+it'\in\mathbb{C}^+$, then

$$\begin{aligned} ||g(s+it) - g(s'+it')|| &\leq ||[T(s)-T(s')]\,[U(t)Ax]|| + ||T(s')[U(t)-U(t')]\,Ax|| \\ &\leq ||[T(s)-T(s')]\,[U(t)Ax]|| + Me^{as'}||[U(t)-U(t')]\,Ax|| \to 0 \end{aligned}$$

when $s'+it'\to s+it$, by (4) and the strong continuity of $T(\cdot)$ and $U(\cdot)$ (cf. [**5**, Theorem 1.1]).

Next, since A generates $T(\cdot)$ and iA generates $U(\cdot)$, it follows from the definition (3) of $W(\cdot)$ and [**5**, Theorem 1.2] that for all $s + it \in \mathbb{C}^+$

$$\frac{\partial}{\partial s} W(s+it)x = T(s)AU(t)x = -iT(s)(iA)U(t)x = -i\frac{\partial}{\partial t} W(s+it)x. \tag{6}$$

Thus, for each $x \in D(A)$ and $x^* \in X^*$, the complex valued function $x^* W(\cdot)x$ satisfies the Cauchy-Riemann equation and has *continuous partial derivatives* (by our observation on the function g) in $\mathbb{C}^+$. Therefore, by (the classical) Theorem 11.2 in [**8**], $x^* W(\cdot)x$ is holomorphic in $\mathbb{C}^+$ for all $x \in D(A)$ and $x^* \in X^*$.

Fix $x^* \in X^*$, and let $0 \neq x \in X$ be *arbitrary*. Since $D(A)$ is dense in X (cf. [**5**, Theorem 1.2]), we may choose a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$ in X and $||x_n|| \leq 2\,||x||$ for all $n \in \mathbb{N}$.

Let H be any compact subset of $\mathbb{C}^+$, and set

$$\sigma = \sigma(H) := \max_{z \in H} \Re z; \quad \tau = \tau(H) := \max_{z \in H} |\Im z|. \tag{7}$$

Then for all $s + it \in H$ and $n \in \mathbb{N}$,

$$|x^* W(s+it)x_n| \leq 2M\,N\,e^{a\sigma + b\tau}\,||x^*||\,||x||,$$

that is, $\{x^* W(\cdot)x_n;\, n \in \mathbb{N}\}$ is a family of holomorphic functions in $\mathbb{C}^+$, uniformly bounded on each compact subset of $\mathbb{C}^+$. It is therefore a *normal family* (cf. [**8**, Theorem 14.6]). Let then $\{x^* W(\cdot)x_{n_k}\}$ be a subsequence converging uniformly on compact subsets of $\mathbb{C}^+$. The limit function, $x^* W(\cdot)x$, is then holomorphic in $\mathbb{C}^+$ (cf. [**8**, Theorem 10.27])). Since this is true for all $x \in X$ and $x^* \in X^*$, it follows from [**3**, Theorem 3.10.1] that the operator valued function $W(\cdot)$ is holomorphic in $\mathbb{C}^+$. For all $s > 0$, we have $W(s) = T(s)U(0) = T(s)$ (by definition), so that $W(\cdot)$ is indeed a holomorphic extension of $T(\cdot)$ to $\mathbb{C}^+$.

For any $w = u + iv$ with $u > 0$ and $v \in \mathbb{R}$, the functions $W(\cdot)W(w)$ and $W(\cdot + w)$ are holomorphic in $\mathbb{C}^+$ and agree on $(0, \infty)$, since for all $s > 0$,

$$W(s)W(w) = T(s)T(u)U(v) = T(s+u)U(v) = W(s+u+iv) = W(s+w).$$

Therefore $W(z)W(w) = W(z+w)$ for all $z, w \in \mathbb{C}^+$.

The strong continuity at 0 of $W(\cdot)$ follows from (4) and the C_0-property of $T(\cdot)$ and $U(\cdot)$. Indeed, for all $x \in X$, $s > 0$, and $t \in \mathbb{R}$, we have

$$\begin{aligned} ||W(s+it)x - x|| &\leq ||T(s)[U(t)x - x]|| + ||T(s)x - x|| \\ &\leq M\,e^{as}||U(t)x - x|| + ||T(s)x - x|| \to 0 \end{aligned}$$

as $s \to 0+$ and $t \to 0$.

We conclude that $W(\cdot)$ is a *holomorphic semigroup* extending $T(\cdot)$ to $\mathbb{C}^+$. It is clearly bounded on the "unit rectangle" Q, since by (3) and (4), for all $z = s + it \in Q$,

$$||W(z)|| \leq ||T(s)||\,||U(t)|| \leq M\,N\,e^{a+b} < \infty.$$

This shows that $T(\cdot)$ is indeed a regular semigroup (with the said extension). □

REMARK. The normal family argument in the proof could be replaced by an application of Morera's theorem. Indeed, fix $x \in X$, $x^* \in X^*$, and let $x_n \in D(A)$ be such that

https://doi.org/10.1017/S0017089507003916 Published online by Cambridge University Press

$x_n \to x$. Then $x^* W(\cdot)x_n$ are holomorphic in $\mathbb{C}^+$ (by the first part of the proof, since $x_n \in D(A)$), and $x^* W(\cdot)x_n \to x^* W(\cdot)x$ pointwise. Let H be any triangular path in $\mathbb{C}^+$, and let $\sigma = \sigma(H)$ and $\tau = \tau(H)$ as in (7). Then

$$||x^* W(\cdot)x|| \leq MN\, e^{a\sigma + b\tau} ||x^*||\, ||x||$$

on H. By dominated convergence and analyticity of $x^* W(\cdot)x_n$ in $\mathbb{C}^+$,

$$\int_H x^* W(z)x\, dz = \lim_n \int_H x^* W(z)x_n\, dz = 0.$$

As in the argument showing the continuity of g in the proof above, it follows from (4) and the C_0 property of $T(\cdot)$ and $U(\cdot)$ that $x^* W(\cdot)x$ is continuous in $\mathbb{C}^+$, and Morera's theorem now implies its analyticity, as desired.

The theorem may be restated as a solution of the extension problem of a given *C_0-group* $U(\cdot)$ to a holomorphic semigroup in $\mathbb{C}^+$, whose boundary group is the group $U(\cdot)$. (Cf. [**3**, Theorem 17.10.1] for a more technical solution).

THEOREM 2.2. *Let $U(\cdot)$ be a C_0-group, and let iA be its generator. Then $U(\cdot)$ is the boundary group associated with a regular semigroup if and only if A generates a C_0-semigroup, $T(\cdot)$.*

In this case, $T(\cdot)$ is the unique regular semigroup with associated boundary group $U(\cdot)$, and the unique holomorphic extension of $U(\cdot)$ to $\mathbb{C}^+$ is given by (3).

3. Characterization of generators of regular semigroups. The characterization of a regular semigroup provided by Theorem 2.1, combined with various versions of the Hille-Yosida theorem (cf. [**5**, Theorem 1.17 and Corollary 1.18]), yield easily to characterizations of generators of regular semigroups.

COROLLARY 3.1. *Let A be an operator on X with domain $D(A)$. Then A generates a regular semigroup if and only if the following conditions* (a)–(c) *are satisfied:*

(a) *$D(A)$ is dense in X;*
(b) *the resolvent set of A contains the rays (a, ∞) and $\pm i(b, \infty)$, for some $a, b \geq 0$;*
(c) $\sup_{s>a;\, n\in\mathbb{N}} ||[(s-a)R(s;A)]^n|| < \infty$; *and* $\sup_{t>b;\, n\in\mathbb{N}} ||[(t-b)R((\pm it;A)]^n|| < \infty$.

Proof. If A generates a regular semigroup, then iA generates a C_0-group, and conditions (a)–(c) follow from the necessity part of the Hille-Yosida theorem for the generators A and iA and the relation

$$R(\lambda; iA) = -iR(-i\lambda; A). \tag{8}$$

Conversely, suppose A satisfies Conditions (a)–(c). Since A has a non-empty resolvent set (by (b)), it is a closed operator. Therefore, by (a), (b), and the first condition in (c), it follows from the Hille-Yosida theorem ([**5**, Theorem 1.17]) that A generates a C_0-semigroup. By (8), the second condition in (c) may be written in the form

$$\sup_{t>b;\, n\in\mathbb{N}} ||(t-b)R(\pm t; iA)]^n|| < \infty.$$

https://doi.org/10.1017/S0017089507003916 Published online by Cambridge University Press

Together with (a) and (b), this implies that iA generates a C_0-group (cf. [**5**, Theorem 1.39]). By Theorem 2.1, we conclude that A generates a regular semigroup. □

COROLLARY 3.2. *Let $T(\cdot)$ be a C_0-semigroup of contractions, and let A be its generator. Then $T(\cdot)$ extends to a holomorphic semigroup of contractions in $\mathbb{C}^+$ if and only if iA generates a C_0-group of contractions.*

(As observed before, since A is not necessarily "sectorial", this corollary does not follow from [**2**, Theorem II.4.6]; cf. Condition (e) in this theorem and [**2**, Definition II.4.1].)

The sufficient (and necessary) condition in Corollary 3.2 should be compared to the condition given in [**5**, Theorem 1.54] (see also [**6**]). There, in the general case of holomorphic extensions to arbitrary sectors

$$S_\theta := \{z = re^{i\phi};\ r > 0,\ -\theta < \phi < \theta\}$$

with $0 < \theta \leq \pi/2$, the (necessary and) sufficient condition is that $e^{i\alpha}A$ generate a C_0-semigroup of contractions for *all* $\alpha \in (-\theta, \theta)$. Here, in the particular case of holomorphic extension to the right halfplane (case $\theta = \pi/2$), we obtained the preceding (necessary and) sufficient condition involving only the endpoint values $\alpha = \pm\pi/2$ (namely, that $\pm iA$ generate C_0-semigroups of contractions).

In Liu [**7**], A is even assumed to have a bounded everywhere defined inverse, a condition that is not needed here.

Proof. If $T(\cdot)$ extends to a holomorphic semigroup of contractions in $\mathbb{C}^+$, it is trivially regular, and therefore iA generates the associated boundary group, that is necessarily a C_0-group of contractions. Conversely, if iA generates a C_0-group of contractions $U(\cdot)$, then by Theorem 2.1, $T(\cdot)$ is regular, and its unique extension as a holomorphic semigroup in $\mathbb{C}^+$ is $W(s + it) = T(s)U(t)$, that consists clearly of contractions. □

COROLLARY 3.3. *The operator A generates a holomorphic contraction semigroup in $\mathbb{C}^+$ if and only if the following conditions* (a) *and* (b) *are satisfied:*

(a) *$D(A)$ is dense in X;*

(b) *$sR(s; A)$ and $tR(\pm it; A)$ exist and are contractions for all $s, t > 0$.*

Proof. This follows from [**5**, Corollary 1.18], Corollary 3.2, and (8). □

The next two corollaries deal with perturbations of regular semigroups generators. They follow almost trivially from Theorem 2.1 and known perturbation theorems for C_0-semigroups, but do not seem to be found in the literature.

COROLLARY 3.4. *Let A generate a regular semigroup, and let $B \in B(X)$. Then $A + B$ generates a regular semigroup.*

Proof. By a special case of the Hille-Phillips perturbation theorem (cf. [**5**, Theorem 1.38]) and the necessity part of Theorem 2.1, the perturbations $A + B$ and $i(A + B) = (iA) + (iB)$ generate a C_0-semigroup and a C_0-group respectively. Therefore $A + B$ generates a regular semigroup, by the sufficiency part of Theorem 2.1. □

In the special case when A generates a holomorphic C_0-semigroup of *contractions* in $\mathbb{C}^+$ (*that is*, a C_0-semigroup of contractions $T(\cdot)$ on $[0,\infty)$ that extends to a holomorphic semigroup of contractions $W(\cdot)$ in $\mathbb{C}^+$), we may apply [**5**, Theorem 1.30] to get the following perturbation result. Recall that the *numerical range* of an operator B is the set

$$\nu(B) := \{x^*Bx;\; x \in D(B),\; x^* \in X^*,\; ||x|| = ||x^*|| = x^*x = 1\}.$$

Corollary 3.5. *Let A generate a holomorphic C_0-semigroup of contractions in $\mathbb{C}^+$. Let B be an operator satisfying the following conditions* (a) *and* (b)*:*

(a) $\nu(B) \subset (-\infty, 0]$*;*

(b) $D(A) \subset D(B)$ *and there exist constants* $0 \le a < 1$ *and* $b \ge 0$ *such that*

$$||Bx|| \le a\,||Ax|| + b\,||x||$$

for all $x \in D(A)$.

Then $A + B$ generates a holomorphic C_0-semigroup of contractions in $\mathbb{C}^+$.

Proof. By [**5**, Theorem 1.30], $A + B$ generates a C_0-semigroup of contractions. By Corollary 3.2, $\pm iA$ generate C_0-semigroups of contractions. By Condition (a),

$$\Re\nu(\pm iB) = \mp\Im\nu(B) = 0,$$

and therefore iB and $-iB$ are trivially *dissipative*. They are also iA*-bounded* and $-iA$-bounded (respectively), with iA*-bound* ($-iA$-bound, respectively) smaller than 1 (by Condition (b)). Consequently, by [**5**, Theorem 1.30], the operators $i(A + B)$ and $-i(A + B)$ generate C_0-semigroups of contractions. We now conclude from Corollary 3.2 that $A + B$ generates a holomorphic C_0-semigroup of contractions. □

Our last corollary gives a growth estimate of a regular semigroup $T(\cdot)$ in terms of any fixed value of the norm $||T(c)||$ (the result may be new).

Corollary 3.6. *Let $T(\cdot)$ be a regular semigroup. Let $U(\cdot)$ be its boundary group, and let $b \ge 0$ and $N \ge 1$ be constants such that $||U(t)|| \le N\,e^{b|t|}$ for all $t \in \mathbb{R}$ (cf. (4)). Fix $c > 0$. Then*

$$||T(s)|| \le N\,e^{bc/2}||T(c)||^{s/c}$$

for all $s > 0$.

Proof. It suffices to prove the estimate for $c = 1$, since the general case follows from this special case applied to the regular semigroup $T'(s) := T(cs)$. Since the estimate is trivial for $s \in \mathbb{N}$ (if $s = n \in \mathbb{N}$, then $||T(s)|| = ||T(1)^n|| \le ||T(1)||^n \le N\,e^{b/2}||T(1)||^s$, since $N \ge 1$ and $b \ge 0$), it suffices to consider non-integral $s > 0$. This will follow in turn from the special case $0 < s < 1$, because writing $s = n + t$ with n a non-negative integer and $0 < t < 1$, we get (from the said special case)

$$\begin{aligned}||T(s)|| &= ||T(1)^n T(t)|| \le ||T(1)||^n ||T(t)|| \\ &\le N\,e^{b/2}||T(1)||^t ||T(1)||^n = N\,e^{b/2}||T(1)||^s.\end{aligned}$$

By Theorem 2.1, the holomorphic extension of $T(\cdot)$ to $\mathbb{C}^+$ is given by $W(s + it) = T(s)U(t)$. Consider the operator-valued continuous function $\Phi(z) = e^{bz^2}\,W(z)$ on $\overline{\mathbb{C}^+}$.

https://doi.org/10.1017/S0017089507003916 Published online by Cambridge University Press

It is holomorphic in $\mathbb{C}^+$, and

$$||\Phi(s+it)|| = e^{b(s^2-t^2)}||T(s)U(t)|| \leq N\, \exp(b[s^2 - t^2 + |t|])\, ||T(s)||$$
$$\leq N\, \exp(b[s^2 + 1/4])\, ||T(s)|| \tag{9}$$

(because $-t^2 + |t| = |t|(1-|t|) \leq 1/4$). By (9), Φ is bounded in the vertical strip $\{s+it;\ 0 \leq s \leq 1,\ t \in \mathbb{R}\}$. Also, for all $t \in \mathbb{R}$,

$$||\Phi(it)|| \leq N\, e^{b/4}; \qquad ||\Phi(1+it)|| \leq N\, e^{b(1+1/4)}\, ||T(1)||. \tag{10}$$

If $s \in (0, 1)$, write s as the convex combination $s = p.0 + (1-p).1 = 1-p$ with $p \in (0, 1)$. By the "Three Lines theorem" for operator-valued holomorphic functions (cf. [**1**, Theorem VI.10.3]), it follows from (10) that for all $s \in (0, 1)$ and $t \in \mathbb{R}$,

$$||\Phi(s+it)|| \leq N\, e^{b/4}\, e^{bs}||T(1)||^s.$$

Taking $t = 0$, it follows from (9) that for all $s \in (0, 1)$

$$||T(s)|| \leq N\, e^{b/4} e^{bs(1-s)}||T(1)||^s \leq N\, e^{b/2}||T(1)||^s$$

(because $s(1-s) \leq 1/4$ for $s \in (0, 1)$). □

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*, Volume 1 (Interscience Publishers, New York, 1958).

2. K. Engel and R. Nagel, *One parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics (Springer-Verlag, 2000).

3. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Coll. Publ. **31** (Providence, R.I., 1957).

4. S. Kantorovitz, *Spectral theory of Banach space operators*, Lecture Notes in Math., Vol. 1012 (Springer-Verlag, 1983).

5. S. Kantorovitz, *Semigroups of operators and spectral theory*, Pitman Research Notes in Math., Vol. 330 (Longman, New York, 1995).

6. S. Kantorovitz, On Liu's analyticity criterion for semigroups, *Semigroup Forum* **53** (1996), 262–265.

7. Y. Liu, An equivalent condition for analytic C_0-semigroups, *J. Math. Anal. Appl.* **180** (1993), 71–78.

8. W. Rudin, *Real and complex analysis* (McGraw-Hill, 1966).